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On the excess of complex exponential systems in $L^2(-a, a)$

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Abstract

In this article we give new criteria for two complex sequences to have the same *excess* in the sense of Paley and Wiener in $L^2(-a, a)$. As a result, we prove that given any positive integer q, a real number $\alpha \in (0, 1/(2\pi))$, and complex numbers

$$\nu_0 = 0, \qquad \nu_n = nq + i\alpha \log|nq|, \quad |n| \geqslant 1,$$

the exponential system $\{t^k e^{it\nu_n}\colon k = 0, 1, \dots, q-1\}_{n=-\infty}^{\infty}$ has excess 0 in $L^2(-\pi, \pi)$.

Keywords: Exponential systems; Excess

1. Introduction

The main sources for this article are the survey papers of R. Redheffer [8] and of A. Sedletskii [14], as well as the excellent expository account [15] of R. Young. Some other interesting results are found in the joint paper of R. Redheffer and R. Young [9]. We shall also assume that the reader is familiar with the theory of Entire Functions of Exponential Type, as treated in the books of Boas [1], Levinson [7], and Levin [5,6].

Let $\mu=\{\mu_n, k_n\}_{n=1}^{\infty}$ be a multiplicity sequence, that is, $\{\mu_n\}$ are distinct complex numbers satisfying $|\mu_n| \leqslant |\mu_{n+1}| \mapsto \infty$ as $n \mapsto \infty$, and k_n are positive integers showing the

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multiplicity of μ_n. We associate with this sequence the exponential system

$$E_\mu = \left\{t^{k-1}e^{i\mu_n t}\colon\ 1 \leqslant k \leqslant k_n\right\}. \tag{1.1}$$

For $1 \leqslant p < \infty$, we say that the system E_μ is complete in $L^p(-a,a)$ if $\overline{\operatorname{span}}\, E_\mu = L^p(-a,a)$. By the Hahn–Banach theorem, incompleteness is equivalent to the existence of a non-trivial entire function $F(z)$ which vanishes on μ and which has the integral representation

$$F(z) = \int_{-a}^{a} e^{izt} f(t)\,dt, \quad f \in L^q(-a,a),\ \frac{1}{p} + \frac{1}{q} = 1. \tag{1.2}$$

Complete systems which become incomplete on the removal of a single term are called *exact*. The most classical example is the trigonometric system $\{e^{int}\}_{n=-\infty}^{\infty}$ which is *exact* in $L^p(-\pi,\pi)$ for all $p \in [1,\infty)$ and whose *excess* in $C[-\pi,\pi]$ is equal to -1. By the term *excess* $E(\mu; p, a)$, we mean the number of terms that have to be removed from (added to) the system in order for E_μ to become *exact* in $L^p(-a,a)$.

We note that in order for a system E_μ to have a finite *excess*, a necessary condition is for μ to belong to (what we shall refer to) the class $\mathcal{B}$. The elements of $\mathcal{B}$ are all the *two-sided* sequences μ, that is, $\mu = \{\mu_n, k_n\}_{-\infty}^{\infty}$ where $\Re\mu_n \geqslant 0$ for $n > 0$ and $\Re\mu_n < 0$ for $n < 0$, that have a finite upper density, their exponent of convergence is equal to 1 and the series $\sum_{n=-\infty}^{\infty} |\Im\mu_n| k_n/|\mu_n|^2$ converges. We shall also denote by $\mathcal{B}'$ the subclass of $\mathcal{B}$, where in addition the terms μ_n of some sequence $\mu \in \mathcal{B}$ satisfy

$(*)$ $|\mu_n - \mu_{n+1}| \leqslant c$ for some $c > 0$, $-\infty < n < \infty$.

Our goal in the present paper is, given a system E_μ with $\mu \in \mathcal{B}'$, to give a general way to generate another system E_ν with the same *excess* in $L^2(-a,a)$. This new sequence ν may have radically different geometric properties. Such an example is provided in Corollary 1.1, where we start with all the terms of μ having multiplicity 1 and construct ν whose terms have multiplicity $q \in \mathbf{N}$. The construction of ν is based on partitioning μ into at most three sets (see the $\mathcal{P}_{\mu,\delta}$ partition below) and then subjecting two of them to a bounded perturbation.

Our method is particularly useful for systems E_μ having a finite *excess*, even if $\sup|\Im\mu_n| = \infty$. In [13, Theorem 3], A. Sedletskii constructed exact systems with unbounded imaginary parts. We state a special form of his result as a theorem.

Theorem 1.1. *If $\mu = \{\mu_n, 1\}_{-\infty}^{\infty}$ with $\mu_0 = 0$ and $\mu_n = n + i\alpha\log|n|$ for $|n| \geqslant 1$ and $\alpha \in (0, 1/(2\pi))$, then the system E_μ is exact in $L^2(-\pi,\pi)$.*

A classical way of deriving equivalent systems is by the Alexander–Redheffer theorem [8, Theorem 14], which states that for all $p \in [1,\infty)$ the *excesses* $E(\nu; p, a)$ and $E(\mu; p, a)$ of two exponential systems E_μ and E_ν are equal, assuming that

$$\sum_{n=1}^{\infty} \frac{|\mu_n - \nu_n|}{1 + |\Im\mu_n| + |\Im\nu_n|} < \infty.$$

This theorem is interesting on its own because it does not assume any regularity in the distribution of either of the individual sequences. However, the necessary condition $|\nu_n - \mu_n| \mapsto 0$ for the convergence does not provide a large class of examples. Nevertheless, this theorem is still the most celebrated result for systems having the same excess. In fact, it has been generalized for spaces of functions on arcs, other than the interval, and on domains, by Bulat Khabibullin [4].

When A. Sedletskii searches for equivalent systems, he usually imposes the condition that μ is a non-concentrated sequence where the $\sup|\Im\mu_n| < \infty$. A sequence μ is called non-concentrated if $n_\mu(t+1) - n_\mu(t) = O(1)$ where $n_\mu(t)$ is the counting function. But the condition $\sup|\Im\mu_n| < \infty$ is obviously a limitation when one wants to derive other *exact* systems from the one in Theorem 1.1.

Thus, these later conditions and the Alexander–Redheffer theorem are inadequate for what we want to prove. As mentioned before, our method yields equivalent systems E_μ and E_ν, with their sequences μ and ν having different geometric properties, even when their imaginary parts are unbounded.

For the rest of this article when we write a sequence $\{p_n\}'$, a series $\sum'$, or a product $\prod'$, we mean that the index n is running through all $n \in \mathbf{Z} \setminus \{0\}$.

1.1. The $\mathcal{P}_{\mu,\delta}$ partition of some $\mu \in \mathcal{B}'$ and the construction of ν

Let $\mu \in \mathcal{B}'$. We will partition μ into at most three sets, not necessarily disjoined:

$$\mu = \{\gamma_n\}' \cup \{\lambda_n\}' \cup \{\rho_n\},$$

where $\{\rho_n\}$ might be infinite, finite or empty. This is done as follows: Fix $\delta > 0$ so that $\delta \geqslant c$ and write μ as $\mu = \{\mu_{n,k}\colon k = 1, 2, \ldots, k_n\}_{n=-\infty}^{\infty}$. Consider the closed disks $\overline{B}_{n,k} = \overline{B}(\mu_{n,k}, \delta) = \{z\colon |\mu_{n,k} - z| \leqslant \delta\}$. Since $\delta \geqslant c$, then in each $\overline{B}_{n,k}$ there are at least two elements of μ. Thus, we pair $\mu_{n,k}$ with at most one other element of μ which is in $\overline{B}_{n,k}$, and once paired together, they cannot be paired with other ones. Thus, two subsets of μ are constructed, not necessarily disjoined and each containing one of the two elements. We call them $\{\gamma_n\}'$ and $\{\lambda_n\}'$, γ_n is paired with λ_n and satisfy $|\gamma_n - \lambda_n| \leqslant \delta$. It is not necessary to have $|\gamma_{|n|}| \leqslant |\gamma_{|n+1|}|$ or $|\lambda_{|n|}| \leqslant |\lambda_{|n+1|}|$ (see Example 1.1). The remaining (if any) terms of μ, we call them $\{\rho_n\}$, are totally independent, that is, they do not participate in the pairing. We shall refer to such a partition by $\mathcal{P}_{\mu,\delta}$.

Then for some two-sided, bounded sequence of complex numbers $\{a_n\}'$, we define the new sequence ν as

$$\nu = \{\gamma_n + a_n\}' \cup \{\lambda_n - a_n\}' \cup \{\rho_n\}. \tag{1.3}$$

The following example illustrates the above construction.

Example 1.1. We present a $\mathcal{P}_{\mu,\delta}$ partition when $\mu = \mathbf{Z}$ and $\delta = 4$. From this we construct a new sequence ν. Let $\{\rho_n\} = \{-2, -1, 1, 2\} \cup \{5n\}_{-\infty}^{\infty}$, that is $\{\rho_n\} = \{0, \pm 1, \pm 2, \pm 5, \pm 10, \ldots\}$, and let

$$\begin{aligned}
&\gamma_1 = 3, \quad \gamma_2 = 4, \quad \gamma_3 = 8, \quad \gamma_4 = 9, \quad \gamma_5 = 13, \quad \gamma_6 = 14, \quad \ldots, \\
&\lambda_1 = 7, \quad \lambda_2 = 6, \quad \lambda_3 = 12, \quad \lambda_4 = 11, \quad \lambda_5 = 17, \quad \lambda_6 = 16, \quad \ldots;
\end{aligned} \tag{1.4}$$

that is, for $n \geqslant 1$ (similarly for $n \leqslant -1$) we put

$$\gamma_n = \begin{cases} \frac{5n+1}{2}, & n \text{ odd}, \\ \frac{5n-2}{2}, & n \text{ even}, \end{cases} \qquad \lambda_n = \begin{cases} \frac{5n+9}{2}, & n \text{ odd}, \\ \frac{5n+2}{2}, & n \text{ even}. \end{cases} \tag{1.5}$$

For $n \geqslant 1$, take $a_{2n-1} = 2$, $a_{2n} = 1$ and for $n \leqslant -1$, take $a_{2n+1} = -2$, $a_{2n} = -1$. Then from (1.3) the new sequence ν is $\{0, \pm 1, \pm 2, \pm 5, \pm 10, \pm 15, \ldots\}$, with all the terms having multiplicity 5, except $\{0, \pm 1, \pm 2\}$ whose multiplicity is 1.

We now state our main result which is the following:

Theorem 1.2. *Let $\mu \in \mathcal{B}'$ and, for some $\delta > 0$ fixed let $\mathcal{P}_{\mu,\delta}$ be a corresponding partition. Let $\{a_n\}'$ be a two-sided bounded sequence of real numbers and ν as in* (1.3). *Then the relation $E(\nu; 2, a) = E(\mu; 2, a)$ holds. If $\inf \Im \mu_n \geqslant u \in \mathbf{R}$, we may choose the $\{a_n\}'$ to be a sequence of complex numbers instead of real.*

Corollary 1.1. *Given any positive integer q, a real number $\alpha \in (0, 1/(2\pi))$ and complex numbers*

$$\nu_0 = 0, \qquad \nu_n = nq + i\alpha \log |nq|, \quad |n| \geqslant 1,$$

then the sequence $\nu = \{\nu_n, q\}_{-\infty}^{\infty}$ yields the following exact system in $L^2(-\pi, \pi)$:

$$\left\{t^k e^{it\nu_n}\colon k = 0, 1, \ldots, q-1\right\}_{n=-\infty}^{\infty}. \tag{1.6}$$

Moreover, we may construct a sequence $\nu = \{\nu_n, k_n\}$ with *different* multiplicities k_n so that for $\alpha \in (0, 1/(2\pi))$ one has the following result:

Corollary 1.2. *The exponential system*

$$\left\{t^k e^{it(8n+i\alpha\log 8n)}\colon k = 0, 1, 2\right\} \cup \left\{t^k e^{it[8n-4+i\alpha\log(8n-4)]}\colon k = 0, 1, 2, 3, 4\right\} \tag{1.7}$$

is exact in $L^2(-\pi, \pi)$.

2. Some additional results

In this section we first prove Corollary 1.1 and then present some further results.

Proof of Corollary 1.1. We consider the case $q = 5$. For other values of q, the proof is similar.

Change the five terms μ_n for $-2 \leqslant n \leqslant 2$ into five zeros. By a theorem of Levinson [7, Theorem VI] the *excess* is not altered. Next, for every n such that $|n| \geqslant 1$, keep the terms μ_{5n} fixed and shift vertically the terms $\mu_{5n+\beta}$ for $\beta \in \{-2, -1, 1, 2\}$, so that their new imaginary part is equal to $\Im \mu_{5n} = \alpha \log |5n|$. Observe that this vertical shifting is bounded, thus from [8, Theorem 17] the *excess* does not change. Write these new terms as $\{5n + \beta + i\alpha \log |5n|\colon \beta \in \{-2, -1, 0, 1, 2\}\}_{|n| \geqslant 1}$. We then proceed with a partition as in relations (1.4) and (1.5), that is, for $|n| \geqslant 1$ keep $5n + i\alpha \log |5n|$ fixed and pair $5n - 2 + i\alpha \log |5n|$

with $5n+2+i\alpha\log|5n|$ and $5n-1+i\alpha\log|5n|$ with $5n+1+i\alpha\log|5n|$. Then carry out the same shifting as in Example 1.1 to get $5n+i\alpha\log|5n|$ with multiplicity 5. By Theorem 1.2, the result is valid. $\square$

Similarly one proves Corollary 1.2.

When $\mu=\mathbf{Z}$, the set of integers, a more general result holds. Although we feel that this result might be known to some people, nevertheless, since we could not trace it in the literature, we state it here.

Theorem 2.1. *Let $\mathcal{P}_{\mathbf{Z},\delta}$ be a partition for some $\delta>0$ fixed. Let $\{a_n\}'$ be a bounded two-sided sequence of complex numbers and define ν as in* (1.3). *Then $E(Z;p,\pi)=E(\nu;p,\pi)$ for all $p\in(1,\infty)$.*

Remark 2.1. The theorem fails for $L^1(-\pi,\pi)$ and $C[-\pi,\pi]$.

Two further results will be proved where the condition $\sup|\Im\mu_n|<\infty$ is once more not essential. The first one generalizes the following recent result from [10].

Theorem 2.2. *Let $\{\mu_n\}_{-\infty}^{\infty}$ be a sequence satisfying $|\mu_n-n|\leqslant c$ for some $c>0$. Let $\lambda_0=\mu_0$,*

$$\lambda_n=\mu_n+\alpha,\quad \lambda_{-n}=\mu_{-n}-\beta,\quad n>0, \tag{2.1}$$

where $\alpha\geqslant 0$ and $\beta\geqslant 0$. Then $E(\lambda;2,\pi)\leqslant E(\mu;2,\pi)$.

The authors of [10] asked whether their result remains true, assuming that $|\Im\mu_n|\leqslant c$ and $|\mu_n+\mu_{-n}|\leqslant 2c$, instead of $|\mu_n-n|\leqslant c$. The answer is affirmative, and in fact the assumption that the imaginary parts are bounded is not required. Our result is as follows:

Theorem 2.3. *Let $\mu\in\mathcal{B}$ and assume that for some $c>0$ the condition $|\mu_n+\mu_{-n}|\leqslant c$ is satisfied for every $n\geqslant 1$. Assume also that the condition $|\Re\mu_n|\geqslant(\Im\mu_n)^2$ holds for $|n|\geqslant 1$ and suppose that $E(\mu;2,a)$ is finite for some $a>0$. Let $\lambda_0=\mu_0$,*

$$\lambda_n=\mu_n+\alpha,\quad \lambda_{-n}=\mu_{-n}-\beta,\quad n>0, \tag{2.2}$$

where $\alpha\geqslant 0$ and $\beta\geqslant 0$. Then $E(\lambda;2,a)\leqslant E(\mu;2,a)$.

We note that another generalization of Theorem 2.2 with unbounded imaginary parts is due to A. Boivin and H. Zhong [3].

If we now combine Theorems 1.2 and 2.3, another more interesting result is obtained. The constants α and β may be replaced by a bounded two-sided real sequence $\{\epsilon_n\}'$, subject to the condition

$$\epsilon_{2n-1}+\epsilon_{2n}=\delta_1\geqslant 0,\quad n\geqslant 1,\qquad \epsilon_{2n+1}+\epsilon_{2n}=\delta_2\geqslant 0,\quad n\leqslant -1. \tag{2.3}$$

Theorem 2.4. *Let μ be as in Theorem* 2.3 *and assume that $\mu\in\mathcal{B}'$ as well. Let $\mathcal{P}_{\mu,\delta}$ be a partition with the set $\{\rho_n\}$ finite and let $\{\epsilon_n\}'$ be a bounded two-sided real sequence satisfying* (2.3). *Then construct the sequence*

$$\nu=\{\gamma_n+\epsilon_{2n-1}\}_1^{\infty}\cup\{\lambda_n+\epsilon_{2n}\}_1^{\infty}\cup\{\gamma_n-\epsilon_{2n+1}\}_{-1}^{-\infty}\cup\{\lambda_n-\epsilon_{2n}\}_{-1}^{-\infty}\cup\{\rho_n\}.$$

Assuming that $E(\mu; 2, a)$ *is finite for some* $a > 0$, *the relation* $E(\nu; 2, a) \leqslant E(\mu; 2, a)$ *holds.*

Proof. As usual, write $\mu = \{\gamma_n\}' \cup \{\lambda_n\}' \cup \{\rho_n\}$. Then construct a new sequence $\tau = \{\tau_n\}' \cup \{\rho_n\}$ so that

$$\tau_{2n-1} = \gamma_n + \frac{\epsilon_{2n-1} - \epsilon_{2n}}{2}, \quad \tau_{2n} = \lambda_n - \frac{\epsilon_{2n-1} - \epsilon_{2n}}{2}, \quad n \geqslant 1,$$

$$\tau_{2n+1} = \gamma_n - \frac{\epsilon_{2n+1} - \epsilon_{2n}}{2}, \quad \tau_{2n} = \lambda_n + \frac{\epsilon_{2n+1} - \epsilon_{2n}}{2}, \quad n \leqslant -1.$$

Since $\{\epsilon_n\}'$ is bounded, the fractions are uniformly bounded also. It follows from Theorem 1.2 that $E(\tau; 2, a) = E(\mu; 2, a)$.

Next, observe that one obtains ν by shifting to the right (left) all the terms of $\{\tau_n\}'$ with positive (negative) index n, by the same amount δ_1 (δ_2). This holds since $\gamma_n + \epsilon_{2n-1} = \tau_{2n-1} + \delta_1/2$ and $\lambda_n + \epsilon_{2n} = \tau_{2n} + \delta_1/2$ for $n \geqslant 1$. Similarly $\gamma_n - \epsilon_{2n+1} = \tau_{2n+1} - \delta_2/2$ and $\lambda_n - \epsilon_{2n} = \tau_{2n} - \delta_2/2$ for $n \leqslant -1$. Then from Theorem 2.3 one has that $E(\nu; 2, a) \leqslant E(\tau; 2, a)$. The relation $E(\nu; 2, a) \leqslant E(\mu; 2, a)$ is now obvious. □

As a special case of Theorem 2.4, let take $\epsilon_{2n-1} = \delta_1$ and $\epsilon_{2n} = 0$. Then only *half* of the terms are shifted and the inequality still holds. We also note that similar results with inequalities, but with *real* sequences, are found in [12, Theorem 1].

The rest of this article is divided into three sections. Our main result, Theorem 1.2, is proved in Section 4. In its proof, a crucial role is played by a meromorphic function whose properties are discussed in Section 3. In Section 4 we also prove Theorem 2.1. Theorem 2.3 is proved in Section 5.

3. Constructing a meromorphic function that replaces frequencies

Throughout Section 3, we assume that $\mu \in \mathcal{B}'$ with $\Im\mu_n \geqslant 0$ for all $n \in \mathbf{Z}$. For $\delta > 0$ fixed, $\mathcal{P}_{\mu,\delta}$ is the partition of μ, $\mu = \{\gamma_n\}' \cup \{\lambda_n\}' \cup \{\rho_n\}$. For the two-sided bounded sequence of complex numbers $\{a_n\}'$, we construct the sequence ν as in (1.3).

A well-known theorem of Plancherel–Polya [15, Theorem 16, p. 79] states that if a function $F(z)$ of exponential type belongs to $L^p(-\infty, \infty)$, then $F(x - it) \in L^p(-\infty, \infty)$ for any $t \in \mathbf{R}$. Motivated by this, we define for every $t \in (0, \infty)$ the function

$$\prod{}' \frac{\left(1 - \frac{z}{\gamma_n + a_n}\right)\left(1 - \frac{z}{\lambda_n - a_n}\right) e^{z/(\gamma_n + a_n) + z/(\lambda_n - a_n)}}{\left(1 - \frac{z - it}{\gamma_n}\right)\left(1 - \frac{z - it}{\lambda_n}\right) e^{(z - it)/\gamma_n + (z - it)/\lambda_n}}. \tag{3.1}$$

Standard calculations show that (3.1) defines a meromorphic function of z in the complex plane with poles at $\{\gamma_n + it\} \cup \{\lambda_n + it\}$. Note also that since the exponent of convergence for μ is 1 and $\{a_n\}'$ is bounded, then the series

$$\sum{}' \left(\frac{1}{\gamma_n} - \frac{1}{\gamma_n + a_n} + \frac{1}{\lambda_n} - \frac{1}{\lambda_n - a_n} \right)$$

converges to some $\omega \in \mathbf{C}$. Thus, multiplication of $e^{\omega z}$ with the function in (3.1) for fixed $t \in (0, \infty)$ yields the meromorphic function of z:

$$\prod{}' \frac{\left(1 - \frac{z}{\gamma_n + a_n}\right)\left(1 - \frac{z}{\lambda_n - a_n}\right)}{\left(1 - \frac{z - it}{\gamma_n}\right)\left(1 - \frac{z - it}{\lambda_n}\right)} e^{it/\gamma_n + it/\lambda_n}. \tag{3.2}$$

We denote this function by $M(z, t)$ and remark that for some $t = t_0$, $M(z, t_0)$ has a certain upper bound on the real line (see Proposition 3.1) which is very crucial for proving Theorem 1.2. The key to all these is the following lemma.

Lemma 3.1. *There exists a positive t_0 such that for any $n \in \mathbf{Z} \setminus \{0\}$ and all $x \in \mathbf{R}$, one has*

$$\left| \frac{(\gamma_n + a_n - x)(\lambda_n - a_n - x)}{(\gamma_n - x + it_0)(\lambda_n - x + it_0)} \right| \leqslant 1. \tag{3.3}$$

Proof. When the $\{a_n'\}$ are imaginary numbers, the proof is rather easy. Thus, we will prove it for the real case, and as a result the complex case follows as well. Let

$$(\mathrm{I}) = \left|(\gamma_n - x + it)(\lambda_n - x + it)\right|^2, \qquad (\mathrm{II}) = \left|(\gamma_n + a_n - x)(\lambda_n - a_n - x)\right|^2.$$

Denote the quantity (I) − (II) by $g_n(x, t)$. Observe that relation (3.3) is proved as soon as we show that there is some $t = t_0 > 0$, independent of n and x, so that $g_n(x, t_0) \geqslant 0$ for any $n \in \mathbf{Z} \setminus \{0\}$ and all $x \in \mathbf{R}$.

One has:

$$\begin{aligned}
(\mathrm{I}) &= \left[(\Re\gamma_n - x)^2 + (\Im\gamma_n + t)^2\right]\left[(\Re\lambda_n - x)^2 + (\Im\lambda_n + t)^2\right] \\
&= (\Re\gamma_n - x)^2(\Re\lambda_n - x)^2 + (\Im\gamma_n + t)^2(\Im\lambda_n + t)^2 \\
&\quad + (\Im\gamma_n + t)^2(\Re\lambda_n - x)^2 + (\Im\lambda_n + t)^2(\Re\gamma_n - x)^2 \\
&= (\Re\gamma_n - x)^2(\Re\lambda_n - x)^2 + \omega_n(t) + \tau_n(t)(\Re\gamma_n - x)^2 + \sigma_n(t)(\Re\lambda_n - x)^2,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
&\omega_n(t) = (\Im\gamma_n + t)^2(\Im\lambda_n + t)^2, \qquad \tau_n(t) = (\Im\lambda_n + t)^2, \\
&\sigma_n(t) = (\Im\gamma_n + t)^2.
\end{aligned} \tag{3.5}$$

Similarly

$$\begin{aligned}
(\mathrm{II}) &= \left[(\Re\gamma_n - x + a_n)^2 + (\Im\gamma_n)^2\right]\left[(\Re\lambda_n - x - a_n)^2 + (\Im\lambda_n)^2\right] \\
&= \left[(\Re\gamma_n - x)^2 + 2a_n(\Re\gamma_n - x) + p_n\right]\left[(\Re\lambda_n - x)^2 - 2a_n(\Re\lambda_n - x) + q_n\right]
\end{aligned}$$

where

$$q_n = a_n^2 + (\Im\lambda_n)^2, \qquad p_n = a_n^2 + (\Im\gamma_n)^2. \tag{3.6}$$

If we expand the terms, we get:

$$\begin{aligned}
(\mathrm{II}) &= (\Re\gamma_n - x)^2(\Re\lambda_n - x)^2 - 2a_n(\Re\lambda_n - x)(\Re\gamma_n - x)^2 + q_n(\Re\gamma_n - x)^2 \\
&\quad + 2a_n(\Re\gamma_n - x)(\Re\lambda_n - x)^2 - 4a_n^2(\Re\gamma_n - x)(\Re\lambda_n - x) + 2a_nq_n(\Re\gamma_n - x) \\
&\quad + p_n(\Re\lambda_n - x)^2 - 2a_np_n(\Re\lambda_n - x) + p_nq_n \\
&= (\Re\gamma_n - x)^2(\Re\lambda_n - x)^2 + q_n(\Re\gamma_n - x)^2 + p_n(\Re\lambda_n - x)^2 \\
&\quad + \xi_n(\Re\gamma_n - x)(\Re\lambda_n - x) + 2a_nq_n(\Re\gamma_n - x) - 2a_np_n(\Re\lambda_n - x) \\
&\quad + p_nq_n
\end{aligned} \tag{3.7}$$

where

$$\xi_n = 2a_n(\Re\lambda_n - \Re\gamma_n) - 4a_n^2.$$

Since $a_n = O(1)$ and $|\lambda_n - \gamma_n| = O(1)$, the $\sup|\xi_n| < \infty$. From now on, we let $t \gg \sup|\xi_n|$.

Since $g_n(x,t) = (\mathrm{I}) - (\mathrm{II})$, from (3.4) and (3.7) one gets

$$\begin{aligned}
g_n(x,t) &= [\tau_n(t) - q_n](\Re\gamma_n - x)^2 + [\sigma_n(t) - p_n](\Re\lambda_n - x)^2 + \omega_n(t) \\
&\quad + \Upsilon_n(x),
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\Upsilon_n(x) &= -\xi_n(\Re\gamma_n - x)(\Re\lambda_n - x) - 2a_nq_n(\Re\gamma_n - x) + 2a_np_n(\Re\lambda_n - x) \\
&\quad - p_nq_n.
\end{aligned} \tag{3.9}$$

Observe now that since $\{a_n\}'$ is bounded, for large t fixed we have

$$\tau_n(t) - q_n \approx t^2 + 2t\Im\lambda_n, \qquad \sigma_n(t) - p_n \approx t^2 + 2t\Im\gamma_n. \tag{3.10}$$

Since $t \gg \sup|\xi_n|$, both quantities above are bigger than the $\sup|\xi_n|$, and this implies that the coefficient of x^2 in (3.8) is positive. Thus for t fixed and large enough, $g_n(x,t)$ has a minimum. Our goal is to prove that, for t fixed, $g_n(x,t)$ is non-negative there, thus everywhere else as well. This suffices to complete the proof.

We differentiate $g_n(x,t)$ with respect to x to get:

$$\begin{aligned}
g_n'(x,t) &= 2(x - \Re\gamma_n)\big(t^2 + 2t\Im\lambda_n\big) + 2(x - \Re\lambda_n)\big(t^2 + 2t\Im\gamma_n\big) - \xi_n(x - \Re\gamma_n) \\
&\quad - \xi_n(x - \Re\lambda_n) + 2a_nq_n - 2a_np_n \\
&= 2(x - \Re\gamma_n)\big[t^2 + 2t\Im\lambda_n - \xi_n\big] + 2(x - \Re\lambda_n)\big(t^2 + 2t\Im\gamma_n - \xi_n\big) \\
&\quad + 2a_nq_n - 2a_np_n.
\end{aligned} \tag{3.11}$$

It follows that $g_n'(x,t) = 0$ when

$$\begin{aligned}
x &= \frac{a_n(p_n - q_n)}{2t^2 - 2\xi_n + 2t\Im\gamma_n + 2t\Im\lambda_n} + \frac{\Re\gamma_n}{2}\left(\frac{t^2 - \xi_n + 2t\Im\lambda_n}{t^2 - \xi_n + t\Im\gamma_n + t\Im\lambda_n}\right) \\
&\quad + \frac{\Re\lambda_n}{2}\left(\frac{t^2 - \xi_n + 2t\Im\gamma_n}{t^2 - \xi_n + t\Im\gamma_n + t\Im\lambda_n}\right).
\end{aligned} \tag{3.12}$$

Consider now the first fraction. From (3.6) one has

$$\frac{a_n(p_n - q_n)}{2t^2 - 2\xi_n + 2t\Im\gamma_n + 2t\Im\lambda_n} = \frac{a_n(\Im\gamma_n - \Im\lambda_n)(\Im\gamma_n + \Im\lambda_n)}{2t^2 - 2\xi_n + 2t(\Im\gamma_n + \Im\lambda_n)}.$$

Since $a_n = O(1)$, $|\gamma_n - \lambda_n| = O(1)$ and $\xi_n = O(1)$, it follows that for large t the fraction is very *small*. Call this fixed $t > 0$, t_0. Since $t_0 \gg \sup|\xi_n|$, then ξ_n has no effect in the other two fractions of (3.12). All these imply that $g_n'(x, t_0)$ takes its minimum value at $x = x_0$ where

$$x_0 \approx \frac{\Re\gamma_n}{2}\left(\frac{t_0 + 2\Im\lambda_n}{t_0 + \Im\gamma_n + \Im\lambda_n}\right) + \frac{\Re\lambda_n}{2}\left(\frac{t_0 + 2\Im\gamma_n}{t_0 + \Im\gamma_n + \Im\lambda_n}\right). \tag{3.13}$$

This implies that

$$\Re\gamma_n - x_0 \approx \left(\frac{\Re\gamma_n - \Re\lambda_n}{2}\right)\left(\frac{t_0 + 2\Im\gamma_n}{t_0 + \Im\gamma_n + \Im\lambda_n}\right) \quad \text{and} \tag{3.14}$$

$$\Re\lambda_n - x_0 \approx \left(\frac{\Re\lambda_n - \Re\gamma_n}{2}\right)\left(\frac{t_0 + 2\Im\lambda_n}{t_0 + \Im\gamma_n + \Im\lambda_n}\right). \tag{3.15}$$

If $\Im\gamma_n \leqslant t_0$ and $\Im\lambda_n \leqslant t_0$, then one gets

$$\frac{1}{3} \leqslant \frac{t_0 + 2\Im\gamma_n}{t_0 + \Im\gamma_n + \Im\lambda_n} \leqslant 3, \qquad \frac{1}{3} \leqslant \frac{t_0 + 2\Im\lambda_n}{t_0 + \Im\gamma_n + \Im\lambda_n} \leqslant 3.$$

If $\Im\gamma_n \geqslant t_0$ then the relation $\Im\gamma_n/2 < \Im\lambda_n < 2\Im\gamma_n$ holds since $\{\gamma_n - \lambda_n\}'$ is bounded and t_0 is large. Similarly if $\Im\lambda_n \geqslant t_0$. Then one gets

$$\frac{1}{4} < \frac{t_0 + 2\Im\gamma_n}{t_0 + \Im\gamma_n + \Im\lambda_n} < 3, \qquad \frac{1}{4} < \frac{t_0 + 2\Im\lambda_n}{t_0 + \Im\gamma_n + \Im\lambda_n} < 5.$$

Either way, substitution in (3.14) and (3.15) yields $|\Re\gamma_n - x_0| < 3\delta$ and $|\Re\lambda_n - x_0| < 3\delta$ since $|\gamma_n - \lambda_n| \leqslant \delta$. Then from (3.6) and the boundedness of $\{a_n\}'$ we deduce that there is some positive constant κ such that $\Upsilon_n(x)$ in (3.9) satisfies

$$|\Upsilon_n(x_0)| \leqslant \kappa + \kappa(\Im\gamma_n)^2 + \kappa(\Im\lambda_n)^2 + (\Im\gamma_n)^2(\Im\lambda_n)^2. \tag{3.16}$$

We now go back to relation (3.8). Observe that

$$g_n(x_0, t_0) \geqslant \omega_n(t_0) + \Upsilon_n(x_0), \tag{3.17}$$

and from (3.5) one has

$$\omega_n(t_0) \geqslant (\Im\gamma_n)^2(\Im\lambda_n)^2 + t_0^4 + t_0^2(\Im\gamma_n)^2 + t_0^2(\Im\lambda_n)^2. \tag{3.18}$$

Since t_0 is large, it follows from (3.16) and (3.18) that $\omega_n(t_0) + \Upsilon_n(x_0) > 0$. Thus $g_n(x_0, t_0) > 0$ and this completes the proof. $\square$

Proposition 3.1. *There exist positive constants A, C such that $\forall x \in \mathbf{R}$ the meromorphic function $M(z, t_0)$ in* (3.2), *where t_0 is as in Lemma* 3.1, *satisfies*

$$|M(x, t_0)| \leqslant Ae^{Ct_0}. \tag{3.19}$$

Proof. Let us write $|M(x, t_0)|$ as

$$\left|M(x, t_0)\right| = \left|\prod{}' \frac{\gamma_n\lambda_n}{(\gamma_n + a_n)(\lambda_n - a_n)} e^{it_0/\gamma_n + it_0/\lambda_n} \frac{(\gamma_n + a_n - x)(\lambda_n - a_n - x)}{(\gamma_n - x + it_0)(\lambda_n - x + it_0)}\right|. \tag{3.20}$$

Since $\{a_n\}'$ is bounded, $|\gamma_n - \lambda_n| \leqslant \delta$, and the exponent of convergence for $\{\gamma_n\}'$ and $\{\lambda_n\}'$ is less than or equal to 1, then one deduces that the series

$$\sum{}' \frac{a_n^2 + a_n(\gamma_n - \lambda_n)}{(\gamma_n + a_n)(\lambda_n - a_n)}$$

converges absolutely. It follows that the infinite product

$$\prod{}' \left| \frac{\gamma_n \lambda_n}{(\gamma_n + a_n)(\lambda_n - a_n)} \right|$$

converges and is bounded above by some positive A. Also

$$\prod{}' \left| e^{(it_0/\gamma_n + it_0/\lambda_n)} \right| = \prod{}' e^{\Re\left(it_0/\gamma_n + it_0/\lambda_n\right)} = e^{t_0 \sum' \Im\gamma_n/|\gamma_n|^2 + \Im\lambda_n/|\lambda_n|^2} = e^{Ct_0}$$

for some $C > 0$ since by definition the series converges. Applying Lemma 3.1 gives the upper bound Ae^{Ct_0} for the product in (3.20). $\square$

4. Proof of Theorems 1.2 and 2.1

Proof of Theorem 1.2. In order to derive equivalent systems, it suffices to prove that incompleteness of any of the two systems implies incompleteness of the other. To achieve this, we need to have *symmetric* conditions with respect to their associated sequences. In our case this holds since the terms a_n of the sequence causing the perturbations have no *no pre-assigned argument*. We compare this with Theorem 2.3 where due to the lack of such conditions (α and β are *positive*), we cannot deduce equivalence.

We assume that E_μ is incomplete. This implies the existence of a non-trivial entire function F of exponential type $\sigma \leqslant a$, which vanishes on some sequence $\tau \supset \mu$ with the properties:

(i) $F \in L^2(-\infty, \infty)$ and so does $F(x - it)$ for all $t \in \mathbf{R}$.
(ii) $F(z) = \int_{-a}^{a} f(t)e^{izt}\,dt$ for some $f \in L^2(-a, a)$.
(iii) The conjugate diagram of F is a vertical line segment of length 2σ, thus its indicator function satisfies $h_F(\pi/2) + h_F(-\pi/2) = 2\sigma$.
(iv) $\sum |\Im\tau_n|/|\tau_n|^2 < \infty$.
(v) $\lim_{r\to\infty} n_+(r,\phi)/r = \lim_{r\to\infty} n_-(r,\phi)/r = \sigma/\pi$ where $n_+(r,\phi)$ and $n_-(r,\phi)$ are the numbers of zeros of F in the sectors $\{z\colon |z| \leqslant r,\ |\arg z| \leqslant \phi\}$ and $\{z\colon |z| \leqslant r,\ |\pi - \arg z| \leqslant \phi\}$, respectively, for $\phi \in (0, \pi)$.

Our goal is to show that there is some function G vanishing on ν with similar properties as F. This will prove incompleteness of E_ν.

For some $d \in \mathbf{C}$, we can write F as

$$F(z) = e^{dz} \prod \left(1 - \frac{z}{w_n}\right) e^{z/w_n} \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{\mu_n}\right)^{k_n} e^{zk_n/\mu_n}, \tag{4.1}$$

where k_n is the multiplicity of μ_n and $\{w_n\} = \tau \setminus \mu$. Note that the set $\{w_n\}$ might be infinite, finite or empty. We can also assume that $\Im\tau_n \geqslant 0$ for all $n \in \mathbf{Z}$. For if $\{\omega_n\} \subset \tau$ and

$\Im\omega_n < 0$, then multiplication of F by a Blaschke product which vanishes on $\{\overline{\omega}_n\}$ and has poles on $\{\omega_n\}$ yields a function in $L^2(-\infty,\infty)$ whose zeros are all in the upper half-plane.

Then proceed with the $\mathcal{P}_{\mu,\delta}$ partition and construct ν as in (1.3). Let $M(z,0)$ be the meromorphic function as in (3.2) for $t=0$. Consider then $t_0>0$ as in Lemma 3.1 and denote by $G(z)$ the function $e^{-idt_0}F(z)M(z,0)$. Then, based on the partition of μ, one expresses $G(z)$ as

$$e^{d(z-it_0)}\prod\left(1-\frac{z}{w_n}\right)e^{z/w_n}{\prod}'\left(1-\frac{z}{\gamma_n+a_n}\right)\left(1-\frac{z}{\lambda_n-a_n}\right)e^{z/\gamma_n+z/\lambda_n},$$

where the $\{\rho_n\}$ terms have been included in $\{w_n\}$.

Note that μ is replaced by ν and this due to the bounded sequence $\{a_n\}'$. It follows that G is of exponential type as well. For the same reason, properties (iv) and (v) do not change, which implies the same for (iii). Then we can assume, without loss of generality, that $h_G(\pm\pi/2)=\sigma$. To complete the proof, we have to show that $G\in L^2(-\infty,\infty)$.

From (4.1) and the partition of μ, we may write $F(x-it_0)/e^{d(x-it_0)}$ as

$$\prod\left(1-\frac{x-it_0}{w_n}\right)e^{(x-it_0)/w_n}{\prod}'\left(1-\frac{x-it_0}{\gamma_n}\right)\left(1-\frac{x-it_0}{\lambda_n}\right)e^{(x-it_0)/\gamma_n}e^{(x-it_0)/\lambda_n}.$$

Then one gets

$$\left|\frac{G(x)}{F(x-it_0)}\right|=\left|\prod\left(\frac{1-\frac{x}{w_n}}{1-\frac{x-it_0}{w_n}}\right)e^{it_0/w_n}{\prod}'\frac{\left(1-\frac{x}{\gamma_n+a_n}\right)\left(1-\frac{x}{\lambda_n-a_n}\right)}{\left(1-\frac{x-it_0}{\gamma_n}\right)\left(1-\frac{x-it_0}{\lambda_n}\right)}e^{it_0/\gamma_n+it_0/\lambda_n}\right|.$$

But the $\prod'$ function is the meromorphic function $M(x,t_0)$. Thus, from Proposition 3.1, there are positive constants A and C such that

$$\left|\frac{G(x)}{F(x-it_0)}\right|\leqslant Ae^{Ct_0}\prod\left|\frac{w_n-x}{w_n-x+it_0}\right|\left|e^{it_0/w_n}\right|,$$

for every $x\in\mathbf{R}$. Since $\Im w_n\geqslant 0$ and $t_0>0$, we also have $|w_n-x|<|w_n-x+it_0|$. Combining this with the convergence of the series $\sum\Im w_n/|w_n|^2$, we deduce that

$$|G(x)|\leqslant\phi(t_0)\left|F(x-it_0)\right|\quad\forall x\in\mathbf{R},\tag{4.2}$$

where ϕ depends only on t_0. This relation implies that $G\in L^2(-\infty,\infty)$. Then by the Paley–Wiener theorem, G admits the integral representation

$$G(z)=\int_{-a}^{a}g(t)e^{izt}\,dt,\quad g\in L^2(-a,a).\tag{4.3}$$

Since G vanishes on ν, this implies that E_ν in incomplete in $L^2(-a,a)$. $\square$

Remark 4.1. We note that for real μ and ν our result follows from [11, Theorem 2].

Proof of Theorem 2.1. Let $\mu=\mathbf{Z}$ and $P_{\mathbf{Z},\delta}$ be its partition with the term $0\in\{\rho_n\}$. Let ν be new sequence and $\nu'=\nu\setminus\{0\}$. Since $\{e^{int}\}_{-\infty}^{\infty}$ is *exact* in $L^2(-\pi,\pi)$, then from Theorem 1.2 one has $E(\nu;2,\pi)=0$ as well. But the *excess* is a decreasing function of p and changes at most by 1 (see [15, p. 98, Problems 1, 2]). Thus

(A) $E(\nu; p, \pi)$ is either 0 or 1 for any $p \in (1, 2)$.
(B) $E(\nu; p, \pi)$ is either 0 or -1 for any $p \in (2, \infty)$.

We will show that in both cases E_ν is *exact*.

Case $1 < p < 2$. Consider the function $F(z) = (\sin \pi z)/z$. Then $F(x) \in L^p(-\infty, \infty)$ for all $p > 1$ and vanishes exactly on $\mathbf{Z} \setminus \{0\}$. Let $M(z, 0)$ be the usual meromorphic function and define $G(z)$ as before. Then $G(z)$ is an entire function of exponential type not exceeding π and vanishes exactly on ν'. As in (4.2), one has that $G(x) \in L^p(-\infty, \infty)$ for all $p > 1$. Consider now any $1 < p_0 < 2$. Then, from [2, Theorem 6.4] G admits the integral representation

$$G(z) = \int_{-a}^{a} g(t)e^{izt}\,dt, \quad g \in L_0^q(-a, a), \qquad p_0^{-1} + q_0^{-1} = 1. \tag{4.4}$$

This implies that $E_{\nu'}$ is incomplete in $L^{p_0}(-\pi, \pi)$, thus $E(\nu'; p_0, \pi) \leqslant -1$. It follows that $E(\nu; p_0, \pi) \leqslant 0$. Combining this with (A), shows that $E(\nu; p_0, \pi) = 0$.

Case $2 < p < \infty$. Assume $E(\nu; p_0, \pi) = -1$ for some $p_0 \in (2, \infty)$. Thus, there exists a non-trivial $f \in L^{q_0}(-\pi, \pi)$, $p_0^{-1} + q_0^{-1} = 1$, such that

$$H(z) = \int_{-\pi}^{\pi} f(t)e^{izt}\,dt$$

is an entire function which vanishes *exactly* on ν. The latter holds since if $H(u) = 0$ for some $u \notin \nu$, then $E_\nu \cup \{e^{iut}\}$ is incomplete contradicting the fact that $E(\nu; p_0, \pi) = -1$. Since $q_0 \in (1, 2)$, from [2, Theorem 6.5] one has that H is of exponential type π and $H \in L^{p_0}(-\infty, \infty)$. Thus $H(z) = ke^{cz} z \prod_{\nu_n \in \nu'} (1 - z/\nu_n)e^{z/\nu_n}$ for some constants $k, c \in \mathbf{C}$.

We then consider the usual meromorphic function $M(z, 0)$, this time with $\{\nu_n + it\}$ as its poles and $\mathbf{Z}$ as its zeros. Define analogously $G(z) = H(z)M(z, 0)$. Then G is an entire function of exponential type, vanishes exactly on $\mathbf{Z}$, and as in (4.2), $G(x) \in L^{p_0}(-\infty, \infty)$. But this implies that $\sin \pi x \in L^{p_0}(-\infty, \infty)$ as well, which is false. Therefore $E(\nu; p_0, \pi) \neq -1$, thus E_ν is *exact*. $\square$

The theorem fails for $L^1(-\pi, \pi)$ and $C[-\pi, \pi]$: Consider the system

$$E_\nu = \{e^{int}\}_{-\infty}^{0} \cup \{e^{it(n+ih(-1)^n)}\}_1^\infty$$

and compare it with the system $\{e^{int}\}_{-\infty}^{\infty}$ which is *exact* for all $1 \leqslant p < \infty$ and whose *excess* equals -1 in $C[-\pi, \pi]$. From what we have already proved, it follows that the *excess* is unaltered for $1 < p < \infty$. However, in [13] A. Sedletskii proved that the excess of E_ν in $L^1(-\pi, \pi)$ is 1, and in the space $C[-\pi, \pi]$ it is 0.

5. Proof of Theorem 2.3

We will follow very closely the steps of the proof of Theorem 2.2, as given by its authors.

First, we need to make the following simplifications to enable us prove the result:

(a) Observe that replacing the sequence $\mu = \{\mu_n\}$ with $\{\mu_n + d\}$ for any $d \in \mathbf{R}$ does not change the completeness properties of the new system. Thus we replace β with 0 and α with $\alpha_0 = \alpha + \beta$. However, in what follows, we treat α as a variable where $0 \leqslant \alpha \leqslant \alpha_0$.
(b) Since $\Re\mu_n \mapsto \infty$ as $n \mapsto \infty$, we assume that $\Re\mu_n \geqslant 1$ for all $n \geqslant 1$. Then the condition $(\Re\mu_n) \geqslant (\Im\mu_n)^2$ yields $(\Re\mu_n)^2 \geqslant (\Im\mu_n)^2$ for all $n \geqslant 1$ as well.
(c) Since $E(\mu; 2, a)$ is finite, we can assume that E_μ is an *exact* system in $L^2(-a, a)$ by adding or removing a finite number of terms. This implies that for $\alpha = 0$ the entire function
$$F(z, \alpha) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{\mu_n + \alpha}\right)\left(1 - \frac{z}{\mu_{-n}}\right)$$
belongs to $L^2(-\infty, \infty)$. This would not have been true if we had retained the factor corresponding to μ_0. We remark that the convergence of the product is justified since $|\mu_n + \mu_{-n}| = O(1)$.

Theorem 2.3 follows as soon as we show that $F(x, \alpha_0) \in L^2(-\infty, \infty)$ as well. In other words, the integral $\int_{-R}^{R} |F(x, \alpha_0)|^2\, dx$ denoted by $S(R, \alpha_0)$ must converge to a real number as $R \mapsto \infty$. Thus we decompose $S(R, \alpha_0)$ into the form
$$S(R, \alpha_0) = \int_{-R}^{0} |F(x, \alpha_0)|^2\, dx + \int_{0}^{\alpha_0+c+1} |F(x, \alpha_0)|^2\, dx + \int_{\alpha_0+c+1}^{R} |F(x, \alpha_0)|^2\, dx,$$
and observe that the middle integral is finite and independent of R. In order to complete the proof, we have to show that the first and third integrals, denoted by $\mathrm{I}(R, \alpha_0)$ and $\mathrm{III}(R, \alpha_0)$, respectively, converge to a real number as $R \mapsto \infty$. Comparison is made with respect to $\mathrm{I}(R, 0)$ and $\mathrm{III}(R, 0)$ which converge since $F(x, 0) \in L^2(-\infty, \infty)$. As already mentioned, we treat α as a variable, with $\alpha \in [0, \alpha_0]$.

We give the proof for $\mathrm{III}(R, \alpha_0)$. After the substitution $u = x - \alpha$, we have
$$\mathrm{III}(R, \alpha) = \int_{c+1}^{R-\alpha} \prod_{n=1}^{\infty}\left|1 - \frac{u + \alpha}{\mu_n + \alpha}\right|^2 \left|1 - \frac{u + \alpha}{\mu_{-n}}\right|^2 du$$
$$= \int_{c+1}^{R-\alpha} \prod_{n=1}^{\infty}\left|\frac{\mu_n - u}{\mu_{-n}}\right|^2 \left|\frac{\mu_{-n} - u - \alpha}{\mu_n + \alpha}\right|^2 du. \tag{5.1}$$
Observe that
$$\left|\frac{\mu_{-n} - u - \alpha}{\mu_n + \alpha}\right|^2 = \frac{(\Re\mu_{-n} - u - \alpha)^2 + (\Im\mu_{-n})^2}{(\Re\mu_n + \alpha)^2 + (\Im\mu_{-n})^2}.$$

Denote the whole fraction by $L_n(u,\alpha)$ and the denominator by $U_n(\alpha)$. For fixed $u \geqslant c+1$, differentiating $L_n(u,\alpha)$ with respect to α gives

$$\begin{aligned} U_n^2(\alpha)L_n'(u,\alpha) &= -2(\Re\mu_{-n}-u-\alpha)\big[(\Re\mu_n+\alpha)^2+(\Im\mu_{-n})^2\big] \\ &\quad -\big[(\Re\mu_{-n}-u-\alpha)^2+(\Im\mu_{-n})^2\big](\Re\mu_n+\alpha) \\ &= -2(\Re\mu_{-n}-u-\alpha)\big[(\Re\mu_n+\alpha)^2+(\Im\mu_{-n})^2 \\ &\quad +(\Re\mu_{-n}-u-\alpha)(\Re\mu_n+\alpha)\big]-2(\Im\mu_{-n})^2(\Re\mu_n+\alpha) \\ &\leqslant -2(\Re\mu_{-n}-u-\alpha)\big[(\Re\mu_n+\alpha)(\Re\mu_n+\Re\mu_{-n}-u)+(\Im\mu_n)^2\big]. \end{aligned} \tag{5.2}$$

We show now that (5.2) is negative. Since $|\Re\mu_n+\Re\mu_{-n}| \leqslant c$ and $u \geqslant c+1$, we have $(\Re\mu_n+\alpha)(\Re\mu_n+\Re\mu_{-n}-u) \leqslant -\Re\mu_n$. Therefore $(\Re\mu_n+\alpha)(\Re\mu_n+\Re\mu_{-n}-u)+(\Im\mu_n)^2 \leqslant -\Re\mu_n+(\Im\mu_n)^2 \leqslant 0$ since $\Re\mu_n \geqslant (\Im\mu_n)^2$. But then $\Re\mu_{-n}-u-\alpha<0$ as well since $u>0$. Thus (5.2) is negative and the same is true for $L_n'(u,\alpha)$. This implies that for fixed $u \geqslant c+1$, $L_n(u,\alpha)$ is a decreasing function of α. Thus $L_n(u,\alpha_0) \leqslant L_n(u,0)$ for all $u \geqslant c+1$. It then follows that $\mathrm{III}(R,\alpha_0)$ converges to a real number as $R \mapsto \infty$.

Similarly, we prove it for $\mathrm{I}(R,\alpha_0)$ using the conditions $x \leqslant 0$ and $(\Re\mu_n)^2 \geqslant (\Im\mu_n)^2$.

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