New expressions for Riemann's functions $\xi(s)$ and $\Xi(t)$

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1. The principal object of this paper is to prove that if the real parts of α and β are positive, and $\alpha\beta = \pi^2$, and t is real, then

$$\begin{aligned}
&\alpha^{-\frac{1}{4}}\left\{\frac{1}{1+t^2} + 4\alpha\int_0^\infty \left(\frac{1}{3^2+t^2} - \frac{\alpha}{1!}\frac{x^2}{7^2+t^2} + \frac{\alpha^2}{2!}\frac{x^4}{11^2+t^2} - \cdots\right)\frac{x\,dx}{e^{2\pi x}-1}\right\}\\
&\quad - \beta^{-\frac{1}{4}}\left\{\frac{1}{1+t^2} + 4\beta\int_0^\infty \left(\frac{1}{3^2+t^2} - \frac{\beta}{1!}\frac{x^2}{7^2+t^2} + \frac{\beta^2}{2!}\frac{x^4}{11^2+t^2} - \cdots\right)\frac{x\,dx}{e^{2\pi x}-1}\right\}\\
&\quad = \frac{\pi^{-\frac{3}{4}}}{4t}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\sin\left(\frac{t}{8}\log\frac{\beta}{\alpha}\right). \qquad (1)
\end{aligned}$$

Consider the integral

$$J(u) = \int_0^\infty \frac{xe^{-\pi u x^2}}{e^{2\pi x}-1}\,dx,$$

where the real part of u is positive. Since

$$\int_0^\infty \frac{\sin \pi n x}{e^{\pi x}-1}\,dx = \frac{1}{e^{2\pi n}-1} + \frac{1}{2} - \frac{1}{2\pi n},$$

we have

$$\begin{aligned}
J(u) + \frac{1}{4\pi u} - \frac{1}{4\pi\sqrt{u}} &= \int_0^\infty xe^{-\pi u x^2}\left(\frac{1}{e^{2\pi x}-1} + \frac{1}{2} - \frac{1}{2\pi x}\right)\,dx\\
&= \int_0^\infty\int_0^\infty xe^{-\pi u x^2}\frac{\sin \pi x y}{e^{\pi y}-1}\,dx\,dy = u^{-\frac{3}{2}}\int_0^\infty \frac{xe^{-\pi x^2/u}}{e^{2\pi x}-1}\,dx; \qquad (2)
\end{aligned}$$

and so

$$J(u) - \frac{1}{4\pi\sqrt{u}} = u^{-\frac{3}{2}}\int_0^\infty xe^{-\pi x^2/u}\left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x}\right)\,dx. \qquad (3)$$

Suppose now that $s = \sigma + it$, where $0 < \sigma < 1$. Then, from (3), we have

$$\int_0^1 u^{\frac{1}{2}(s-1)} \left\{ J(nu) - \frac{1}{4\pi\sqrt{(nu)}} \right\} du$$

$$= n^{-\frac{3}{2}} \int_0^1 u^{\frac{1}{2}(s-4)} \, du \int_0^\infty x e^{-\pi x^2/nu} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) dx. \tag{4}$$

Changing u into $1/v$, we obtain

$$n^{-\frac{3}{2}} \int_1^\infty v^{-\frac{1}{2}s} \, dv \int_0^\infty x e^{-\pi v x^2/n} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) dx$$

$$= n^{-\frac{3}{2}} \left\{ \int_0^\infty v^{-\frac{1}{2}s} \, dv \int_0^\infty x e^{-\pi v x^2/n} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) dx \right.$$

$$\left. - \int_0^1 v^{-\frac{1}{2}s} \, dv \int_0^\infty x e^{-\pi v x^2/n} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) dx \right\}.$$

But

$$\int_0^1 u^{\frac{1}{2}(s-1)} \left\{ J(nu) - \frac{1}{4\pi\sqrt{(nu)}} \right\} du$$

$$= \int_0^1 u^{\frac{1}{2}(s-1)} J(nu) \, du - \frac{1}{2\pi s\sqrt{n}}$$

$$= -\frac{1}{2\pi s\sqrt{n}} + \int_0^1 u^{\frac{1}{2}(s-1)} \, du \int_0^\infty \frac{x e^{-\pi n u x^2}}{e^{2\pi x}-1} \, dx$$

$$= -\frac{1}{2\pi s\sqrt{n}} + \int_0^\infty \frac{x \, dx}{e^{2\pi x}-1} \int_0^1 u^{\frac{1}{2}(s-1)} e^{-\pi n u x^2} \, du$$

$$= -\frac{1}{2\pi s\sqrt{n}} + 2\int_0^\infty \left\{ \frac{1}{1+s} - \frac{\pi n x^2}{1!(3+s)} + \frac{(\pi n x^2)^2}{2!(5+s)} - \cdots \right\} \frac{x \, dx}{e^{2\pi x}-1}. \tag{5}$$

Also

$$n^{-\frac{3}{2}} \int_0^\infty v^{-\frac{1}{2}s} \, dv \int_0^\infty x e^{-\pi v x^2/n} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) dx$$

$$= n^{-\frac{3}{2}} \int_0^\infty x \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) \, dx \int_0^\infty v^{-\frac{1}{2}s} e^{-\pi v x^2/n} \, dv$$

$$= \pi^{\frac{1}{2}(s-2)} n^{-\frac{1}{2}(s+1)} \Gamma(1-\frac{1}{2}s) \int_0^\infty x^{s-1} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) \, dx$$

$$= -\frac{n^{-\frac{1}{2}(s+1)}}{4\pi\sqrt{\pi}} \Gamma\left(-\frac{s}{2}\right) \Gamma\left(\frac{s-1}{2}\right) \xi(s), \tag{6}$$

where

$$\xi(s) = (s-1)\Gamma(1+\frac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s).$$

Finally

$$n^{-\frac{3}{2}} \int_0^1 v^{-\frac{1}{2}s} \, dv \int_0^\infty x e^{-\pi v x^2/n} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) \, dx$$

$$= -\frac{n^{-\frac{3}{2}}}{2\pi} \int_0^1 v^{-\frac{1}{2}s} \, dv \int_0^\infty e^{-\pi v x^2/n} \, dx + n^{-\frac{3}{2}} \int_0^1 v^{-\frac{1}{2}s} \, dv \int_0^\infty \frac{x e^{-\pi v x^2/n}}{e^{2\pi x}-1} \, dx$$

$$= -\frac{1}{4\pi n} \int_0^1 v^{-\frac{1}{2}(1+s)} \, dv + n^{-\frac{3}{2}} \int_0^\infty \frac{x \, dx}{e^{2\pi x}-1} \int_0^1 v^{-\frac{1}{2}s} e^{-\pi v x^2/n} \, dv$$

$$= -\frac{1}{2\pi n(1-s)} + 2n^{-\frac{3}{2}} \int_0^\infty \left\{ \frac{1}{2-s} - \frac{\pi x^2/n}{1!(4-s)} + \frac{(\pi x^2/n)^2}{2!(6-s)} - \cdots \right\} \frac{x \, dx}{e^{2\pi x}-1}. \tag{7}$$

All the inversions of the order of integration, effected in the preceding argument, are easily justified, since every integral remains convergent when the subject of integration is replaced by its modulus.
It follows from (4) - (7) that, if the real parts of α and β are positive, and $\alpha\beta = \pi^2$, then

$$\alpha^{-\frac{1}{4}} \left\{ \frac{1}{1-s} - 4\alpha \int_0^\infty \left(\frac{1}{1+s} - \frac{\alpha}{1!} \frac{x^2}{3+s} + \frac{\alpha^2}{2!} \frac{x^4}{5+s} - \cdots \right) \frac{x \, dx}{e^{2\pi x}-1} \right\}$$

$$+\beta^{-\frac{1}{4}} \left\{ \frac{1}{s} - 4\beta \int_0^\infty \left(\frac{1}{2-s} - \frac{\beta}{1!} \frac{x^2}{4-s} + \frac{\beta^2}{2!} \frac{x^4}{6-s} - \cdots \right) \frac{x \, dx}{e^{2\pi x}-1} \right\}$$

$$= \frac{1}{2}\pi^{-\frac{3}{4}} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{8}-\frac{1}{4}s} \Gamma\left(-\frac{s}{2}\right) \Gamma\left(\frac{s-1}{2}\right) \xi(s). \tag{8}$$

Changing s to $\frac{1}{2}(1+it)$ in (8), and writing as usual

$$\xi(\frac{1}{2}+\frac{1}{2}it)=\Xi(\frac{1}{2}t),$$

and equating the real and imaginary parts, we obtain the formula (1), and also the formula

$$\alpha^{-\frac{1}{4}}\left\{\frac{1}{1+t^2}-4\alpha\int\limits_0^\infty\left(\frac{3}{3^2+t^2}-\frac{\alpha}{1!}\frac{7x^2}{7^2+t^2}+\frac{\alpha^2}{2!}\frac{11x^4}{11^2+t^2}-\cdots\right)\frac{x\,dx}{e^{2\pi x}-1}\right\}$$
$$+\beta^{-\frac{1}{4}}\left\{\frac{1}{1+t^2}-4\beta\int\limits_0^\infty\left(\frac{3}{3^2+t^2}-\frac{\beta}{1!}\frac{7x^2}{7^2+t^2}+\frac{\beta^2}{2!}\frac{11x^4}{11^2+t^2}-\cdots\right)\frac{x\,dx}{e^{2\pi x}-1}\right\}$$
$$=\frac{1}{4}\pi^{-\frac{3}{4}}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi(\frac{1}{2}t)\cos\left(\frac{t}{8}\log\frac{\alpha}{\beta}\right). \tag{9}$$

2. We have proved (8) on the assumption that $0<\sigma<1$. But it can be shewn that the formula is true for all values of s other than integral values.
Suppose first that $-1>\sigma<0$. The formula (3) is equivalent to

$$J(u)=u^{-\frac{3}{2}}\int\limits_0^\infty xe^{-\pi x^2/u}\left(\frac{1}{e^{2\pi x}-1}-\frac{1}{2\pi x}+\frac{1}{2}\right)\,dx. \tag{10}$$

Using this formula as we used (3) in the previous section, we can shew that (8) is true in the strip $-1<\sigma<0$ also. In the right-hand side of (3), the first term in the expansion of $1/(e^{2\pi x}-1)$, viz. $1/(2\pi x)$, is removed, and in that of (10) two terms are removed. By considering the corresponding formulæ in which more and more terms in the expansion of $1/(e^{2\pi x}-1)$, viz.

$$\frac{1}{2\pi x}-\frac{1}{2}+\frac{\pi x}{6}-\frac{\pi^3x^3}{90}+\frac{\pi^5x^5}{945}-\frac{\pi^7x^7}{9450}+\frac{\pi^9x^9}{93555}-\cdots,$$

are removed,we can shew that the formula (8) is true in the strips $-2<\sigma<1, -3<\sigma<-2$, and so on. That it is also true in the strips $1<\sigma<2, 2<\sigma<3,\ldots$ is easily deduced from the functional equation $\xi(s)=\xi(1-s)$.
The formula also holds on the lines which divide the strips, except at the special points $s=k$, where k is an integer. This follows at once from the continuity of $\xi(s)$ and the uniform convergence of the integrals in question.

3. As a particular case of (9) we have, when $\alpha=\beta=\pi$,

$$\frac{1}{1+t^2}-4\pi\int\limits_0^\infty\left(\frac{3}{3^2+t^2}-\frac{\pi}{1!}\frac{7x^2}{7^2+t^2}+\frac{\pi^2}{2!}\frac{11x^4}{11^2+t^2}-\cdots\right)\frac{x\,dx}{e^{2\pi x}-1}$$

$$= \frac{1}{8\sqrt{\pi}}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi(\frac{1}{2}t). \tag{11}$$

But the left-hand side of (11) is equal to

$$\int_0^\infty \left\{e^{-z} - 4\pi\int_0^\infty \left(e^{-3z} - \frac{\pi x^2}{1!}e^{-7z} + \frac{\pi^2 x^4}{2!}e^{-11z} - \cdots\right)\frac{x\ dx}{e^{2\pi x}-1}\right\}\cos tz\ dz.$$

Hence,

$$\int_0^\infty \left\{e^{-z} - 4\pi\int_0^\infty \frac{xe^{-3z-\pi x^2 e^{-4z}}}{e^{2\pi x}-1}\ dx\right\}\cos tz\ dz$$
$$= \frac{1}{8\sqrt{\pi}}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi(\frac{1}{2}t). \tag{12}$$

It follows from this and Fourier's theorem that

$$e^{-n} - 4\pi e^{-3n}\int_0^\infty \frac{xe^{-\pi x^2 e^{-4n}}}{e^{2\pi x}-1}\ dx$$
$$= \frac{1}{4\pi\sqrt{\pi}}\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi(\frac{1}{2}t)\cos nt\ dt. \tag{13}$$

But it is easily seen from (2) that, if α and β are positive and $\alpha\beta = \pi^2$, then

$$\alpha^{-\frac{1}{4}}\left\{1 + 4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\ dx\right\} = \beta^{-\frac{1}{4}}\left\{1 + 4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\ dx\right\}. \tag{14}$$

From this it follows that the left-hand side of (13) is an even function of n. and so the formula (13) is true for all real values of n.

4. It can easily be shewn that, if $\alpha\beta = 4\pi^2$ and $R(s)$, where $R(s)$ is the real part of s, is greater than -1, then

$$\frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\alpha^{\frac{1}{2}(s-1)} + \frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\alpha^{\frac{1}{2}(s+1)}$$
$$+ \alpha^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty \frac{x^s \sin\alpha xy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\ dx\ dy$$
$$= \frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\beta^{\frac{1}{2}(s-1)} + \frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\beta^{\frac{1}{2}(s+1)}$$

$$+\beta^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty \frac{x^s\sin\beta xy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\,dx\,dy. \tag{15}$$

From this we can shew, by arguments similar to those of §§ 1- 2, that if $\alpha\beta = 4\pi^2$ and $R(s) > -1$ then

$$\frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\frac{\alpha^{\frac{1}{2}(s-1)}}{s-1-t}+\frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\frac{\alpha^{\frac{1}{2}(s+1)}}{s+1-t}+\alpha^{\frac{1}{2}(s+1)}\int_0^\infty\!\!\int_0^\infty\left\{\frac{\alpha xy}{1!(s+3-t)}-\frac{(\alpha xy)^3}{3!(s+7-t)}\right.$$

$$\left.+\frac{(\alpha xy)^5}{5!(s+11-t)}-\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}+\frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\frac{\beta^{\frac{1}{2}(s-1)}}{s-1+t}+\frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\frac{\beta^{\frac{1}{2}(s+1)}}{(s+1+t)}$$

$$+\beta^{\frac{1}{2}(s+1)}\int_0^\infty\!\!\int_0^\infty\left\{\frac{\beta xy}{1!(s+3+t)}-\frac{(\beta xy)^3}{3!(s+7+t)}+\frac{(\beta xy)^5}{5!(s+11+t)}-\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$

$$=\left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}t}\frac{2^{\frac{1}{2}(s-3)}\Gamma\{\frac{1}{4}(s-1+t)\}\Gamma\{\frac{1}{4}(s-1-t)\}}{\pi\quad (s+1)^2-t^2}\times\xi\left(\frac{1+s+t}{2}\right)\xi\left(\frac{1+s-t}{2}\right). \tag{16}$$

From this we deduce that, if $\alpha\beta = 4\pi^2$ and $R(s) > -1$, then

$$\frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\frac{s-1}{(s-1)^2+t^2}\{\alpha^{\frac{1}{2}(s-1)}+\beta^{\frac{1}{2}(s-1)}\}+\frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\frac{s+1}{(s+1)^2+t^2}\{\alpha^{\frac{1}{2}(s+1)}+\beta^{\frac{1}{2}(s+1)}\}$$

$$+\alpha^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty\left\{\frac{\alpha xy}{1!}\frac{s+3}{(s+3)^2+t^2}-\frac{(\alpha xy)^3}{3!}\frac{s+7}{(s+7)^2+t^2}+\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$

$$+\beta^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty\left\{\frac{\beta xy}{1!}\frac{s+3}{(s+3)^2+t^2}-\frac{(\beta xy)^3}{3!}\frac{s+7}{(s+7)^2+t^2}+\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$

$$=\frac{2^{\frac{1}{2}(s-3)}}{\pi}\frac{\Gamma\{\frac{1}{4}(s-1+it)\}\Gamma\{\frac{1}{4}(s-1-it)\}}{(s+1)^2+t^2}$$

$$\times\ \Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\cos\left(\frac{1}{4}t\log\frac{\alpha}{\beta}\right); \tag{17}$$

and

$$\frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\frac{1}{(s-1)^2+t^2}\{\alpha^{\frac{1}{2}(s-1)}-\beta^{\frac{1}{2}(s-1)}\}+\frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\frac{1}{(s+1)^2+t^2}\{\alpha^{\frac{1}{2}(s+1)}-\beta^{\frac{1}{2}(s+1)}\}$$

$$+\alpha^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty\left\{\frac{\alpha xy}{1!}\frac{1}{(s+3)^2+t^2}-\frac{(\alpha xy)^3}{3!}\frac{1}{(s+7)^2+t^2}+\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$

$$-\beta^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty\left\{\frac{\beta xy}{1!}\frac{1}{(s+3)^2+t^2}-\frac{(\beta xy)^3}{3!}\frac{1}{(s+7)^2+t^2}+\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$

$$=\frac{2^{\frac{1}{2}(s-3)}}{\pi}\frac{\Gamma\{\frac{1}{4}(s-1+it)\}\Gamma\{\frac{1}{4}(s-1-it)\}}{(s+1)^2+t^2}$$

$$\times\;\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\sin\left(\frac{1}{4}t\log\frac{\alpha}{\beta}\right). \tag{18}$$

5. Proceeding as in § 3 we can shew that, if n is real, and

$$F(n)=\int_0^\infty\frac{\Gamma\{\frac{1}{4}(s-1+it)\}\Gamma\{\frac{1}{4}(s-1-it)\}}{(s+1)^2+t^2}\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\cos ntdt,$$

then, if $R(s)>1$,

$$F(n)=\frac{1}{8}(4\pi)^{-\frac{1}{2}(s-3)}\left\{\int_0^\infty\frac{x^s\,dx}{(e^{xe^n}-1)(e^{xe^{-n}-1})}-2\Gamma(s)\zeta(s)\cosh n(1-s)\right\}; \tag{19}$$

if $-1<R(s)<1$,

$$F(n)=\frac{1}{8}(4\pi)^{-\frac{1}{2}(s-3)}\int_0^\infty x^s\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)\,dx; \tag{20}$$

if $-3<R(s)<-1$,

$$F(n)=\frac{1}{8}(4\pi)^{-\frac{1}{2}(s-3)}\left\{\int_0^\infty x^s\left\{\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}+\frac{1}{2}\right)\right.$$

$$\left.\times\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}+\frac{1}{2}\right)\,dx-\Gamma(1+s)\zeta(1+s)\cosh n(1+s)\right\}; \tag{21}$$

and so on. If, in particular, we put $s=0$ in (20), we obtain

$$\int_0^\infty\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\{\Xi(\frac{1}{2}t)\}^2\frac{\cos nt}{1+t^2}\,dt$$

$$=\pi\sqrt{\pi}\int_0^\infty\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)\,dx. \tag{22}$$