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TWO CONSEQUENCES OF THE BEURLING-MALLIAVIN THEORY[1]

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ABSTRACT. If $(1/\lambda_n)-(1/\mu_n)$ forms an absolutely convergent series, then $\{\exp(i\lambda_n x)\}$ and $\{\exp(i\mu_n x)\}$ have the same completeness interval. This follows from a new formula for the completeness radius which is simpler than the well-known formula of Beurling and Malliavin.

1. **Summary of results.** Throughout this note $\lambda=\{\lambda_n\}$ and $\mu=\{\mu_n\}$ are sequences of complex numbers none of which is zero. As in [1], $I(\lambda)$ denotes the completeness interval of $\{e^{i\lambda_n x}\}$ and $E(\lambda)$ denotes the L^p excess on that interval. The quantities $I(\mu)$ and $E(\mu)$ are similarly related to $\{\mu_n\}$.

In [1] it is shown that

$$\sum |\lambda_n - \mu_n| < \infty \Rightarrow E(\lambda) = E(\mu). \tag{1}$$

Here we are going to show that

$$\sum \left| \frac{1}{\lambda_n} - \frac{1}{\mu_n} \right| < \infty \Rightarrow I(\lambda) = I(\mu). \tag{2}$$

This was conjectured by the author in a seminar at the University of California at Los Angeles (April 1961), later set as a research problem [4], and finally announced as a theorem [6].

Since (1) is easy to establish, it might be thought that (2) is also easy. However, (2) is deep; the appearance of simplicity is deceptive. We shall sketch a proof that (2) is equivalent to a theorem of Beurling and Malliavin, in the sense that either can be deduced from the other. An independent proof of (2) would therefore give a new solution to the completeness problem. In this connection it is interesting to note that (2) implies, as a special case,

$$\sum \left| \frac{1}{\lambda_n} \right| < \infty \Rightarrow I(\lambda) = 0. \tag{3}$$

Direct proof of (3) is trivial [5].

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Equation (2) follows from a somewhat novel form of the completeness criterion. We say that the positive number c *belongs to* λ if there exists a sequence $\{\nu_k\}$ of distinct integers such that

$$\sum\left|\frac{1}{\lambda_n}-\frac{c}{\nu_n}\right|<\infty. \tag{4}$$

It is shown below that the set of all c belonging to λ is a semi-infinite interval of the positive real axis. Let the left-hand endpoint be denoted by $D(\lambda)$, so that $D(\lambda)=\infty$ if no c belongs to λ, otherwise

$$D(\lambda)=\inf c \mid c \text{ belongs to } \lambda.$$

Our completeness criterion is

$$I(\lambda)=2\pi D(\lambda). \tag{5}$$

The "two consequences" referred to in the title are (2) and (5).

If the series (2) converges, then any c belonging to λ also belongs to μ, and vice versa. Hence (5) implies (2). Most of the rest of this paper is devoted to the proof of (5).

2. **Some results of Beurling and Malliavin.** The Beurling-Malliavin solution of the completeness problem for real sequences depends on a certain exterior density, $A_e(d\mu)$, whose definition **[2]** is too long for inclusion here. To extend this to complex sets satisfying

$$\sum\left|\operatorname{Im}\frac{1}{\lambda_n}\right|<\infty, \tag{6}$$

Beurling and Malliavin consider the mapping taking λ_n into λ_n^*,

$$\frac{1}{\lambda_n^*}=\frac{1}{2}\left(\frac{1}{\lambda_n}+\frac{1}{\bar{\lambda}_n}\right). \tag{7}$$

The Beurling-Malliavin completeness criterion is then

$$I(\lambda)=2\pi A_e(d\Lambda^*) \tag{8}$$

where Λ^* is the (signed) counting function for $\{\lambda_n^*\}=\lambda^*$.

Since (6) is obvious if any c belongs to λ, the result (5) is equivalent to the assertion that

$$A_e(d\Lambda^*)=D(\lambda). \tag{9}$$

Equation (9) has interest apart from the objectives of this paper, since the exterior density $A_e(d\Lambda^*)$ is connected with deep properties of harmonic and entire functions **[2]**.

Direct proof of (9) is by no means an easy task. However the early work of Beurling and Malliavin contains another completeness criterion which is more tractable. Let λ_n be real and have counting function $\Lambda(u)$. Consider all functions $h(t)$ such that

$$(10) \qquad \int_{-\infty}^{\infty} \frac{|\Lambda(t) - h(t)|}{t^2}\, dt < \infty, \qquad 0 \leqq h'(t) \leqq c.$$

The *effective density*, $B(\lambda)$, is defined to be the inf of all c for which such an $h(t)$ can be found. If there is no such c, then $B(\lambda)=\infty$. With these definitions,

$$(11) \qquad I(\lambda) = 2\pi B(\lambda), \quad \lambda \text{ real}.$$

This result is an early form of the Beurling-Malliavin theorem. Although it was not published officially by them, an account of it has been given by Kahane [3], and further insight into the relation of $B(\lambda)$ and $A_e(d\Lambda)$ is given by Lemma II.2 of [2]. It will be found that (5) follows with ease from (11).

3. **Proof of the completeness criterion.**[2] As already noted, we can assume (6), since in the contrary case no c belongs to λ and (5) gives the correct result, $I(\lambda)=\infty$. By (6) and (7), any c belonging to λ belongs to λ^* and vice versa. Since the work of Beurling and Malliavin establishes $I(\lambda)=I(\lambda^*)$ under the hypothesis (6), we can assume that λ is real.

LEMMA. *Let $\{\lambda_n\}$ be a real sequence with counting function Λ. Then the following statements are equivalent*:

(i) *c belongs to λ*;

(ii) *equation* (10) *holds for some h.*

We first show that (ii)⇒(i). If (10) holds we can approximate the curve $y=h(t)$ by the counting function, $M(t)$, for a sequence of the form $\{v_n/c\}$ where the v_n are integers. Convergence of the integral in (10) makes $\Lambda(t)=O(t)$, since Λ is increasing, and partial integration gives (4). This shows that c belongs to λ.

To show that (i)⇒(ii), let c belong to λ and set $\mu_n=v_n/c$, so that the series (2) converges. It may happen that some positive λ's have been correlated with negative μ's and vice versa. However, the hypothesis makes (3) hold for such terms and hence we can drop the corresponding set of λ's and μ's without affecting completeness. That being done, the series consists of a sum in which both λ_n and μ_n are positive, together with

[2] The original proof [6] has been revised somewhat, following helpful suggestions of the referee.

another sum in which both are negative. It suffices to consider the former only.

Since statement (i) is invariant under permutation of the λ's we may assume $\lambda_{n+1} \geqq \lambda_n$. The elementary inequality

$$\left|\frac{1}{\lambda} - \frac{1}{\mu'}\right| + \left|\frac{1}{\lambda'} - \frac{1}{\mu}\right| \geqq \left|\frac{1}{\lambda} - \frac{1}{\mu}\right| + \left|\frac{1}{\lambda'} - \frac{1}{\mu'}\right|$$

holds for $0 < \lambda \leqq \lambda'$ and $0 < \mu \leqq \mu'$, and shows that we may also assume $\mu_{n+1} \geqq \mu_n$.

Suppose now that, for some $u > 0$, $\Lambda(u) - M(u) \geqq m > 0$, where $M(u)$ is the counting function for μ. In this case

$$\sum_{k \in S} \left|\frac{1}{\lambda_k} - \frac{1}{\mu_k}\right| \geqq \sum_{j=0}^{m-1} \left(\frac{1}{u} - \frac{1}{u + j/c}\right)$$
$$\geqq \frac{m}{u} - c \log\left(1 + \frac{m}{cu}\right) + O\left(\frac{1}{u}\right)$$

where S is the set of indices k for which $\lambda_k \leqq u$ and $\mu_k > u$. A similar calculation can be made if $\Lambda(u) - M(u) \leqq -m < 0$ for some u. The two calculations together show that convergence of the series (2) makes $m/u \to 0$ as $u \to \infty$, that is,

$$(12) \qquad \sum_{k=1}^{\infty} \left|\frac{1}{\lambda_k} - \frac{1}{\mu_k}\right| < \infty \Rightarrow \Lambda(u) - M(u) = o(u), \qquad u \to \infty.$$

In order to get estimates for $|\Lambda(u) - M(u)|$ it is convenient to distinguish the terms of the series in which $\lambda_n \leqq \mu_n$ from those in which $\lambda_n > \mu_n$. We consider the former only. The sum is over a set of indices m_n rather than n, but for ease of writing we use λ_n instead of λ_{m_n} and similarly for μ. Thus the series has the appearance (12) with the additional condition $\lambda_n \leqq \mu_n$. Since λ_n and μ_n increase with n,

$$\sum_{k=1}^{n} \left(\frac{1}{\lambda_k} - \frac{1}{\mu_k}\right) = \int_0^{\lambda_n} \frac{d\Lambda_1(t)}{t} - \int_0^{\mu_n} \frac{dM_1(t)}{t}$$

where Λ_1 and M_1 pertain to the subsequences $\{\lambda_n\}$ and $\{\mu_n\}$ being considered now. With $u = \lambda_n$ and $v = \mu_n$, the above result can be written

$$(13) \qquad \int_0^u \frac{d\Lambda_1(t) - dM_1(t)}{t} dt - \int_u^v \frac{dM_1(t)}{t}.$$

Since $M(t)=O(t)$ by inspection and $\Lambda(t)=O(t)$ by (12), partial integration gives

$$\int_0^u \frac{d\Lambda_1(t) - dM_1(t)}{t}\,dt = \int_0^u \frac{\Lambda_1(t) - M_1(t)}{t^2}\,dt + O(1).$$

The second term of (13) is $\sum (1/\mu_k)$ for μ_k on the interval (u, v). We show that this term is also $O(1)$. Indeed, if the number of terms in this sum is m, a calculation similar to (12) gives $m/u=O(1)$. Since $\mu_k \geqq u$ we have, for the terms in question, $\sum (1/\mu_k) \leqq \sum_1^m (1/u)=O(1)$.

By the above estimates,

$$\int_0^\infty \frac{|\Lambda_1(t) - M_1(t)|}{t^2}\,dt < \infty. \tag{14}$$

A similar discussion can be given for the terms of the original series in which $\lambda_n > \mu_n$ and, in fact, this case is somewhat simpler. The result of the analysis is that if the counting function of these λ's is $\Lambda_2(t)$ and the counting function of these μ's is $M_2(t)$ then the integral (14) for $\Lambda_2 - M_2$ converges. Since

$$\Lambda(t) = \Lambda_1(t) + \Lambda_2(t), \qquad M(t) = M_1(t) + M_2(t)$$

it follows that

$$\int_{-\infty}^\infty \frac{|\Lambda(t) - M(t)|}{t^2}\,dt < \infty.$$

The lower limit can be taken as $-\infty$ rather than 0 because a similar calculation could have been made for the terms with $\lambda_k < 0$ and $\mu_k < 0$.

The counting function $M(t)$ for the sequence $\{\nu_k/c\}$ can be approximated within 1 by a curve $y=h(t)$ such that $0 \leqq h' \leqq c$. This shows that (i)$\Rightarrow$(ii). In view of the lemma, $B(\lambda)=D(\lambda)$ for real λ, and hence (5) follows from (11).

4. **Concluding remarks.** It was stated above that the set of c belonging to λ is a semi-infinite interval of the real axis. In other words, if a belongs to λ, and $b > a$, then b belongs to λ. For proof, since $b > a$ every interval $[k/a, (k+1)/a]$ with k integral contains a number of form j/b. This means that, given the sequence $\{\nu_n\}$ associated with a, we can find another sequence of distinct integers, $\{\sigma_n\}$, such that $|\nu_n/a - \sigma_n/b| < 1/a$. Multiplication by $ab/\nu_n\sigma_n$ and use of $|\sigma_n|/b \geqq |\nu_n|/a - 1/a$ give

$$\left| \frac{a}{\nu_n} - \frac{b}{\sigma_n} \right| < \frac{b}{\nu_n\sigma_n} \leqq \frac{a}{|\nu_n|\,(|\nu_n| - 1)} \qquad (|\nu_n| \neq 1).$$

If we sum on n the resulting series is, essentially, a rearrangement of a subseries of the absolutely convergent series $\sum (a/n(n-1))$. Since

$$\left|\frac{1}{\lambda_n}-\frac{b}{\sigma_n}\right| \leqq \left|\frac{1}{\lambda_n}-\frac{a}{\nu_n}\right| + \left|\frac{a}{\nu_n}-\frac{b}{\sigma_n}\right|,$$

it follows that b belongs to λ.

It was stated above that a proof of (2) would solve the completeness problem, and we shall show why this is so. From (2), (6) and (7) follows $I(\lambda)=I(\lambda^*)$ and thus (2) reduces the completeness problem to the problem of real sequences. For real sequences the inequality $I(\lambda)\geqq 2\pi D(\lambda)$ is elementary; it follows from the well-known fact that

$$I(\lambda) \geqq 2\pi \liminf_{n\to\infty} \frac{\Lambda(x_n+y_n)-\Lambda(x_n)}{y_n} \qquad (\lambda \text{ real}) \tag{15}$$

where (x_n, x_n+y_n) are nonoverlapping intervals such that

$$\sum \left(\frac{y_n}{x_n}\right)^2 = \infty. \tag{16}$$

This is an easy theorem of Beurling and Malliavin, also found independently by the author; a simple proof is given in [5]. In fact, in the presence of (2) we need establish $I(\lambda)\geqq 2\pi D(\lambda)$ only for sequences $\lambda=\{\nu_n/c\}$ which satisfy a separation condition, $\lambda_{n+1}-\lambda_n\geqq 1/c$. For such sequences (15) is even easier [5].

Suppose, then, that $0<c<D(\lambda)$, where λ is real. We want to use (15) to show that $I(\lambda)\geqq 2\pi c$. The first positive zero, λ_1, is denoted by $x_1>0$ and we consider a line of slope c through $(x_1, 0)$. If the horizontal part of the graph of $\Lambda(u)$ does not intersect this line for $u>0$ then $\liminf \Lambda(u)/u\geqq c$ as $u\to\infty$ and $I(\lambda)\geqq 2\pi c$ is trivial. We can suppose, then, that the first "horizontal" crossing point is at x_1+y_1. Since $D(\lambda)>0$, clearly λ is infinite and so there is a zero beyond x_1+y_1. The first such is denoted by x_2. A line of slope c is drawn through the point $(x_2, \Lambda(x_2-))$, and the next intersection of this line with the horizontal part of the graph of $\Lambda(u)$ is x_2+y_2. Continuing in this fashion we get a set of nonoverlapping intervals (x_n, x_n+y_n) such that $\Lambda(x_n+y_n)-\Lambda(x_n)=cy_n$.

Let μ be constructed by taking all numbers of the form ν_n/c, where ν_n are integers, on the semiclosed intervals $[x_n, x_n+y_n)$. Then for λ_n and μ_n on this interval the term

$$\left|\frac{1}{\lambda_n}-\frac{1}{\mu_n}\right| = \left|\frac{\mu_n-\lambda_n}{\lambda_n\mu_n}\right|$$

has the order of magnitude y_n/x_n^2. The number of terms is of the order y_n, and hence if the series (16) converges, the corresponding series for $u<0$ must diverge. (Otherwise c would belong to λ, contradicting the hypothesis $c<D(\lambda)$.) Equation (15) therefore gives $I(\lambda)\geqq 2\pi c$, which implies $I(\lambda)\geqq 2\pi D(\lambda)$.

Thus the main depth of the completeness problem consists in showing that $I(\lambda)\leqq 2\pi D(\lambda)$. However, this follows from (2) almost by inspection, and indeed, for complex as well as real λ. If $\mu_n=\nu_n/c$, where the ν_n are distinct integers, then the function $\sin \pi zc$ vanishes at μ_n, and hence $I(\mu)\leqq 2\pi c$. Equation (2) gives $I(\lambda)=I(\mu)\leqq 2\pi c$ whenever c belongs to λ, and therefore $I(\lambda)\leqq 2\pi D(\lambda)$.

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