# RIEMANN'S HYPOTHESIS 

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#### Abstract

We examine the rich history of Riemann's 1859 hypothesis and some of the attempts to prove it and the partial progress resulting from these efforts.


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## 1. Introduction

On Christmas Eve 1849 Gauss wrote a letter to his former student Encke in which he described his thoughts about the number of primes $\pi(x)$ less than or equal to $x$. Gauss had developed his ideas around 1792 when he was 15 or 16 years old. His conclusion was that up to a small error term $\pi(x)$ was close to $\mathrm{li}(x)$ the logarithmic integral

$$
\operatorname{li}(x)=\int_{2}^{x} \frac{d t}{\log t} .
$$

The strikingly good approximation was computed over and over by Gauss at intervals up to 3 million, all computed by Gauss himself who could determine the number of primes in a chiliad (block of one thousand numbers) in 15 minutes.

Riemann, in 1859, in a paper [R] written on the occasion of his admission to the Berlin Academy of Sciences and read to the Academy by none other than Encke, devised an analytic way to understand the error term in Gauss' approximation, via the zeros of the zeta-function

$$
\zeta(s)=\sum_{n=1}^{\infty} \frac{1}{n^{s}} .
$$

The connection of $\zeta(s)$ with prime numbers was found by Euler via his product formula

$$
\sum_{n=1}^{\infty} \frac{1}{n^{s}}=\prod_{p}\left(1-p^{-s}\right)^{-1}
$$

where there is one factor for each prime number $p$. This formula encodes the fundamental theorem of arithmetic that every integer is a product of primes in a unique way. Riemann saw that the zeros of what we now call the Riemann zeta-function were the key to an analytic expression for $\pi(x)$. Riemann observes that

$$
\pi^{-s / 2} \Gamma(s / 2) \zeta(s)=\int_{0}^{\infty} \psi(x) x^{s-1} d x
$$

where

$$
\psi(x)=\sum_{n=1}^{\infty} e^{-\pi n^{2} x}
$$

and then uses

$$
2 \psi(x)+1=x^{-\frac{1}{2}}\left(2 \psi\left(\frac{1}{x}\right)+1\right)
$$

which follows from a formula of Jacobi, to transform the part of the integral on $0 \leq x \leq 1$. In this way he finds that $\zeta(s)$ is a meromorphic function of $s$ with its only a pole a simple pole at $s=1$ and that

$$
\xi(s)=\frac{1}{2} s(s-1) \pi^{-s / 2} \Gamma(s / 2) \zeta(s)
$$

is an entire function of order 1 which satisfies the functional equation

$$
\xi(s)=\xi(1-s)
$$

As a consequence, $\zeta(s)$ has zeros at $s=-2 n$ for $n \in \mathbb{Z}_{+}$, these are the so called trivial zeros, as well as a denser infinite sequence of zeros in the "critical strip"

$$
0 \leq \Re s \leq 1
$$

The Euler product precludes any zeros with real part larger than 1. Also, for any non-trivial zero $\rho=\beta+i \gamma$ there is a dual zero $1-\rho$ by the functional equation. Also, $\bar{\rho}$ and $1-\bar{\rho}$ are zeros since $\zeta(s)$ is real for real $s$ but note that $\bar{\rho}$ coincides with $1-\rho$ whenever $\beta=1 / 2$. Riemann
used Stirling's formula and the functional equation to evaluate the number of non-trivial zeros in the critical strip as

$$
N(T):=\#\{\rho=\beta+i \gamma: 0<\gamma \leq T\}=\frac{T}{2 \pi} \log \frac{T}{2 \pi e}+7 / 8+S(T)+O(1 / T)
$$

where

$$
S(T)=\frac{1}{\pi} \arg \zeta(1 / 2+i T)
$$

where the argument is determined by beginning with $\arg \zeta(2)=1$ and continuous variation along line segments from 2 to $2+i T$ and then to $1 / 2+i T$, taking appropriate action if a zero is on the path. Riemann implied that $S(T)=O(\log T)$ a fact that was later proven rigorously by Backlund [Bac18]. Thus, the zeros get denser as one moves up the critical strip.

The functional equation together with Riemann's formula for the number of zeros of $\zeta(s)$ up to a height $T$ help give us a picture of $\zeta(1 / 2+i t)$. In particular Hardy defined a function $Z(t)$ which is a real function of a real variable having the property that $|\zeta(1 / 2+i t)|=|Z(t)|$. It may be defined by

$$
Z(t)=\chi(1 / 2-i t)^{1 / 2} \zeta(1 / 2+i t)
$$

where $\chi(s)$ is the factor from the functional equation which may be written in asymmetric form as

$$
\chi(1-s)=2(2 \pi)^{-s} \gamma(s) \cos \frac{\pi s}{2} .
$$

Here are some graphs of $Z(t)$ :

1.1. Riemann's formula for primes. Riemann found an exact formula for $\pi(x)$. If we invert Euler's formula we find

$$
\frac{1}{\zeta(s)}=\prod_{p}\left(1-p^{-s}\right)=1-2^{-s}-3^{-s}-5^{-s}+6^{-s}+\cdots=\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}}
$$

where $\mu$ is known as the Möbius mu-function. A simple way to explain the value of $\mu(n)$ is that it is 0 if $n$ is divisible by the square of any prime, while if $n$ is squarefree then it is +1 if $n$ has an even number of prime divisors and -1 if $n$ has an odd number of prime divisors. Riemann's formula is

$$
\pi(x)=\sum_{n=1}^{\infty} \frac{\mu(n)}{n} f\left(x^{1 / n}\right)
$$

where

$$
f(x)=\operatorname{li}(x)-\sum_{\rho} \operatorname{li}\left(x^{\rho}\right)-\ln 2+\int_{x}^{\infty} \frac{d t}{t\left(t^{2}-1\right) \log t}
$$

Here the $\rho=\beta+i \gamma$ are the zeros of $\zeta(s)$ and the sum over the $\rho$ is to be taken symmetrically, i.e. to pair the zero $\rho$ with its dual $1-\rho$ as the sum is performed. Thus the difference between Riemann's formula and Gauss' conjecture is, to a first estimation, about $\operatorname{li}\left(x^{\beta_{0}}\right)$ where $\beta_{0}$ is the largest or the supremum of the real parts of the zeros. Riemann conjectured that all of the zeros have real part $\beta=1 / 2$ so that the error term is of size $x^{1 / 2} \log x$. This assertion of the perfect balance of the zeros, and so of the primes, is Riemann's Hypothesis.

In 1896 Hadamard and de la Vallée Poussin independently proved that $\zeta(1+i t) \neq 0$ and concluded that

$$
\pi(N) \sim \operatorname{li}(N)
$$

a theorem which is known as the prime number theorem.

## 2. Riemann and the zeros

After his evaluation of $N(T) \sim \frac{T}{2 \pi} \log T$ he asserted that we find about this many real zeros of

$$
\Xi(t):=\xi(1 / 2+i t)
$$

in $0<t \leq T$. This is an assertion which is still unproven and is the subject of speculation. His memoir "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" is only 8 pages. But in the early 1930s his Nachlass was delivered from the library at Göttingen to Princeton where C. L. Siegel [?] looked over Riemann's notes at the Institute for Advanced Study. In the notes were found an "approximate functional" equation, which had been independently found by Hardy and Littlewood [HL29]:

$$
\zeta(s)=\sum_{n \leq \frac{t}{2 \pi}} \frac{1}{n^{s}}+\chi(s) \sum_{n \leq \frac{t}{2 \pi}} \frac{1}{n^{1-s}}+O\left(t^{-\sigma / 2}\right)
$$

for $s=\sigma+i t$. Here $\chi(s)$ is the factor from the asymmetric form of the functional equation

$$
\zeta(s)=\chi(s) \zeta(1-s)
$$

with

$$
\chi(s)=\frac{\pi^{-(1-s) / 2} \Gamma((1-s) / 2)}{\pi^{-s / 2} \Gamma(s / 2)}=2(2 \pi)^{s-1} \Gamma(1-s) \sin \frac{\pi s}{2}
$$

Now $|\chi(1 / 2+i t)|=1$; in fact $\chi(1 / 2+i t)=e^{i t \log t / 2 \pi}(1+O(1 / t))$. One might be led to believe that $1+\chi(s)$ is a reasonable approximation to $\zeta(s)$, i.e. that the contributions from the oscillatory terms $2^{-s}$ etc. might be small overall. This approximation has zeros on $s=1 / 2+i t$ at a rate sufficient to produce asymptotically all of the zeros of $\zeta(s)$, so it seems reasonable to conclude that almost all of the zeros are on this line, and to go on and conjecture that ALL of the zeros are on the one-line. But we have found it hard to make this reasoning precise.

Riemann computed the first few zeros:

$$
1 / 2+i 14.13 \ldots, 1 / 2+i 21.02 \ldots, 1 / 2+i 25.01 \ldots, \ldots
$$

A good way to be convinced that these are indeed zeros is to use the easily proven formula

$$
\left(1-2^{1-s}\right) \zeta(s)=1-\frac{1}{2^{s}}+\frac{1}{3^{s}}-\frac{1}{4^{s}} \pm \ldots
$$

The alternating series on the right converges for $\Re s>0$ and so, for example,

$$
s=1 / 2+i 14.1347251417346937904572519835624 \ldots
$$

can be substituted into a truncation of this series (using a computer algebra system) to see that it is very close to 0 . (See www.lmfdb.org to find a list of high precision zeros of $\zeta(s)$ as well as a wealth of information about $\zeta(s)$ and similar functions called L-functions.)

## 3. Elementary equivalents of the Riemann Hypothesis

We've mentioned that the Riemann Hypothesis implies a good error bound for the prime number theorem. The converse is also true: the Riemann Hypothesis is equivalent to

$$
\pi(x):=\sum_{p \leq x} 1=\int_{2}^{x} \frac{d u}{\log u}+O\left(x^{1 / 2} \log x\right)
$$

and to

$$
\psi(x):=\sum_{p^{k} \leq x} \log p=x+O\left(x^{1 / 2} \log ^{2} x\right)
$$

Equivalences may also be phrased in terms of the Möbius function $\mu(n)$ where

$$
\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}}
$$

It is not difficult to show that the Riemann Hypothesis is equivalent to the assertion that this series is (conditionally) convergent for any $s$ with $1 / 2<\sigma<1$.

The Riemann Hypothesis is also equivalent to each of

$$
M(x):=\sum_{n \leq x} \mu(n)=O\left(x^{1 / 2+\epsilon}\right)
$$

and

$$
\int_{1}^{X}(\psi(x)-x)^{2} \frac{d x}{x^{2}} \sim C \log X
$$

The assertion that

$$
\int_{1}^{X} M(x)^{2} \frac{d x}{x^{2}} \sim C \log X
$$

implies the Riemann Hypothesis and that all of the zeros are simple. A question is whether the converse is true.

Stieltjes thought that he had found a proof that $M(x)=O\left(x^{1 / 2}\right)$ and so of the Riemann Hypothesis. His claim appeared in Comptes Rendus Mathematique. Consequently de la Hadamard was somewhat apologetic about his inconsequential offering in his own paper [Had96] which proves the prime number theorem!


Figure 1. A plot of $M(x)$ versus $\pm \sqrt{x}$

## 4. The general distribution of the zeros

An immediate consequence of Euler's product formula

$$
\zeta(s)=\prod_{p}\left(1-p^{-s}\right)^{-1}
$$

is that $\zeta(s) \neq 0$ if $\Re s>1$. A subsequent consequence of Riemann's functional equation is that $\zeta(s) \neq 0$ if $\Re s<0$ except at $s=-2,-4,-6, \ldots$, the so-called trivial zeros. The prime number theorem

$$
\pi(x) \sim \frac{x}{\log x}
$$

is equivalent to the assertion that $\zeta(1+i t) \neq 0$; equivalently $\zeta(i t) \neq 0$. In order to be precise about the error term in the prime number theorem it is necessary to prove that there is a
region near the line $\sigma=1$ in which there are no zeros. It was shown by de la Vallée Poussin in 1899 [Val96] that

$$
\zeta(\sigma+i t) \neq 0
$$

for $\sigma>\frac{c}{\log (2+|t|)}$ for a specific $c$. This is known as a zero-free region. The best known shape of the zero-free region is due to Korobov [Kor58] and Vinogradov [Vin58] in 1958: $\zeta(\sigma+i t)$ is free of zeros when

$$
\sigma>1-\frac{C}{(\log t)^{2 / 3}(\log \log t)^{1 / 3}}
$$

The best explicit value of $C$ is due to Kevin Ford [For00] who showed that $C=1 / 54.57$ is admissible.
4.1. Density results. Bounds for the quantity

$$
N(\sigma, T):=\#\{\rho=\beta+i \gamma: \beta \geq \sigma \text { and } 0<\gamma \leq T\}
$$

are known as density estimates. Near to $\sigma=1$ we have [For00]

$$
N(\sigma, T) \ll T^{58.05(1-\sigma)^{3 / 2}}(\log T)^{15}
$$

As we move away from the line $\sigma=1$ our estimates get weaker but are still pretty good. Bounds often take the shape

$$
N(\sigma, T) \ll T^{k(\sigma)+\epsilon}
$$

there are many forms of admissible $k(\sigma)$. A strong classical one due to Ingham in 1940 [Ing40] is that

$$
N(\sigma, T)=O\left(T^{3(1-\sigma) /(2-\sigma)} \log ^{5} T\right) ;
$$

this is still the best bound when $1 / 2<\sigma<3 / 4$. It is known that $k(\sigma)=3 / 2-\sigma$ is also admissible.

The unproven "Density Hypothesis" is that the above holds with $k(\sigma)=2(1-\sigma)$.
It is known that an estimate of the sort

$$
\zeta(1 / 2+i t) \ll t^{c}(\log t)^{c^{\prime}}
$$

implies that

$$
N(\sigma, T) \ll T^{2(1+2 c)(1-\sigma)} \log ^{5} T
$$

see [Tit86]. Thus, the Density Hypothesis is a consequence of the Lindelöf Hypothesis (for which see below).

A consequence of the Density Hypothesis is that for any $\epsilon>0$ there is a $C(\epsilon)$ such that

$$
p_{n+1}-p_{n} \leq C(\epsilon) n^{1 / 2+\epsilon}
$$

where $p_{n}$ denotes the $n$th prime. This estimate is not quite strong enough to conclude that there is always a prime between consecutive squares.

Here is a plot of the exponent in density theorems (the minimum of the two graphs is an admissible density exponent):

4.2. Zeros near the $\mathbf{1} / \mathbf{2}$-line. It has been known for quite some time that almost all of the zeros are near the $1 / 2$-line. For example at least $99 \%$ of the zeros $\rho=\beta+i \gamma$ satisfy

$$
|\beta-1 / 2|<\frac{8}{\log \gamma}
$$

and almost all are within $\phi(\gamma) / \log \gamma$ of the critical line where $\phi$ is any function which goes to infinity. Thus, we know that the zeros cluster around the critical line.
4.3. Zeros on the critical line. Many people have worked on verifying the Riemann Hypothesis. Today it is known that the first ten trillion zeros are all on the critical line $\Re s=1 / 2$ !

Hardy was the first one to show that there are infinitely many zeros on the $1 / 2$-line. He and Littlewood [HL18] later gave proofs that the number of zeros on the $1 / 2$-line up to a height $T$ is more than a positive constant times $T$. In 1942 Selberg [Sel42] proved that a positive proportion of the zeros are on the critical line. In 1973 N. Levinson [Lev74] proved that at least $1 / 3$ of the zeros are on the half-line. This was improved in 1989 to at least $2 / 5$ of the zeros are on the line. The current record is Feng [Fen12] with 0.412 ; for simple zeros the record proportion is due to Bui, Conrey, and Young [BCY11] who show that at least 0.405 of the zeros of $\zeta(s)$ are on the critical line and simple.

It follows from the Riemann Hypothesis that all of the zeros of all of the derivatives $\xi^{(k)}(s)$ are on the critical line. Along these lines it can be shown, for example, that more than $4 / 5$ of the zeros of $\xi^{\prime}(s)$ are on the critical line and more than $99 \%$ of the zeros of $\xi^{(5)}(s)$ are on the critical line, see [Con83].

## 5. The Lindelöf Hypothesis

The assertion that for any $\epsilon>0$,

$$
\zeta(1 / 2+i t) \ll_{\epsilon} t^{\epsilon}
$$

is known as the Lindelöf Hypothesis and is a consequence of the Riemann Hypothesis. It is a consequence of the functional equation, trivial bounds for $\zeta(i t)$ and $\zeta(1+i t)$, and general
principles of the growth of analytic functions that

$$
\zeta(1 / 2+i t) \ll_{\epsilon} t^{\frac{1}{4}+\epsilon}
$$

this is known as the convexity bound. Weyl, using exponential sums, improved the bound to

$$
\zeta(1 / 2+i t) \ll \epsilon_{\epsilon} t^{\frac{1}{6}+\epsilon}
$$

Bombieri and Iwaniec [BI86] used some novel ideas to show

$$
\zeta(1 / 2+i t) \ll t^{89 / 560+\epsilon}
$$

Huxley [Hux05] obtained

$$
\zeta(1 / 2+i t) \ll t^{32 / 205} \log ^{c} t
$$

Recently Bourgain [Bou14] announced that

$$
\zeta(1 / 2+i t) \ll t^{53 / 342+\epsilon}
$$

5.1. Estimates for $\zeta(s)$ near the 1-line. Richert [Ric67] proved the important estimate that for an explicit $c>0$,

$$
|\zeta(\sigma+i t)|<c t^{100(1-\sigma)^{3 / 2}} \log ^{2 / 3} t
$$

for $1 / 2 \leq \sigma \leq 1, t \geq 2$. Such a bound is useful for zero-free regions, the error term in the prime number theorem, and zero density results near 1. K. Ford [For02] has improved these made the constants explicit:

$$
|\zeta(\sigma+i t)|<76.2 t^{4.45(1-\sigma)^{3 / 2}} \log ^{2 / 3} t
$$

## 5.2. $\mathbf{1}$ versus 2. RH implies that

$$
|\zeta(1 / 2+i t)| \ll \exp \left(\frac{\log 2}{2} \frac{\log t}{\log \log t}+O\left(\frac{\log t \log \log \mid \log t}{(\log \log t)^{2}}\right)\right)
$$

see [ChS11] . It can be proven, see [Sou08] that every interval $[T, 2 T]$ contains a $t$ for which

$$
|\zeta(1 / 2+i t)| \geq \exp \left((1+o(1)) \frac{(\log t)^{1 / 2}}{(\log \log t)^{1 / 2}}\right)
$$

Which of these is closer to the true largest order of magnitude of $\zeta$ on the $1 / 2$-line? It is difficult to say, though most people (not the author!) think that the lower bound ( $\Omega$-result) is closer to the truth. Farmer, Gonek, and Hughes [FGH07] conjecture that

$$
|\zeta(1 / 2+i t)| \leq \exp \left(\sqrt{\left(\frac{1}{2}+o(1)\right)(\log t)(\log \log t)}\right)
$$

## 6. Computations

Turing was the first to use a computer to calculate the zeros of $\zeta(s)$. He proposed an efficient rigorous method to verify RH up to a given height, or indeed within an interval. It involves using a precise version of the approximate functional equation, known as the Riemann - Siegel formula, to evaluate $Z(t)$ and detect sign changes, together with an explicit bound for the average of $S(t)$ namely if $t_{2}>t_{1}>168 \pi$ then

$$
\int_{t_{1}}^{t_{2}} S(t) d t=\Theta\left(2.30+0.128 \log \frac{t_{2}}{2 \pi}\right)
$$

to verify that all of the zeros are accounted for. (Here $\Theta$ represents a number that is at most 1 in absolute value.) Goldfeld has pointed out that if $\zeta(s)$ had a double zero somewhere up the line, the computational verification of RH would come to a halt because it would be impossible to distinguish a double zero from two very close zeros either on or off the line.

Here is one of Turing's versions of the Riemann-Siegel formula:
Theorem 1. Let $m$ and $\xi$ be respectively the integral and non-integral parts of $\tau^{1 / 2}$ and

$$
\begin{gathered}
\tau \geq 64, \\
\kappa(\tau)=\frac{1}{4 \pi i} \log \frac{\Gamma\left(\frac{1}{4}+\pi i \tau\right)}{\Gamma\left(\frac{1}{4}-\pi i \tau\right)}-\frac{1}{4} \tau \log \pi, \\
Z(\tau)=\zeta(1 / 2+2 \pi i \tau) e^{-2 \pi \kappa(\tau)}, \\
\kappa_{1}(\tau)=\frac{1}{2}\left(\tau \log \tau-\tau-\frac{1}{2}\right), \\
h(\xi)=\frac{\cos 2 \pi\left(\xi^{2}-\xi-\frac{1}{16}\right)}{\cos 2 \pi \xi} .
\end{gathered}
$$

Then $Z(\tau)$ is real and

$$
\begin{gathered}
Z(\tau)=2 \sum_{n=1}^{m} n^{-\frac{1}{2}} \cos 2 \pi\{\tau \log n-\kappa(\tau)\}+(-1)^{m+1} \tau^{-\frac{1}{4}} h(\xi)+\Theta\left(1.09 \tau^{-\frac{3}{4}}\right. \\
\kappa(\tau)=\kappa_{1}(\tau)+\Theta\left(0.006 \tau^{-1}\right)
\end{gathered}
$$

In 1988, Andrew Odlyzko and Schönhage [OS88] invented an algorithm which allowed for the very speedy calculation of many values of $\zeta(s)$ at once. The Riemann-Siegel allows for a single computation of $\zeta(1 / 2+i t)$ with $T<t<T+T^{1 / 2}$ in time $T^{1 / 2+\epsilon}$. The Odlyzko Schönhage algorithm allows for a single computation in time $T^{\epsilon}$ after a pre-computation of time $T^{1 / 2+\epsilon}$. This led Odlyzko to compile extensive statistics about the zeros at enormous heights - up to $10^{23}$ and higher. His famous graphs showed an incredible match between data for zeros of $\zeta(s)$ and for the proven statistical distributions for random matrices.

Here is a list of contributors to verifying RH in an initial segment of the $1 / 2$-line and the year they did the work.

| G. H. B. Riemann | 3 | 1859 |
| :--- | :--- | :--- |
| J. P. Gram | 15 | 1903 |
| R. J. Backlund | 79 | 1914 |
| E. C. Titchmarsh | 1041 | 1935 |
| A. M. Turing | 1104 | 1953 |
| D. H. Lehmer | 15000 | 1956 |
| D. H. Lehmer | 25000 | 1956 |
| N. A. Meller | 35337 | 1958 |
| R S. Lehman | 250000 | 1966 |
| J. B. Rosser, J. M. Yohe, L. Schoenfeld | 3500000 | 1968 |
| R. P. Brent | 40000000 | 1977 |
| R. P. Brent | 81000000 | 1979 |
| R. P. Brent, J. van de Lune, H. J. J. te Riele, D. T. Winter | 200000001 | 1982 |
| J. van de Lune, H. J. J. te Riele | 300000001 | 1983 |
| J. van de Lune, H. J. J. te Riele, D. T. Winter | 1500000001 | 1986 |
| J. van de Lune | 10000000000 | 2001 |
| S. Wedeniwski | 900000000000 | 2004 |
| X. Gourdon, P. Demichel | 10000000000000 | 2004 |

Ghaith Hiary [Hia11] has improved these algorithms. He can compute one value of $\zeta(1 / 2+$ it) in time $T^{1 / 3+\epsilon}$ using an algorithm that has been implemented by Jonathan Bober and Hiary; he has a more complicated algorithm that will work in time $T^{\frac{4}{13}+\epsilon}$. They have verified RH in some small ranges around the $10^{33}$ zero!

Bober's website [Bob14] has some great pictures of large values of $Z(t)$.

## 7. Why do we think RH is true?

The main reason is because of the beauty of the conjecture. It strikes our sensibilities as appropriate that something so incredibly symmetric should be true in mathematics. The second reason is that the first 10 trillion zeros are all on the line. If there were a counterexample it should have shown itself by now. A third reason is that the numerical evidence for all L-functions ever computed lead to this conclusion; some have thought that a counterexample to RH might show itself when computing zeros of L-functions associated with Maass forms because these have no arithmetic-geometry interpretation (eg. their coefficients are generally believed to be transcendental); however the computations reveal that the zeros are still on the $1 / 2$-line. A fourth reason is probabilistic. RH is known to be equivalent to the assertion that $M(x):=\sum_{n \leq x} \mu(n) \ll x^{1 / 2} \log ^{2} x$. This sum represents the difference between the number of squarefree integers up to $x$ with an even number of prime factors and the number with an odd number of prime factors. It is similar to the difference in the number of heads and tails when one flips $x$ coins, and so should be around $x^{1 / 2}$.

Here is another more elaborate reason. Suppose that a Dirichlet series $F(s)=\sum_{n=1}^{\infty} a_{n} n^{-s}$ converges for $\sigma>0$, and suppose that it has a zero with real part $\beta>1 / 2$. We might reasonably expect it then to have $\gg T$ zeros in $\sigma>\beta-\epsilon, 0<t<T$ for any large $T$ by
"almost periodicity." But zero density results tell us that there are $\ll T^{1-\delta}$ zeros in $\sigma \geq \sigma_{0}$ and $t<T$.
7.1. Almost periodicity. As just mentioned a possible strategy is to try to prove that if $\zeta(s)$ has one zero off the line then it has infinitely many off the line. Bombieri [Bom00] has come closest to achieving this.

Here is a conjecture that attempts to encapsulate this idea:
Conjecture 1. Suppose that the Dirichlet series

$$
F(s)=\sum_{n=1}^{\infty} a_{n} n^{-s}
$$

converges for $\sigma>0$ and has a zero in the half-plane $\sigma>1 / 2$. Then there is a number $C_{F}>0$ such that $F(s)$ has $>C_{F} T$ zeros in $\sigma>1 / 2,|t| \leq T$.

This seemingly innocent conjecture implies the Riemann Hypothesis for virtually any primitive L-function (except curiously possibly the Riemann zeta-function itself!). And it seems that the Euler product condition has already been used (in the density result above); i.e. the hard part is already done. Note that the " $1 / 2$ " in the conjecture needs to be there as the example

$$
\sum_{n=1}^{\infty} \frac{\mu(n) / n^{1 / 2}}{n^{s}}
$$

demonstrates. Assuming RH, this series converges for $\sigma>0$ and its lone zero is at $s=1 / 2$. This example is possibly at the boundary of what is possible.

## 8. A spectral interpretation

Hilbert and Pólya are reputed to have suggested that the zeros of $\zeta(s)$ should be interpreted as eigenvalues of an appropriate operator.

Odlyzko wrote to Pólya to ask about this. Here is the text of Odlyzko's letter, dated Dec. 8, 1981.

Dear Professor Pólya:
I have heard on several occasions that you and Hilbert had independently conjectured that the zeros of the Riemann zeta function correspond to the eigenvalues of a self-adjoint hermitian operator. Could you provide me with any references? Could you also tell me when this conjecture was made, and what was your reasoning behind this conjecture at that time?

The reason for my questions is that I am planning to write a survey paper on the distribution of zeros of the zeta function. In addition to some theoretical results, I have performed extensive computations of the zeros of the zeta function, comparing their distribution to that of random hermitian matrices, which have been studied very seriously by physicists. If a hermitian operator associated to the zeta function exists, then in some respects we might expect it to behave like a random hermitian operator, which in turn ught to resemble a random hermitian matrix. I have discovered that the distribution of zeros of
the zeta function does indeed resemble the distribution of eigenvalues of random hermitian matrices of unitary type.

Any information or comments you might care to provide would be greatly appreciated.
Sincerely yours,
Andrew Odlyzko
and Pólya's response, dated January 3, 1982.
Dear Mr. Odlyzko,
Many thanks for your letter of Dec. 8. I can only tell you what happened to me.
I spent two years in Göttingen ending around the beginning of 1914. I tried to learn analytic number theory from Landau. He asked me one day: "You know some physics. Do you know a physical reason that the Riemann Hypothesis should be true?"

This would be the case, I answered, if the non-trivial zeros of the $\zeta$ function were so connected with the physical problem that the Riemann Hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.

I never published this remark, but somehow it became known and it is still remembered.
With best regards.
Yours sincerely,
George, Pólya

## 9. The vertical spacing of zeros

In the 1950s physicists predicted that excited nuclear particles emit energy at levels which are distributed like the eigenvalues of random matrices. This was verified experimentally in the 1970s and 1980s; Oriol Bohigas was the first to put this data together in a way that demonstrated this law. Figure 2 shows 96 zeros of $\zeta(s)$ starting at a height $T=1200$ "wrapped" once around a circle for the purposes of comparing with the eigenvalues of a randomly chosen $96 \times 96$ unitary matrix, and with 96 points chosen randomly independently on a circle (Poisson). It should be clear that the zeros of $\zeta(s)$ do not have a Poisson distribution (and would have been clear to anyone looking carefully at them, say in the mid 1900's!).

In 1972 Hugh Montgomery, then a graduate student at Cambridge, delivered a lecture at a symposium on analytic number theory in St. Louis, outlining his work on the spacings between zeros of the Riemann zeta-function. This was the first time anyone had considered such a question. On his flight back to Cambridge he stopped over in Princeton to show his work to Selberg. At afternoon tea at the Institute for Advanced Study, Chowla insisted that Montgomery meet the famous physicist - and former number theorist - Freeman Dyson. When Montgomery explained to Dyson the kernel he had found that seemed to govern the spacings of pairs of zeros, Dyson immediately responded that it was the same kernel that


Figure 2. 96 points of three different types of spacings
governs pairs of eigenvalues of random matrices. Montgomery [Mon73] proved that

$$
\begin{aligned}
& \sum_{\gamma_{1}, \gamma_{2} \in[0, T]} w\left(\gamma_{1}-\gamma_{2}\right) f\left(\frac{\log T}{2 \pi}\left(\gamma_{1}-\gamma_{2}\right)\right) \\
&=\frac{T \log T}{2 \pi}\left(f(0)+\int_{-\infty}^{\infty} f(u)\left[1-\left(\frac{\sin (\pi u)}{\pi u}\right)^{2}\right] d u+o(1)\right)
\end{aligned}
$$

assuming the Riemann Hypothesis and that the Fourier transform $\hat{f}$ of $f$ vanishes outside of $[-1,1]$ and $w(x)=4 /\left(4+x^{2}\right)$. The sum here is over pairs of zeros $1 / 2+i \gamma_{1}$ and $1 / 2+i \gamma_{2}$. The conjecture is that the assumption on the support of $\hat{f}$ is not necessary.

Odlyzko did extensive numerical calculations to test this conjecture; the numerics are stunning! The pair-correlation function in Figure 3 is

$$
1-\left(\frac{\sin \pi x}{\pi x}\right)^{2}
$$

The nearest-neighbor density function is more complicated. It may be given as

$$
\frac{1}{4} \frac{d^{2}}{d t^{2}} \exp \left(\int_{0}^{t} \frac{\sigma(2 u)}{u} d u\right)
$$

where $\sigma=\sigma(s)$ is a solution of a Painlevé equation:

$$
\left(s \sigma^{\prime \prime}\right)^{2}+4\left(s \sigma^{\prime}-\sigma\right)\left(\left(\sigma^{\prime}\right)^{2}-\sigma+s \sigma^{\prime}\right)=0
$$

with a boundary condition

$$
\sigma(s) \sim-\frac{s}{\pi}-\frac{s^{2}}{\pi^{2}} \quad \text { as } s \rightarrow 0
$$

as discovered by Jimbo, Miwa, Mori, and Sato, see [JMMS80].


Figure 3. Odlyzko's graphics

Now we have the challenge of not only explaining why all of the zeros are on a straight line, but also why they are distributed on this line the way they are! The connections with Random Matrix theory first discovered by Montgomery and Dyson have received a great deal of support from seminal papers of Katz and Sarnak [KaSa99] and Keating and Snaith [KS00]. The last 15 years have seen an explosion of work around these ideas. In particular, it definitely seems like there should be a spectral interpretation of the zeros à la Hilbert and Pólya.

## 10. Some initial thoughts about proving RH

10.1. Fourier integrals with all real zeros. Riemann proved that

$$
\Xi(t):=\xi(1 / 2+i t)=\int_{-\infty}^{\infty} \Phi(u) e^{i u t} d u
$$

where

$$
\Phi(u)=\sum_{n=1}^{\infty}\left(4 \pi^{2} n^{4} e^{9 u / 2}-6 n^{2} \pi e^{5 u / 2}\right) \exp \left(-n^{2} \pi e^{2 u}\right)
$$

It is known that $\Phi(u)$ is even, is positive for real $u$ and is (rapidly!) decreasing for $u>0$. Consequently, we can write

$$
\begin{aligned}
\Xi(t) & =2 \int_{0}^{\infty} \Phi(u) \cos u t d u \\
& =\sum_{n=0}^{\infty} \frac{(-1)^{n} b_{n}}{(2 n)!} t^{2 n}
\end{aligned}
$$

where

$$
b_{n}:=\int_{-\infty}^{\infty} \Phi(u) u^{2 n} d u
$$

The Riemann Hypothesis is the assertion that all of the zeros of $\Xi(t)$ are real. This has prompted investigations into Fourier integrals with all real zeros. Polya [Pol27] and deBruijn [deB50] spent a lot of time with such investigations. A sample theorem is

Theorem 2. Let $f(u)$ be an even nonconstant entire function of $u, f(u) \geq 0$ for real $u$, and such that $f^{\prime}(u)=\exp \left(\gamma u^{2}\right) g(u)$, where $\gamma \geq 0$ and $g(u)$ is an entire function of genus $\leq 1$ with purely imaginary zeros only. Then $\Psi(z)=\int_{-\infty}^{\infty} \exp \{-f(u)\} e^{i z u} d t$ has real zeros only.

Now $\Phi(u)>0$ for all real $u$ and $\Phi^{\prime}(u)<0$ for $u \geq 0$. Thus, we can write $\Phi(u)=e^{-f(u)}$. The functional equation for $\zeta(s)$ is equivalent to the assertion that $\Phi(u)$ is even.

In particular, it was shown by Pólya [Pol26] that all of the zeros of the Fourier transform of a first approximation $\Phi^{*}(u)$ to $\Phi(u)$

$$
\Phi^{*}(u)=(2 \pi \cosh (9 u / 2)-3 \cosh 5 u / 2) \exp (-2 \pi \cosh 2 u)
$$

are real. These ideas have been further explored by deBruijn, Newman [New76], Hejhal, Haseo Ki [KK02], [KK03] and others. Hejhal [Hej90] has shown that almost all of the zeros of the Fourier transform of any partial sum of $\Phi(u)$ are real.

A goal of this approach is to determine necessary and sufficient conditions that describe the Fourier transform of a function all of whose zeros are real.
10.2. Jensen's inequalities. In section 14.32 of [Tit86], we find the assertion that RH is equivalent to

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\alpha) \Phi(\beta) e^{i(\alpha+\beta) x} e^{(\alpha-\beta) y}(\alpha-\beta)^{2} d \alpha d \beta \geq 0
$$

for all real $x$ and $y$ where $\Phi(u)$ is as in the last section.
We quote a passage from Pólya's collected works, volume I, page 427, written by M. Marden commenting on the paper of Pólya.

In this paper Professor Pólya reports his findings on examining the "Nachlass" of the Danish mathematician J. L. W. V. Jensen who died in 1925. Fourteen years earlier Jensen had announced that he would publish a paper regarding his algebraic-function theoretic research on the Riemann $\xi$-function. In view of Jensen's well-known interest in the zeros of polynomials and entire functions, expectations were high that Jensen would contribute to the solution of the Riemann hypothesis problem regading the zeros of the $\xi$-function. However, this paper was never published, and so on Jensen's death it was a matter of paramount importance to have his papers examined by an expert in this area. Professor Pólya undertook this task, but after an arduous examination he found no clue to any progress that Jensen may have made towards the Riemann hypothesis.

Professor Pólya does sketch Jensen's algebraic-function-theoretic investigations, many of which were advanced considerably by Pólya's own work.
In this paper, Pólya gives two more necessary and sufficient conditions for RH. RH is equivalent to

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\alpha) \Phi(\beta) e^{i(\alpha+\beta) x}(\alpha-\beta)^{2 n} d \alpha d \beta \geq 0
$$

for all real values of $x$ and $n=0,1,2, \ldots$; and finally RH is equivalent to

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\alpha) \Phi(\beta)(x+i \alpha)^{n}(x+i \beta)^{n}(\alpha-\beta)^{2} d \alpha d \beta \geq 0
$$

for all real values of $x$ and $n=0,1,2, \ldots$.
Pólya points out that the first equivalence to RH follows immediately from the more general theorem that all of the zeros of a real entire function $F(z)$ of genus at most 1 are real if and only if

$$
\frac{\partial^{2}}{\partial y^{2}}|F(z)|^{2} \geq 0
$$

for all $z=x+i y$. To see that this condition is necessary for polynomials suppose that $F(z)=\prod_{j=1}^{J}\left(z-r_{j}\right)$ and let $f(x, y)=|F(z)|^{2}$. Then $\log f=\sum_{j=1}^{J} \log \left(z-r_{j}\right)+\log \left(\bar{z}-\overline{r_{j}}\right)$ so that

$$
\frac{f_{y}}{f}=\sum_{j=1}^{J}\left(\frac{i}{z-r_{j}}-\frac{i}{\bar{z}-\overline{r_{j}}}\right)=2 \sum_{j=1}^{J} \frac{\Im\left(z-r_{j}\right)}{\left|z-r_{j}\right|^{2}}
$$

Taking another partial with respect to $y$ leads to

$$
\frac{f_{y y}-\left(f_{y}\right)^{2}}{f^{2}}=\sum_{j=1}^{J}\left(\frac{1}{\left(z-r_{j}\right)^{2}}+\frac{1}{\left(\bar{z}-\overline{r_{j}}\right)^{2}}\right)=2 \sum_{j=1}^{J} \frac{\Re\left(z-r_{j}\right)^{2}}{\left|z-r_{j}\right|^{4}}
$$

If all of the $r_{j}$ are real we have

$$
\frac{f_{y y}}{f}=4 y^{2}\left(\sum_{j=1}^{J} \frac{1}{\left|z-r_{j}\right|^{2}}\right)^{2}+2 \sum_{j=1}^{J} \frac{\left(x-r_{j}\right)^{2}}{\left|z-r_{j}\right|^{4}}-2 y^{2} \sum_{j=1}^{J} \frac{1}{\left|z-r_{j}\right|^{4}} .
$$

The middle term is clearly positive and the first term is clearly greater than $4 y^{2} \sum_{j=1}^{J} \frac{1}{\left|z-r_{j}\right|^{4}}$ which is twice the third term in absolute value. Thus the condition is necessary.

The second equivalent to RH is a consequence of the fact that if for each real $x$ the function $f(x, y)=|F(x+i y)|^{2}$ is expanded into a power series in $y$ then all of the coefficients should be non-negative. To see this, again for polynomials, let the notation be as above. We have

$$
\left.\left(\frac{\partial}{\partial y}\right)^{n}\left(\frac{f_{y}}{f}\right)\right|_{y=0}=i^{n-1} n!\sum_{j=1}^{J}\left(\frac{1}{\left(z-r_{j}\right)^{n+1}}+\frac{(-1)^{n+1}}{\left(\bar{z}-\overline{r_{j}}\right)^{n+1}}\right)
$$

Now $f$ is even in $y$ so $f_{y} / f$ is odd in $y$. Thus $\left.\left(f_{y} / f\right)^{(n)}\right|_{y=0}$ is 0 when $n$ is even. For odd $n$ we have

$$
\left.\left(f_{y} / f\right)^{(n)}\right|_{y=0}=2(-1)^{(n-1) / 2} n!\sum_{j=1}^{J}\left(x-r_{j}\right)^{-n-1}
$$

(we have used the fact that each $r_{j}$ has a conjugate that is also a root). Suppose that all of the $r_{j}$ are real. Letting $\Sigma_{k}=\Sigma_{k}(x)=\sum_{j=1}^{J}\left(x-r_{j}\right)^{-k}$, we are led to

$$
\begin{aligned}
\frac{f_{y y}}{f} & =2 \Sigma_{2}=2!E_{1} \\
\frac{f^{(4)}}{f} & =12\left(\Sigma_{2}^{2}-\Sigma_{4}\right)=4!E_{2} \\
\frac{f^{(6)}}{f} & =120\left(2 \Sigma_{6}+\Sigma_{2}^{3}-3 \Sigma_{2} \Sigma_{4}\right)=6!E_{3} \\
\frac{f^{(8)}}{f} & =1680\left(-6 \Sigma_{8}+\Sigma_{2}^{4}-6 \Sigma_{2}^{2} \Sigma_{4}+3 \Sigma_{4}^{2}+8 \Sigma_{2} \Sigma_{6}\right)=8!E_{4}
\end{aligned}
$$

where $E_{n}=E_{n}(x)$ is the $n$th elementary symmetric function of the $\left(x-r_{j}\right)^{-2}$. Thus we see that $\frac{\partial^{n}}{\partial y^{n}} f(x, y) \geq 0$ for each $n$ and all $x$ in the case of all real roots $r_{j}$.

The final equivalence is a consequence of the assertion that if

$$
F(z)=a_{0}+\frac{a_{1}}{1!} z+\frac{a_{2}}{2!} z^{2}+\ldots
$$

and if

$$
F_{n}(z):=a_{0} z^{n}+\binom{n}{1} a_{1} z^{n-1}+\binom{n}{2} a_{2} z^{n-2}+\cdots+a_{n}
$$

then for all real $x$ and $n=1,2, \ldots$ the inequality

$$
F_{n}^{2}(x)-F_{n-1}(x) F_{n+1}(x)>0
$$

holds. The application of these to RH comes about because of the formulae

$$
\begin{aligned}
|\Xi(z)|^{2} & =\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\alpha) \Phi(\beta) e^{i(\alpha+\beta) x} e^{(\alpha-\beta) y} d \alpha d \beta \\
& =\sum_{n=0}^{\infty} \frac{y^{2 n}}{(2 n)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\alpha) \Phi(\beta) e^{i(\alpha+\beta) x}(\alpha-\beta)^{2 n} d \alpha d \beta
\end{aligned}
$$

and

$$
\Xi_{n}(z)=\int_{-\infty}^{\infty} \Phi(u)(z+i u)^{n} d u
$$

Note, for example, the third equivalence with $n=2$ implies that if RH is true then it must be the case that

$$
b_{0} b_{1} X^{4}+\left(3 b_{1}^{2}-b_{0} b_{2}\right) X^{2}+b_{1} b_{2}>0
$$

for all real $X$ where we are using the notation

$$
b_{n}=\int_{-\infty}^{\infty} \Phi(u) u^{2 n} d u
$$

from above. This inequality holds in turn if the discriminant of the quadratic in $X^{2}$ is negative:

$$
9 b_{1}^{4}-10 b_{0}^{2} b_{1} b_{2}+b_{0}^{2} b_{2}^{2}<0
$$

i.e.

$$
\left(9 b_{1}^{2}-b_{0} b_{2}\right)\left(b_{1}^{2}-b_{0} b_{2}\right)<0 .
$$

A consequence is that $b_{0} b_{2}<9 b_{1}^{2}$. (The Turan inequalities, see below, imply that $3 b_{1}^{2}>b_{0} b_{2}$, that $5 b_{2}^{2}>3 b_{1} b_{3}$, that $7 b_{3}^{2}>5 b_{2} b_{4}$ etc. and Cauchy's inequality implies that $b_{n}^{2} \leq b_{n-a} b_{n+a}$ for $a=1,2, \ldots, n$ so in particular

$$
3 b_{1}^{2}>b_{0} b_{2}>b_{1}^{2}
$$

In fact it is easily calculated that the ratio $\frac{b_{0} b_{2}}{b_{1}^{2}}=2.79 \ldots$. Note that the Karlin-Nuttall inequality below would have this ratio smaller than 6 .)

For $n=3$ the Jensen inequality implies that

$$
b_{0} b_{1} X^{6}+\left(6 b_{1}^{2}-3 b_{0} b_{2}\right) X^{4}+3 b_{1} b_{2} X^{2}+b_{2}^{2}>0
$$

for all $X$. The discriminant of this cubic in $X^{2}$ is

$$
-746496 b_{0} b_{2}^{6}\left(b_{0}^{3} b_{2}^{3}-7 b_{0}^{2} b_{1}^{2} b_{2}^{2}+11 b_{0} b_{1}^{4} b_{2}-5 b_{1}^{6}\right)^{2}<0
$$

so that the cubic has only one real root. Since the value at $x=0$ is positive, the real root is negative and so the third Jensen inequality is always true. For $n=4$ the condition is

$$
b_{0} b_{1} X^{8}+\left(10 b_{1}^{2}-6 b_{0} b_{2}\right) X^{6}+\left(5 b_{1} b_{2}+b_{0} b_{3}\right) X^{4}+\left(10 b_{2}^{2}-6 b_{1} b_{3}\right) X^{2}+b_{2} b_{3}>0
$$

for all $X$.

## 11. Grommer inequalities

In 1914 Grommer [Gro14] found a necessary and sufficient condition for the reality of the zeros of an entire function. We describe how it applies to the Riemann Hypothesis. Let $\Xi(t)=\xi(1 / 2+i t)$ so that RH is the assertion that all zeros of $\Xi$ are real. Now the functional equation for $\zeta$ is equivalent to the fact that $\Xi(t)$ is even. Let $Y(t)=\Xi(\sqrt{t})$ and let

$$
-\frac{Y^{\prime}}{Y}(t)=s_{1}+s_{2} t+s_{3} t^{2}+\ldots
$$

Then RH is equivalent to the assertion that for each $n$,

$$
D_{n}=\operatorname{det}\left(\begin{array}{llll}
s_{2} & s_{3} & \ldots & s_{n+1} \\
s_{3} & s_{4} & \ldots & s_{n+2} \\
\vdots & \vdots & & \vdots \\
s_{n+1} & s_{n+2} & \ldots & s_{2 n}
\end{array}\right)>0
$$

The collection of inequalities for $n=1$ applied to $Y(t)=\Xi(\sqrt{t})$ and all of its derivatives are sometimes known as the Turan inequalities.

Here is a proof of the necessity of Grommer's criterion. First off we consider the polynomial case. Assume that $P$ is a polynomial with $P(0) \neq 0$. Let

$$
P(z)=\prod_{r=1}^{n}\left(z-1 / z_{r}\right)
$$

be a polynomial with real coefficients. We have

$$
\frac{P^{\prime}}{P}(z)=\sum_{r=1}^{n} \frac{1}{z-1 / z_{r}}=-\sum_{r=1}^{n} \frac{z_{r}}{1-z z_{r}}=-\sum_{r=1}^{n} \sum_{m=0}^{\infty} z^{m} z_{r}{ }^{m+1}
$$

so that

$$
-\frac{P^{\prime}}{P}(z)=s_{1}+s_{2} z+s_{3} z^{2}+\ldots
$$

where

$$
s_{m}=\sum_{r=1}^{n} z_{r}^{m}
$$

is the sum of the $m$ th powers of the reciprocal roots. Let $D_{m}$ be the $m \times m$ Grommer determinant as above. The key observation is that

$$
D_{n}=\Delta\left(z_{1}, \ldots, z_{n}\right)^{2} \prod_{r=1}^{n} z_{r}^{2}
$$

where $\Delta$ is the Vandermonde determinant for which we have the formula

$$
\Delta\left(z_{1}, \ldots, z_{n}\right)=\prod_{1 \leq i<j \leq n}\left(z_{j}-z_{i}\right)
$$

More generally, if $m \leq n$, then

$$
D_{m}=\sum_{\substack{z \subset\left\{z_{1}, \ldots, z_{n}\right\} \\|Z|=m}}\left(\prod_{z_{r} \in Z} z_{r}^{2}\right) \Delta(Z)^{2}
$$

and $D_{m}=0$ if $m>n$. (Note that whereas $\Delta(Z)$ has an ambiguous sign, the notation $\Delta(Z)^{2}$ makes sense.) Thus, it is clear that if all of the $z_{r}$ are real, then all of the $D_{m} \geq 0$, so Grommer's condition is a necessary condition for the reality of the zeros of $P$.

We can show that the condition is sufficient if there are an odd number of conjugate pairs of non-real zeros. If only one pair, say $z_{1}, z_{2}$ with $z_{2}=\overline{z_{1}}$ is complex, and all of the rest are distinct reals, then

$$
D_{n}=\left|z_{1}\right|^{2} \prod_{r=3}^{n} z_{r}^{2} \Delta\left(z_{3}, \ldots, z_{n}\right)^{2}\left(z_{1}-z_{2}\right)^{2} \prod_{r=3}^{n}\left|z_{r}-z_{1}\right|^{2}
$$

All of the factors here are positive with the exception of $\left(z_{1}-z_{2}\right)^{2}=-4\left(\Im z_{1}\right)^{2}<0$. Thus, $D_{n}<0$. The same argument works anytime there are an odd number of pairs of complex zeros.

If there are an even number of pairs of non-real complex conjugate pairs of zeros, say $m$ of them, then it seems that $D_{n-m}<0$ but we don't see how to prove this.

Grommer's argument proceeds via the Euler-Stieltjes theory of continued fractions, which study contains the genesis of the theory of orthogonal polynomials.

The second set of Grommer inequalities asserts that

$$
10 b_{0}^{2} b_{2} b_{4}-21 b_{0}^{2} b_{3}^{2}-30 b_{0} b_{1}^{2} b_{4}+350 b_{0} b_{1} b_{2} b_{3}-350 b_{0} b_{2}^{3}-420 b_{1}^{3} b_{3}+525 b_{1}^{2} b_{2}^{2}>0
$$

## 12. Turan inequalities

The entire function $\Xi(t)$ can be expanded into an everywhere convergent power series:

$$
\Xi(t)=\sum_{n=0}^{\infty} \frac{(-1)^{n} b_{n} t^{2 n}}{(2 n)!}
$$

where

$$
b_{n}=\int_{-\infty}^{\infty} \Phi(u) u^{2 n} d u
$$

Let

$$
Y(t)=\Xi(\sqrt{t})=\sum_{n=0}^{\infty} \frac{(-1)^{n} b_{n} t^{n}}{(2 n)!}
$$

Then $Y$ is entire of order $1 / 2$ and the Riemann Hypothesis implies that all of its zeros are real, and in addition, that all of the zeros of all derivatives $Y^{(m)}(t)$ are real. From the Grommer inequalities, a necessary condition for all of the zeros of $Y(t)$ to be real is that $s_{2}>0$ where

$$
-\frac{Y^{\prime}}{Y}(t)=s_{1}+s_{2} t^{2}+s_{3} t^{3}+\ldots
$$

in other words

$$
\left(\frac{Y^{\prime}}{Y}\right)^{\prime}(0)<0
$$

Thus, RH implies that

$$
\left(\frac{Y^{(m+1)}}{Y^{(m)}}\right)^{\prime}(0)<0
$$

for $m=0,2,4, \ldots$ It is easy to check that this condition translates to

$$
b_{m}^{2}>\frac{2 m-1}{2 m+1} b_{m-1} b_{m+1} \quad(m=1,2, \ldots)
$$

these are known as the Turan inequalities and give a necessary but not sufficient condition for the reality of all of the zeros of $\Xi(t)$. Matiyasevich [Mat82] and Csordas, Norfolk, and

Varga [CNV86] proved the Turan inequalities for $\Xi$. Conrey and Ghosh [CoGh94] considered these for the $\xi$ function associated with the Ramanujan $\tau$-function.

In conjunction with this, they show
Theorem 3. Let $F \in C^{3}(\mathbf{R})$. Let $F(u)$ be positive, even, and decreasing for positive $u$, and suppose that $F^{\prime} / F$ is decreasing and concave for $u>0$. Suppose that $F$ is rapidly decreasing so that

$$
X(t)=\int_{-\infty}^{\infty} F(u) e^{i t u} d u
$$

is an entire function of $t$. Then $X(t)$ satisfies the Turán inequalities.
12.1. Karlin and Nuttall. We let $\Phi(u)$ be Riemann's function as earlier. We let

$$
\Xi(t)=\sum_{n=0}^{\infty}(-1)^{n} \frac{b_{n}}{(2 n)!} t^{2 n}
$$

where

$$
b_{n}=\int_{-\infty}^{\infty} \Phi(u) u^{2 n} d u
$$

as before. Define

$$
B(i, j)= \begin{cases}\frac{b_{j-i}}{(2 j-2 i)!} & \text { if } i \leq j \\ 0 & \text { if } i>j\end{cases}
$$

Then RH is equivalent to $D(n, r)>0$ for all positive $r$ and non-negative $n$ where

$$
D(n, r)=\left.\operatorname{det}_{r \times r} B(i, j+n)\right|_{i, j=1, r}
$$

(see [Kar68] chapter 8). The case $r=1$ here is clear since $F(u)>0$. The case $r=2$ is slightly weaker than the Turan inequalities; it asserts that

$$
b_{m}^{2}>\frac{m}{m+1} \frac{(2 m-1)}{(2 m+1)} b_{m-1} b_{m+1} .
$$

Nuttall [Nut13] has established the case $r=3$ which asserts that

$$
\begin{gathered}
\frac{b_{m}^{3}}{((2 m)!)^{3}}-\frac{2 b_{m-1} b_{m+1} b_{m}}{(2 m)!(2 m-2)!(2 m+2)!}-\frac{b_{m-2} b_{m+2} b_{m}}{(2 m)!(2 m-4)!(2 m+4)!} \\
\quad+\frac{b_{m-1}^{2} b_{m+2}}{((2 m-2)!)^{2}(2 m+4)!}+\frac{b_{m-2} b_{m+1}^{2}}{(2 m-4)!((2 m+2)!)^{2}}>0
\end{gathered}
$$

for all $m \geq 2$. For $m=2$ this is

$$
b_{2}^{3}-\frac{4}{5} b_{1} b_{3} b_{2}-\frac{1}{70} b_{0} b_{4} b_{2}+\frac{2}{75} b_{0} b_{3}^{2}+\frac{3}{35} b_{1}^{2} b_{4}>0 .
$$

## 13. Turan inequalities, 2

Ramanujan's tau-function may be defined by equating coefficients of the power series on both sides of

$$
\sum_{n=1}^{\infty} \tau(n) x^{n}=x \prod_{n=1}^{\infty}\left(1-x^{n}\right)^{24}
$$

The associated Dirichlet series is

$$
L(s)=L_{\tau}(s)=\sum_{n=1}^{\infty} \tau(n) n^{-s}
$$

This series is absolutely convergent for $\sigma=\Re s>13 / 2$. The xi-function for $\tau$ is given by

$$
\xi_{\tau}(s)=(2 \pi)^{-s} \Gamma(s) L(s)
$$

and it satisfies the functional equation

$$
\xi_{\tau}(s)=\xi_{\tau}(12-s)
$$

This functional equation is equivalent to the fact that

$$
\Delta(z)=\sum_{n=1}^{\infty} \tau(n) e(n z)
$$

is a holomorphic cusp form of weight 12 for the full modular group which in turn is equivalent to: (i) $\Delta(z)$ is expressible in terms of a Fourier series in $z$ in which coefficients of $e(n z)$ with $n \leq 0$ vanish and (ii) $\Delta$ satisfies the transformation formula

$$
\Delta(-1 / z)=z^{12} \Delta(z)
$$

It is believed that all of the zeros of $\xi(s)$ are on the line $\Re s=6$; this is the Riemann Hypothesis for $L_{\tau}$. See Hardy [Har78], Chapter X for introductory information about $\tau$.

Now

$$
\xi_{\tau}(s)=\int_{0}^{\infty} \Delta(i y) y^{s} \frac{d y}{y}
$$

so that

$$
\Xi_{\tau}(t)=\xi_{\tau}(6+i t)=\int_{-\infty}^{\infty} \Delta\left(i e^{u}\right) e^{6 u} e^{i u t} d u
$$

is an entire even function of $t$. We define

$$
\Phi_{\tau}(u)=\Delta\left(i e^{u}\right) e^{6 u}
$$

We see that $\Phi_{\tau}(u)$ is an even function of $u$ by the functional equation for $\xi_{\tau}$. The fact that $\Phi_{\tau}(u)>0$ for real $u$ is immediately obvious from the product formula for $\Delta$ :

$$
\Phi_{\tau}(u)=e^{6 u} e^{-2 \pi e^{u}} \prod_{n=1}^{\infty}\left(1-e^{-2 \pi n e^{u}}\right)^{24}
$$

We can also see that $\Phi_{\tau}(u)$ is decreasing for positive $u$ by calculating the logarithmic derivative. We first observe that

$$
y \frac{d}{d y} \sum_{n=1}^{\infty} \log \left(1-y^{n}\right)=-y \frac{d}{d y} \sum_{m, n=1}^{\infty} \frac{y^{m n}}{m}=-\sum_{m, n=1}^{\infty} n y^{m n}=-\sum_{n=1}^{\infty} \sigma(n) y^{n}
$$

where

$$
\sigma(n)=\sum_{d \mid n} d
$$

is the sum of divisors of $n$. Let $x=2 \pi e^{u}$ and $y=e^{-x}$. Then

$$
\Phi_{\tau}(u)=e^{6 u} y \prod_{n=1}^{\infty}\left(1-y^{n}\right)^{24}
$$

so that

$$
\begin{aligned}
\frac{\Phi_{\tau}^{\prime}}{\Phi_{\tau}}(u) & =6+\left(\frac{1}{y}-24 \sum_{n=1}^{\infty} \sigma(n) y^{n}\right) \frac{d y}{d u} \\
& =6-x\left(1-\Sigma_{0}(x)\right)
\end{aligned}
$$

where

$$
\Sigma_{k}(x)=24 \sum_{n=1}^{\infty} n^{k} \sigma(n) y^{n}
$$

(The expansion of $\Phi_{\tau}^{\prime} / \Phi_{\tau}$ above is related to the Fourier expansion of the Eisenstein series $E_{2}$ :

$$
E_{2}(z)=1-24 \sum_{n=1}^{\infty} \sigma(n) e(n z)
$$

$E_{2}$ is not a modular form of weight 2 ; it transforms according to the formulae

$$
E_{2}(-1 / z)=z^{2} E_{2}(z)+\frac{12 z}{2 \pi i}
$$

and $E_{2}(z+1)=E_{2}(z)$. Note also that

$$
P(y)=E_{2}\left(e^{2 y}\right)
$$

satisfies the Chazy equation

$$
P^{\prime \prime \prime}-2 P P^{\prime \prime}+3\left(P^{\prime}\right)^{2}=0
$$

the Chazy equation is related to a Painlevé equation.)
Now $\Phi_{\tau}^{\prime} / \Phi_{\tau}$ is an odd function of $u$ so that $\Phi_{\tau}^{\prime} / \Phi_{\tau}(0)=0$. Thus, to show that $\Phi_{\tau}^{\prime}(u)<0$ for $u>0$ it suffices to prove that

$$
\left(\frac{\Phi_{\tau}^{\prime}}{\Phi_{\tau}}\right)^{\prime}(u)<0
$$

for $u>0$. But

$$
\begin{aligned}
\left(\frac{\Phi_{\tau}^{\prime}}{\Phi_{\tau}}\right)^{\prime}(u) & =\left(-1+\Sigma_{0}(x)+x \Sigma_{0}^{\prime}(x)\right) \frac{d x}{d u} \\
& =-x+x \Sigma_{0}(x)-x^{2} \Sigma_{1}(x)
\end{aligned}
$$

since $\Sigma_{k}^{\prime}(x)=-\Sigma_{k+1}(x)$ and this then is

$$
=-x\left(1-24 \sum_{n=1}^{\infty} \sigma(n) y^{n}(1-n x)\right)
$$

Since $u \geq 0$ corresponds to $x \geq 2 \pi$ each of the terms $1-n x<0$ so that the whole expression is negative.

Arguing in the same way we see that

$$
\left(\frac{\Phi_{\tau}^{\prime}}{\Phi_{\tau}}\right)^{\prime \prime}(u)
$$

is odd and

$$
\begin{aligned}
\left(\frac{\Phi_{\tau}^{\prime}}{\Phi_{\tau}}\right)^{\prime \prime \prime}(u) & =-x\left(x^{3} \Sigma_{3}(x)-6 x^{2} \Sigma_{2}(x)+7 x \Sigma_{1}(x)-\Sigma_{0}(x)+1\right) \\
& =-x\left(1-24 \sum_{n=1}^{\infty} \sigma(n) y^{n} P_{3}(n x)\right)
\end{aligned}
$$

where

$$
P_{3}(x)=1-7 x+6 x^{2}-x^{3}<0
$$

for $x>6$. Thus we conclude that

$$
\left(\frac{\Phi_{\tau}^{\prime}}{\Phi_{\tau}}\right)^{\prime \prime}(u)<0
$$

for $u>0$, i.e. that $\frac{\Phi_{\tau}^{\prime}}{\Phi_{\tau}}$ is concave for $u>0$; see [CoGh94] for more details.

### 13.1. A difficulty with classifying functions whose Fourier transforms have real

 zeros. Let$$
g(u)=-\frac{u}{4}+\frac{\pi e^{u}}{12}+\sum_{n=1}^{\infty} \sigma_{-1}(n) e^{-2 \pi n e^{u}}
$$

Here $\sigma_{-1}(n)=\sum_{d \mid n} d^{-1}$ is the sum of the reciprocals of the positive divisors of $n$.
Then $g(u)$ is positive, even, decreasing, and its logarithmic derivative is decreasing and concave for $u>0$. So

$$
\Xi_{k}(t)=\int_{-\infty}^{\infty} e^{-k g(u)} e^{i u t} d u
$$

might seem to be a good candidate for a function to have only real zeros. In fact $k=24$ is the case we've just been discussing about the Ramanujan tau-function. And $k=1$ corresponds to the Xi-function associated with the Dirichlet L-function associated to the unique primitive character of modulus 12, and so all of its zeros should be real by the Riemann Hypothesis for
that L-function. We believe for $k=1,2,3,4,6,8,12,24$ that all of the zeros of $\Xi_{k}(t)$ should be real, and for all other values of $k>0$ that there will be non-real zeros. In [CoGh94] it is proven that $\Xi_{48}$ has non-real zeros.

This example illustrates a difficulty with trying to give conditions for a function $f(u)$ to have all of the zeros of its Fourier transform be real. Conditions only involving the positivity of linear combinations of products and quotients of $f(u)$ and its derivatives will fail as this example shows.

By contrast, if $f(t)$ is twice continuously differentiable, $f(t)>0, f^{\prime}(t)<0$, and $f^{\prime \prime}(t)<0$ for $0 \leq t \leq 1$ then all of the zeros of the even entire function

$$
F(z)=\int_{0}^{1} f(t) \cos z t d t
$$

are real. See [PS98], Part V, problem 173.

## 14. Hardy and Littlewood, Riesz, Baez-Duarte

M. Riesz [Rie16] proved that RH is true if and only if

$$
R(x)=x \sum_{k=0}^{\infty} \frac{(-x)^{k}}{k!\zeta(2 k+2)} \ll x^{1 / 4+\epsilon}
$$

and Hardy and Littlewood showed that RH holds if and only if

$$
\sum_{k=1}^{\infty} \frac{(-x)^{k}}{k!\zeta(2 k+1)} \ll x^{-1 / 4}
$$

To see Riesz' theorem, observe that

$$
R(x)=\frac{i}{2} \int_{(3 / 4)} \frac{x^{s}}{\Gamma(s) \zeta(2 s) \sin \pi s} d s=\frac{-1}{2 \pi i} \int_{(3 / 4)} \frac{\Gamma(1-s) x^{s}}{\zeta(2 s)} d s
$$

If RH is true we move the path of integration to $(1 / 4+\epsilon)$ and obtain the upper bound. Conversely, if the upper bound is true, then by the inverse Mellin transform we have that

$$
\frac{\Gamma(1-s)}{\zeta(2 s)}=-\int_{0}^{\infty} R(x) x^{-1-s} d s
$$

is analytic for $\Re s>1 / 4$ so that RH is true.
In a somewhat similar vein, Baez-Duarte [B-D05] has shown that RH is equivalent to the estimate

$$
c_{k}:=\sum_{j=0}^{k} \frac{(-1)^{j}\binom{k}{j}}{\zeta(2 j+2)}<_{\epsilon} k^{-3 / 4+\epsilon}
$$

He initially suggested that $c_{k} k^{3 / 4} \log ^{2} k$ might have a limit. However, further investigations indicate that $c_{k}$ oscillates between $\pm c k^{-3 / 4}$ which, if true, implies that all of the zeros are simple and on the one-half line.

An interesting feature is that, like the Riesz and the Hardy-Littlewood criteria, this condition only involves values of $\zeta(s)$ to the right of the critical strip. The first mentioned criteria involve a whole interval of estimates whereas this is one estimate. Note also that an alternate formula for $c_{k}$ is

$$
c_{k}=\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2}}\left(1-\frac{1}{n^{2}}\right)^{k} .
$$

Baez-Duarte remarks that it is easy to show that

$$
\left|c_{k}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{2}}\left(1-\frac{1}{n^{2}}\right)^{k} \ll k^{-1 / 2}
$$

so that the representation

$$
\frac{1}{\zeta(s)}=\sum_{k=0}^{\infty} c_{k} \prod_{r=1}^{k}\left(1-\frac{s}{2 r}\right)
$$

holds for $\Re s>1$ because of the estimate for $|s| \leq A$,

$$
\prod_{r=1}^{k}\left(1-\frac{s}{2 r}\right) \ll_{A} k^{-\sigma / 2}
$$

A connection with the Riesz function appears through

$$
\sum_{k=0}^{\infty} \frac{c_{k} x^{k}}{k!}=e^{x} \sum_{k=0}^{\infty} \frac{(-1)^{k} x^{k}}{k!\zeta(2 k+2)}=\frac{e^{x}}{x} R(x) .
$$

Cislo and Wolf [CW06] point out that

$$
R(x)=x \sum_{n=1}^{\infty} \frac{\mu(n) e^{-x / n^{2}}}{n^{2}} .
$$

They show that for $\delta<3 / 2$ it is the case that $c_{k} \ll k^{-\delta}$ if and only if $R(x) \ll x^{1-\delta}$.
Marek Wolf [Wol06] has some very nice graphics about this sequence which show the oscillations. He discusses his struggle with computing this sequence only to find out later that representations as sums over zeros are known, and these make computation easier. An example of such (which assumes that the zeros are simple) is:

$$
c_{k-1}=\frac{1}{2 k} \sum_{\rho} \frac{\Gamma(1-\rho / 2)}{\zeta^{\prime}(\rho)} k^{\rho / 2}+o(1 / k)
$$

In light of this rewriting of $c_{k}$ it prompts the problem:
Let

$$
f(x, c):=\sum_{\rho} c_{\rho} x^{\rho}
$$

where, say $\sum_{\rho}\left|c_{\rho}\right| \ll 1$. Clearly, then, RH implies that $f(x) \ll x^{1 / 2}$ as $x \rightarrow \infty$. On the other hand, in certain circumstances one can use Landau's theorem to prove that if there is a zero $\rho=\beta+i \gamma$ with $\beta>1 / 2$ then for every $\beta_{0}<\beta$ it is the case that

$$
f(x)=\Omega\left(x^{\beta_{0}}\right)
$$

Thus, the estimate

$$
f(x) \ll_{\epsilon} x^{1 / 2+\epsilon}
$$

would be equivalent to RH. So, the question is to understand the set of sequences $c_{\rho}$ for which this works; is there some kind of optimal such sequence?

## 15. Speiser's equivalence

Speiser [Spe35] proved that RH is equivalent to the assertion the $\zeta^{\prime}(s)$ has no compex zeros with real parts smaller than $1 / 2$. Levinson and Montgomery [LM74] made an interesting study of the zeros of $\zeta^{\prime}(s)$; and in particular proved that there is essentially a one-to-one correspondence between zeros of $\zeta^{\prime}(s)$ with real part smaller than $1 / 2$ and zeros of $\zeta(s)$ with real part smaller than $1 / 2$. This study was the point of departure for Levinson's proof [Lev74] that at least one-third of the zeros of $\zeta(s)$ are on the critical line.

Speiser's argument is purely geometric. Here we reproduce the argument of Levinson and Montgomery. The Hadamard product for the Riemann $\xi$-function $(\xi(s)=(1 / 2) s(s-$ 1) $\left.\pi^{-s / 2} \Gamma(s / 2) \zeta(s)\right)$ is

$$
\xi(s)=\frac{1}{2} e^{b_{0} s} \prod_{\rho}\left(1-\frac{s}{\rho}\right) e^{\frac{s}{\rho}}
$$

where

$$
b_{0}=\frac{1}{2} \log 2 \pi-1-\frac{1}{2} \gamma .
$$

The logarithmic derivative of this formula yields

$$
\Re \frac{\zeta^{\prime}}{\zeta}(s)=-\Re \frac{1}{s-1}+\frac{1}{2} \log \pi-\frac{1}{2} \Re \frac{\Gamma^{\prime}}{\Gamma}\left(\frac{s}{2}+1\right)+\Re \sum_{\rho} \frac{1}{s-\rho}
$$

Upon pairing the zero $\rho$ with the zero $1-\bar{\rho}$ we have that the sum over zeros is

$$
=-(\sigma-1 / 2) I_{1}
$$

where

$$
I_{1}=2 \sum_{\beta<1 / 2} \frac{(t-\gamma)^{2}+(\sigma-1 / 2)^{2}-(1 / 2-\beta)^{2}}{|s-\rho|^{2}|s-1+\bar{\rho}|^{2}}+\sum_{\beta=1 / 2} \frac{1}{|s-\rho|^{2}}
$$

Now using Stirling's formula they conclude that

$$
\Re \frac{\zeta^{\prime}}{\zeta}(\sigma+10 i)<0
$$

for $0 \leq \sigma \leq 1$; that $\Re \frac{\zeta^{\prime}}{\zeta}(i t)<0$ for $t \geq 10$ and that $\Re \frac{\zeta^{\prime}}{\zeta}(\sigma+i t)<0$ on an appropriately indented to the left path up the $1 / 2$-line. With a little more work they establish that there is
essentially a one-to-one correspondence between zeros of $\zeta(s)$ and zeros of $\zeta^{\prime}(s)$ to the right of one-half; this is the subsequent starting point for Levinson's proof that at least one-third of the zeros of $\zeta(s)$ are on the critical line $\sigma=\Re s=1 / 2$. The figure below was made by Sarah Froelich.


Figure 4. Zeros of $\zeta(s)$ in green and of $\zeta^{\prime}(s)$ in blue

## 16. Weil's explicit formula and positivity criterion

André Weil [Wei52] (see also [Gui42]) proved the following formula which is a generalization of Riemann's formula mentioned above and which specifically illustrates the dependence between primes and zeros:

$$
\begin{aligned}
\mathcal{T}(f):=\sum_{\rho} \tilde{f}(\rho)= & \int_{0}^{\infty} f(x) d x+\int_{0}^{\infty} f^{*}(x) d x-\sum_{n=1}^{\infty} \Lambda(n)\left(f(n)+f^{*}(n)\right. \\
& -(\log 4 \pi+\gamma) f(1)-\int_{1}^{\infty}\left(f(x)-f^{*}(x)-\frac{2}{x} f(1)\right) \frac{x d x}{x^{2}-1}
\end{aligned}
$$

holds whenever $f \in C_{0}^{\infty}(0, \infty)$ where $f^{*}(x)=\frac{1}{x} f\left(\frac{1}{x}\right)$ and $\tilde{f}(s)=\int_{0}^{\infty} f(x) x^{s-1} d x$.
Using this Weil gave a criterion for RH. As stated by Bombieri [Bom00] it is as follows:
The Riemann Hypothesis holds if and only if

$$
\sum_{\rho} \tilde{g}(\rho) \tilde{\bar{g}}^{*}(1-\rho)>0
$$

for every complex-valued $g(x) \in C_{0}^{\infty}(0, \infty)$ which is not identically 0 .

## 17. Li's CRITERION

Xian-Jin Li [Li97] has given a criterion which, in effect, says that one may restrict attention in Weil's criterion to a specific sequence $g_{n}$.

Li proved that the Riemann Hypothesis is true if and only if $\lambda_{n} \geq 0$ for each $n=1,2, \ldots$ where

$$
\lambda_{n}=\sum_{\rho}\left(1-(1-1 / \rho)^{n}\right) .
$$

Note that

$$
\left(1-(1-1 / \rho)^{n}\right)+\left(1-(1-1 /(1-\rho))^{n}\right)=\left(1-(1-1 / \rho)^{n}\right)\left(1-(1-1 /(1-\rho))^{n}\right)
$$

so that the sum of the right hand side over $\rho$ can be identified with $\sum_{\rho} \tilde{g}_{n}(\rho) \tilde{\bar{g}}_{n}(1-\rho)$ where $\tilde{g}_{n}(s)=(1-(1-1 / s))^{n}$ and

$$
g_{n}(x)=\frac{1}{2 \pi i} \int_{(1 / 2)}\left(1-(1-1 / s)^{n}\right) x^{-s} d s= \begin{cases}P_{n}(\log x) & \text { if } 0<x<1 \\ n / 2 & \text { if } x=1 \\ 0 & \text { if } x>1\end{cases}
$$

where

$$
P_{n}(x)=\sum_{j=1}^{n}\binom{n}{j} \frac{x^{j-1}}{(j-1)!} .
$$

The sequence $g_{n}$ doesn't satisfy the hypotheses of Weil's theorem, but Bombieri shows how the converse part of Weil's theorem can be proven using suitable approximations to this sequence.

Another expression for $\lambda_{n}$ is given by

$$
\lambda_{n}=\left.\frac{1}{(n-1)!} \frac{d^{n}}{d s^{n}}\left(s^{n-1} \log \xi(s)\right)\right|_{s=1}
$$

and

$$
\xi(s)=\frac{1}{2} s(s-1) \Gamma(s / 2) \zeta(s) .
$$



Bombieri and Lagarias [BL99] have pointed out that for any multiset $\mathcal{R}$ of complex numbers $\rho$ for which $1 \notin \mathcal{R}$ and $\sum_{\rho} \frac{1+|\Re \rho|}{1+|\rho|^{2}}$ is finite the following are equivalent:

- $\Re \rho \leq 1 / 2$ for all $\rho \in \mathcal{R}$;
- $\sum_{\rho} \Re\left(1-(1-1 / \rho)^{-n}\right) \geq 0$ for $n=0,1,2, \ldots$;
- for every $\epsilon>0$ there is a $c(\epsilon)>0$ such that $\sum_{\rho} \Re\left(1-(1-1 / \rho)^{-n}\right) \geq-c(\epsilon) e^{\epsilon n}$ for $n=1,2,3, \ldots$.
Coffey [Cof05] has shown that

$$
\lambda_{n}=S_{1}(n)+S_{2}(n)+1-\frac{n}{2}(\gamma+\log \pi+2 \log 2)
$$

where

$$
S_{1}(n):=\sum_{k=2}^{n}(-1)^{k}\binom{n}{k}\left(1-2^{-k}\right) \zeta(k)
$$

and

$$
S_{2}(n):=-\sum_{k=1}^{n}\binom{n}{k} \eta_{k-1}
$$

with

$$
\eta_{k}:=\frac{(-1)^{k}}{k!} \lim _{N \rightarrow \infty}\left(\sum_{m=1}^{N} \frac{\Lambda(m) \log ^{k} m}{m}-\frac{\log ^{k+1} N}{k+1}\right)
$$

Regarding $S_{1}$, he has proven that

$$
S_{1}(n)=\frac{1}{2} n \log n-\frac{1}{2}(1-\gamma) n+O(1)
$$

regarding $S_{2}$, he conjectures that

$$
S_{2}(n) \ll n^{1 / 2+\epsilon}
$$

which of course would imply RH.

## 18. Function field zeta-Functions

See [Ros02] for an introduction to this subject. Let $\mathbf{F}_{q}$ be a field with $q$ elements. A variety over $\mathbf{F}_{q}$ has an associated zeta-function obtained by counting points on the variety in extensions $\mathbf{F}_{q^{n}}$. The zeta-function has a functional equation and Euler product. Weil conjectured that the analogue of the Riemann Hypothesis holds for such zeta-functions. Deligne [Del74] proved Weil's conjecture. This result stands today as a beacon for researchers trying to understand the classical Riemann Hypothesis, but attempts to mimic the proof have gone awry. In the case that the variety is a curve Stepanov [Ste69] gave a proof different from Deligne and in the spirit and flavor of work in transcendental number theory. See Bombieri's account [Bom74] of Stepanov's method; also [IK04] gives a nice account of special cases of this proof.

A simple example of the kind of zeta-function we are talking about is as follows. For a monic polynomial $f \in \mathbf{F}_{q}[x]$ let $N(f)=q^{\operatorname{deg}(f)}$ where $\operatorname{deg}(f)$ is just the degree of $f$. We think of the monic polynomials $f$ as being like the positive integers and form the zeta-function

$$
Z(s)=\sum_{f \text { monic }} \frac{1}{N(f)^{s}}
$$

This has an Euler product

$$
Z(s)=\prod_{P \text { irred. }}\left(1-\frac{1}{N(P)^{s}}\right)^{-1}
$$

where the product is over the monic irreducible polynomials $P$. It turns out that both the sum and the product are absolutely convergent for $\Re s>1$. In fact, the number of monic polynomials of degree $d$ is precisely $q^{d}$ and so,

$$
Z(s)=\sum_{d=0}^{\infty} \frac{q^{d}}{q^{d s}}=\frac{1}{1-q^{1-s}}
$$

There is a functional equation

$$
\frac{1}{1-q^{s}} Z(s)=\Phi(s)=\Phi(1-s) .
$$

In general, we can repeat the above situation, but with the integral domain $\mathbf{F}_{q}[x]$ replaced by $\mathbf{F}_{q}[x, y] /(g(x, y))$ for some irreducible polynomial $g$. We have to define a notion of degree so that we will have a multiplicative norm, but the same thing goes through and we have a zeta-function and an Euler product. The general shape of the zeta-function is

$$
\frac{H\left(1 / q^{s}\right)}{1-q^{1-s}}
$$

where $H$ is a polynomial. There is a functional equation, which is the same thing as saying that the roots of the polynomial $H(t)$ are invariant under $t \rightarrow q / t$. And there is a Riemann Hypothesis, which is the assertion that all of the zeros of $H(s)$ have real part equal to $1 / 2$; equivalently, the roots of $H(x)=0$ have absolute value $q^{1 / 2}$.

Patterson, in his book [Pat88] on the zeta-function, gives examples of such "zeta-like" functions which have an Euler product and functional equation but do not satisfy the Riemann Hypothesis. His conclusion is that this indicates that a purely analytic proof of the Riemann Hypothesis is unlikely and that one needs to find some kind of an inner-product structure that will give a positive pairing that will lead to the Riemann Hypothesis.

We want to counter that by observing that, in fact, the analogue of the Selberg axioms for this class of zeta-functions actually does imply the Riemann Hypothesis.

## 19. Hilbert spaces of entire functions

Since the mid 1980's Louis de Branges ([deBra86] and [deBra92] ) has advocated for proving the Riemann Hypothesis by studying Hilbert spaces of entire functions.

Let $E(z)$ be an entire function satisfying $|E(\bar{z})|<|E(z)|$ for $z$ in the upper half-plane. A Hilbert space of entire functions $\mathcal{H}(E)$ is the set of all entire functions $F(z)$ such that $F(z) / E(z)$ is square integrable on the real axis and such that

$$
|F(z)|^{2} \leqslant\|F\|_{\mathcal{H}(E)}^{2} K(z, z)
$$

for all complex $z$, where the inner product of the space is given by

$$
\langle F(z), G(z)\rangle_{\mathcal{H}(E)}=\int_{-\infty}^{\infty} \frac{F(x) \bar{G}(x)}{|E(x)|^{2}} d x
$$

for all elements $F, G \in \mathcal{H}(E)$ and where

$$
K(w, z)=\frac{E(z) \bar{E}(w)-\bar{E}(\bar{z}) E(\bar{w})}{2 \pi i(\bar{w}-z)}
$$

is the reproducing kernel function of the space $\mathcal{H}(E)$, that is, the identity

$$
F(w)=\langle F(z), K(w, z)\rangle_{\mathcal{H}(E)}
$$

holds for every complex $w$ and for every element $F \in \mathcal{H}(E)$. This identity is obtained by using Cauchy's integration formula in the upper half-plane, and the condition is made so that Cauchy's formula applies to all functions in the space $\mathcal{H}(E)$.

The following theorem is essentially due to de Branges.
Theorem 4. Let $E(z)$ be an entire function having no real zeros such that $|E(\bar{z})|<|E(z)|$ for $\Im z>0$, such that $\bar{E}(\bar{z})=\epsilon E(z-i)$ for a constant $\epsilon$ of absolute value one, and such that $|E(x+i y)|$ is a strictly increasing function of $y>0$ for each fixed real $x$. If $\Re\langle F(z), F(z+$ $i)\rangle_{\mathcal{H}(E)} \geqslant 0$ for every element $F(z) \in \mathcal{H}(E)$ with $F(z+i) \in \mathcal{H}(E)$, then the zeros of $E(z)$ lie on the line $\Im z=-1 / 2$, and $\Re\left\{\overline{E^{\prime}}(w) E(w+i) / 2 \pi i\right\} \geqslant 0$ when $w$ is a zero of $E(z)$.

Let $E(z)=\xi(1-i z)$. Then the Riemann hypothesis is that the zeros of $E(z)$ lie on the line $\Im z=-1 / 2$, and the functional identity $\xi(s)=\xi(1-s)$ can be written as $\bar{E}(\bar{z})=E(z-i)$. If $\rho$ is a nontrivial zero of $\zeta(s)$, then $0<\Re \rho<1$. Since

$$
|E(z)|^{2}=\prod\left|1-\frac{i z}{\rho}\right|^{2}=\prod \frac{(\Re \rho+y)^{2}+(\Im \rho-x)^{2}}{|\rho|^{2}}
$$

for $z=x+i y$, we see that $|E(x-i y)|<|E(x+i y)|$ for $y>0$, and that $|E(x+i y)|$ is a strictly increasing function of $y$ on $(0, \infty)$ for each fixed real $x$.

In view of this theorem, it is natural to ask whether the Hilbert space of entire functions $\mathcal{H}(E)$ satisfies the condition that

$$
\Re\langle F(z), F(z+i)\rangle_{\mathcal{H}(E)} \geqslant 0
$$

for every element $F(z)$ of $\mathcal{H}(E)$ such that $F(z+i) \in \mathcal{H}(E)$, because the nontrivial zeros of the Riemann zeta function $\zeta(s)$ would then lie on the critical line $\Re s=1 / 2$ under this condition. However, this is not true as the following example shows.

Let $\rho=1 / 2+i 111.0295355431696745 \cdots$ be the 34 th zero of the Riemann zeta function in the upper half-plane. By using MATHEMATICA, we compute that

$$
-\Re\left\{\xi^{\prime}(\rho) \xi(1+\rho)\right\}=-5.389100507182945 \cdots \times 10^{-69}<0 .
$$

Write $\rho=1-i w$. Then $E(w)=0$, and $\bar{E}^{\prime}(w) E(w+i) / i=-\xi^{\prime}(\rho) \xi(1+\rho)$. Thus, we have

$$
\Re\left\{\overline{E^{\prime}}(w) E(w+i) / 2 \pi i\right\}<0
$$

The conclusion is that $E(z)=\xi(1-i z)$ is not a structure function of a de Branges space.
Lagarias [Lag06] has written an account of some of these investigations. He has shown that

Theorem 5. Let

$$
E_{\zeta}(z)=\xi(1 / 2-i z)+\xi^{\prime}(1 / 2-i z)
$$

Then $E_{\zeta}(z)$ is the structure function of a de Branges space $\mathcal{H}\left(E_{\zeta}(z)\right)$ if and only if the Riemann Hypothesis is true.

## 20. Selberg's Trace Formula

Selberg, perhaps looking for a spectral interpretation of the zeros of $\zeta(s)$, proved a trace formula for the Laplace operator acting on the space of real analytic functions defined on the upper half-plane $\mathcal{H}=\{x+i y: y>0\}$ and invariant under the group $S L(2, \mathbb{Z})$ of linear fractional transformations with integer entries, and determinant one, which acts discontinuosly on $\mathcal{H}$. This invariance is expressed as

$$
f\left(\frac{a z+b}{c z+d}\right)=f(z)
$$

the Laplace operator in this case is

$$
\Delta=-y^{2}\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial x^{2}}\right)
$$

The spectrum of $\Delta$ splits into a continuous part and a discrete part. The eigenvalues $\lambda$ are all positive and, by convention, are usually expressed as $\lambda=s(1-s)$. The continuous part consists of all $s=1 / 2+i t, t \geq 0$ and the discrete part we write as $s_{j}=\frac{1}{2}+i r_{j}$. Then

$$
\begin{aligned}
& \sum_{j=1}^{\infty} h\left(r_{j}\right)=-h(0)-g(0) \log \frac{\pi}{2}-\frac{1}{2 \pi} \int_{-\infty}^{\infty} h(r) G(r) d r+2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} g(2 \log n) \\
& \quad+\sum_{P} \sum_{\ell=1}^{\infty} \frac{g(\ell \log P) \log P}{P^{\ell / 2}-P^{-\ell / 2}}
\end{aligned}
$$

where $g$ and $h$ are as in Weil's formula and

$$
G(r)=\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{2}+i r\right)+\frac{\Gamma^{\prime}}{\Gamma}(1+i r)-\frac{\pi}{6} r \tanh \pi r+\frac{\pi}{\cosh \pi r}\left(\frac{1}{8}+\frac{\sqrt{3}}{9} \cosh \frac{\pi r}{3}\right)
$$

Also, the sum is over the norms $P$ of prime geodesics of $\Gamma \backslash \mathcal{H}$. The values taken on by $P$ are of the form $\left(n+\sqrt{n^{2}-4}\right)^{2} / 4$ with $n \geq 3$ with certain multiplicities (the class number $\left.h\left(n^{2}-4\right)\right)$. H. Haas was one of the first people to compute the eigenvalues $r_{1}=9.533 \ldots, r_{2}=$ $12.173 \ldots, r_{3}=13.779 \ldots$ of $S L_{2}(Z)$ in 1977 in his University of Heidelberg Diplomarbeit. Soon after, Hejhal was visiting San Diego, and Audrey Terras pointed out to him that Haas' list contained the numbers $14.134 \ldots, 21.022 \ldots$; the ordinates of the first few zeros of $\zeta(s)$ were lurking amongst the eigenvalues! Hejhal discovered the ordinates of the zeros of $L\left(s, \chi_{3}\right)$ (see section 7 ) on the list, too. He unraveled this perplexing mystery about 6 months later. It turned out that the spurious eigenvalues were associated to "pseudo cusp forms" and appeared because of the method of computation used. If the zeros had appeared legitimately, RH would have followed because $\lambda=\rho(1-\rho)$ is positive. (The 1979 IHES preprint by P. Cartier and D. Hejhal contains additional details.)

The trace formula resembles the explicit formula in certain ways. Many researchers have attempted to interpret Weil's explict formula in terms of Selberg's trace formula.

## 21. A trace formula in noncommutative geometry

Alain Connes' approach (see [Conn99]) is to construct a space and an operator for which the zeros of the Riemann zeta-function on the critical line are the eigenvalues. Then analysis via the explicit formula of Weil would analyze the trace of this operator and reveal that in fact all of the zeros are in the spectrum.

As a naive example:
We know RH is equivalent to

$$
\sum_{\rho} \frac{1}{|\rho|^{2}}=\sum_{\rho} \frac{1}{\rho(1-\rho)}=2+\gamma-\log 4 \pi
$$

So, we try to evaluate $\sum 1 /|\rho(1-\rho)|$ by using Weil's explicit formula. (The test function in Weil does not have to be analytic.) We do an adelic version of Weil and pay particular attention to what happens at all of the primes. In the end we end up with a formula for our sum. If it is equal to the answer we knew from the start them we have proven RH!
In Connes' construction the space was a Hilbert space and eigenvalues were the zeros of $\zeta(s)$ on the line. Ralf Meyer has amended the construction to give an operator on a space of rapidly decaying functions in which the eigenvalues are all of the zeros of $\zeta(s)$; thus the explicit formula appears as a trace formula. However, it is not clear how to prove the positivity. See [Conn99] and [Mey04] for more details.

See also [Wat02], and [Lac04] for explicit descriptions of Connes' approach.

## 22. Dynamical systems approaches

In dynamical systems one begins with a classical Hamiltonian $H(x, p)$ where $x$ is position and $p$ is momentum and $H$ is the total energy of the system. Hamilton's equations are

$$
\left\{\begin{array}{lll}
\frac{d x}{d t} & =\frac{\partial H}{\partial p} \\
\frac{d p}{d t} & =-\frac{\partial H}{\partial x}
\end{array}\right.
$$

In the quantized dynamical system one has the Schrödinger equation

$$
\hat{H} \psi=E \psi
$$

Here $\psi$ is a wave function and $\hat{H}$ is an operator, the quantum Hamilton, which is obtained from $H$ by replacing $p$ with $-\frac{i \partial}{\partial x}$. For example if

$$
H(x, p)=\frac{x^{2}}{2 m}+V(x)
$$

then

$$
\hat{H}=\frac{1}{2 m} \frac{\partial^{2}}{\partial x^{2}}+V(x)
$$

and the Schrödinger equation is

$$
\frac{1}{2 m} \psi^{\prime \prime}(x)+V(x) \psi(x)=E \psi(x) .
$$

Here $E$ is a constant, energy. One wants to know about the eigenvalues of this equation. The challenge is to construct such a system in which the eigenvalues are the zeros of $\Xi(t)$. In one dimension on a finite interval the eigenvalues are well-spaced; this is the situation of Sturm-Liouville operators for which many of the special functions of classical physics have all of their zeros well-spaced on a line. In particular this situation cannot give the right density of zeros. In two dimensions, one does conjecturally get Random Matrix statistics (eg quantum billiards) but here we have way too many eigenvalues.

Berry and Keating [BK99], see also [BK11], have looked at the dynamical system $x p$ on the positive real line (i.e. not compact). Here one has

$$
H(x, p)=\frac{1}{2} x p+\frac{1}{2} p x
$$

and

$$
\hat{H}=\frac{-i}{2} x \frac{\partial}{\partial x}-\frac{-i}{2} \frac{\partial}{\partial x} x
$$

and the Shrödinger equation is

$$
-i\left(\frac{1}{2}+x \frac{\partial}{\partial x}\right) \psi=E \psi
$$

This has all of it's eigenvalues on the $1 / 2$-line and eigenfunctions

$$
\psi(x)=\frac{1}{x^{1 / 2+I E}}
$$

With the boundary condition

$$
\sum_{n=1}^{\infty} \psi(n x)=0
$$

one would then might expect the eigenvalues to be the zeros of $\zeta(s)$. However the operator is not self-adjoint with respect to this boundary condition.

## 23. The Lee-Yang theorem

Although not directly connected to the Riemann Hypothesis, the Lee-Yang theorem [LY52] is of considerable interest in the study of zeros. Basically it says that the zeros of the partition function of a ferromagnetic Ising model are all on the unit circle.

Mark Kac in his comment on Polya's Bemerkung über die Intergaldarstellung der Riemannschen $\xi$-Funktion [Pol26] writes

Although this beautiful paper takes one to within a hair's breadth of Riemann's Hypothesis it does not seem to have inspired much further work and references to it in the subsequent mathematical literature are rather scant.

Because of this it may be of interest to related that the paper did play a small, but perhaps not wholly negligible, part in the development of an interesting and important chapter in Statistical Mechanics.

In the fall of 1951 and the spring of 1952 C. N. Yang and T. D. Lee were developing their theory of phase transitions which has since become justly celebrated. To illustrate their theory they introduced the concept of a "lattice gas" and they were led to a remarkable conjecture which (not quite in its most general form) can be stated as follows:

Let

$$
G_{N}(z)=\sum \exp \left(\sum_{k, \ell=1}^{N} J_{k, \ell} \mu_{k} \mu_{\ell}\right) \exp \left(i z \sum_{k=1}^{N} \mu_{k}\right)
$$

where $J_{k, \ell} \geq 0$ and the summation is over all $2^{N}$ sequences $\mu_{1}, \mu_{2}, \ldots, \mu_{N}$ with each $\mu_{k}$ assuming only values $p m 1$.

Then $G_{N}(z)$ has only real zeros.
When I first heard of this conjecture I tried the simplest case

$$
J_{k, \ell}=\nu / 2
$$

for all $k$ and $\ell$ and somehow Hilfsatz II of Pólya's paper came to mind.
Kac goes on to describe how one can prove this special case via a slight modification of Pólya's proof. Kac showed the proof to Yang and Lee and within a coule of weeks they had produced the proof of their general theorem [LY52].

A question now is: Is $\zeta(s)$ the partition function of some spin system?

## 24. Newman's conjecture

Newman found a very general form of the Lee-Yang theorem, see [New76].
In subsequent work he found an interesting approach to RH.
It is known that $\Phi(u)$ decays very rapidly. In fact, doubly exponentially:

$$
\Phi(u) \ll e^{9|u| / 2} e^{-\pi e^{2|u|}} .
$$

Thus,

$$
H(\lambda, z):=\int_{-\infty}^{\infty} \Phi(u) e^{\lambda u^{2}} e^{i z u} d u
$$

is rapidly convergent for any real $\lambda$. Also, $H(0, z)=\Xi(z)$. It follows from a theorem of deBruijn that if for some $\lambda_{0}$ all of the zeros of $H\left(\lambda_{0}, z\right)$ are real, then the same is true of $H(\lambda, z)$ whenever $\lambda>\lambda_{0}$. Newman [New91] proved that there does exist such a $\lambda_{0}$ and that $\lambda_{0} \geq 1 / 8$. He also proved that there exists a $\lambda_{1}$ such that $H\left(\lambda_{1}, z\right)$ has a non-real zero. Thus, $\lambda_{0}$ is bounded below. RH is the assertion that $\lambda_{0} \leq 0$. Newman conjectures that $\lambda_{0}=0$. Odlyzko has shown that $H\left(-2.7 \times 10^{-9}, z\right)$ has a non-real zero. Therefore, as Newman says, "...the Riemann hypothesis, if true, is only barely so."

## 25. Stable polynomials

The remarkable work of Branden and Borcea, see [BB09], [BB09b], and [BB09c], about the generalization to several variables of their solution of the Polya-Schur conjecture is worth noting, especially since it has just played an important role in the solution of the KadisonSinger problem by Marcus, Spielman, and Srivastava [MSS14].

We briefly describe their results. First of all a polynomial $f$ in $z_{1}, \ldots, z_{n}$ is stable if $f\left(z_{1}, \ldots, z_{n}\right) \neq 0$ for any $n$-tuple with all $\Im z_{j}>0$. If the coefficients are real then $f$ is called real stable. Real stable polynomials in one variable have only real zeros.

Branden and Borcea characterize real stable polynomials in two variables as those expressible as

$$
f(z, w)= \pm \operatorname{det}(z A+w B+C)
$$

where $A$ and $B$ are positive definite and $C$ is symmetric.
In his 1988 Gibbs lecture [Rue88] David Ruelle proclaimed about the Lee-Yang theorem: "I have called this beautiful result a failure because, while it has important applications in physics, it remains at this time isolated in mathematics. One might think of a connection with zeta-functions (and the Weil conjectures); the idea of such a connection is not absurd, as our second example will show. But the miracle has not happened: one still does not know what to do with the circle theorem."

Lieb and Sokal [LS81] reduced the generalized Lee-Yang theorem to the assertion: if $P, Q \in \mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$ are non-vanishing when all of the variables are in the open right halfplane then the polynomial

$$
P\left(\frac{\partial}{\partial z_{1}}, \ldots, \frac{\partial}{\partial z_{n}}\right) Q\left(z_{1}, \ldots, z_{n}\right)
$$

also has this property. Thus, to better understand Lee-Yang-type theorems one is naturally led to consider the problems of describing linear operators on polynomial spaces that preserve the property of being nonvanishing when the variables are in prescribed subsets of $\mathbb{C}^{n}$.

## 26. Nyman - Beurling approach

This approach begins with the theorem of Nyman [Nym50], a student of Beurling. The work of Beurling and Nyman [Beu55] implies that RH can be recast as an approximation problem in a certain Hilbert space. Let $\{x\}$ denote the fractional part of $x$. One considers functions of the form

$$
f(x)=\sum_{k=1}^{n} c_{k}\left\{\theta_{k} / x\right\}
$$

where $0<\theta_{k} \leq 1$ and $c_{k}$ are complex numbers and asks whether the characteristic function $\chi(x)=\chi_{(0,1]}(x)$ can be approximated by such $f$ on the positive real line. Their theorem is that the Riemann Hypothesis holds if and only if

$$
\lim _{n \rightarrow \infty} \inf _{c_{k}, \theta_{k}} \int_{0}^{\infty}\left|\chi(x)-\sum_{k=1}^{n} c_{k}\left\{\theta_{k} / x\right\}\right|^{2} d x=0
$$

This theorem has been extended by Baez-Duarte, [B-D02] and [B-D03b], who showed that one may take $\theta_{k}=1 / k$. So, let

$$
d_{n}:=\inf _{\left\{c_{1}, \ldots, c_{n}\right\}} \int_{0}^{\infty}\left|\chi(x)-\sum_{k=1}^{n} c_{k}\{1 /(k x)\}\right|^{2} d x
$$

Thus, the Riemann Hypothesis holds if and only if $\lim _{n \rightarrow \infty} d_{n}=0$.
It is conjectured that $d_{n} \sim \frac{C}{\log n}$ where $C=\sum_{\rho} \frac{1}{|\rho|^{2}}$. Burnol [Bur03] has proven that

$$
\frac{1}{\log n} \sum_{\rho \text { on the line }} \frac{m_{\rho}}{|\rho|^{2}}
$$

is a lower bound. If RH holds and all the zeros are simple, then clearly these two bounds are the same.

Note that it is easy to see that

$$
\sum_{\rho} \frac{1}{\rho(1-\rho)}=2+\gamma-\log 4 \pi=0.04619 \ldots
$$

Just begin with

$$
I=\frac{1}{2 \pi i} \int_{(2)} \frac{\zeta^{\prime}}{\zeta}(s) \frac{d s}{s(1-s)}
$$

On the one hand, this integral is 0 as can be seen by moving the path arbitrarily far to the right; on the other hand the integral can be evaluated by moving the path arbitrarily far to the left and accounting for residues of poles at $s=1,0, \rho,-2 n$. In this way we get the result.

An easy exercise shows that the Riemann Hypothesis is equivalent to

$$
\sum_{\rho} \frac{1}{|\rho|^{2}}=2+\gamma-\log 4 \pi
$$

A rephrasing of the Baez-Duarte theorem (by Balazard and Saias, see [BS98], [BS00], and [BS04]) arises from taking the Mellin transform of $f$. In this way one finds that the Riemann Hypothesis holds if and only if

$$
\lim _{N \rightarrow \infty} \inf _{A_{N}} \int_{-\infty}^{\infty}\left|1-A_{N}(1 / 2+i t) \zeta(1 / 2+i t)\right|^{2} \frac{d t}{\frac{1}{4}+t^{2}}=0
$$

where the infimum is over all Dirichlet polynomials $A_{N}(s)=\sum_{h=1}^{N} \frac{a_{h}}{h^{s}}$ of length $N$. Now the problem looks like a mollification problem.

Bagchi [Bag06] has written a very nice exposition explaining this complicated circle of ideas.

### 26.1. The Vasyunin sums. Consider

$$
I_{N}(\vec{a})=\int_{-\infty}^{\infty}\left|1-A_{N}(1 / 2+i t) \zeta(1 / 2+i t)\right|^{2} \frac{d t}{\frac{1}{4}+t^{2}}
$$

Let's assume for convenience that the coefficients $a_{h}$ are real. Squaring out, we have

$$
\begin{aligned}
I_{N}(\vec{a}) & =\int_{-\infty}^{\infty} \frac{d t}{\frac{1}{4}+t^{2}}-\frac{2}{i} \int_{(1 / 2)} \zeta(s) A_{N}(s) \frac{d s}{s(1-s)}+\int_{-\infty}^{\infty}\left|\zeta A_{N}(1 / 2+i t)\right|^{2} \frac{d t}{\frac{1}{4}+t^{2}} \\
& =2 \pi+4 \pi\left((1-\gamma) A_{N}(1)-A_{N}^{\prime}(1)\right)+2 \pi \sum_{h, k=1}^{N} a_{h} a_{k} b_{h, k}
\end{aligned}
$$

where

$$
b_{h, k}=\frac{1}{\sqrt{h k}} \int_{-\infty}^{\infty}|\zeta(1 / 2+i t)|^{2} \frac{(h / k)^{i t}}{\frac{1}{4}+t^{2}} d t
$$

Writing this as a complex integral, we see that

$$
\frac{b_{h, k}}{2 \pi}=\frac{1}{2 \pi i} \int_{(1 / 2)} \frac{\zeta(s)}{s h^{s}} \frac{\zeta(1-s)}{(1-s) k^{1-s}} d s
$$

We recognize this as a convolution of Mellin transforms, and calculate that

$$
\begin{aligned}
\frac{1}{2 \pi i} \int_{(1 / 2)} \frac{\zeta(s)}{s h^{s}} u^{-s} d s & =-\frac{1}{u h}+\sum_{n=1}^{\infty} \frac{1}{2 \pi i} \int_{(1 / 2)} \frac{(n h u)^{-s}}{s} d s \\
& =-\frac{1}{u h}+\sum_{n u h \leq 1} 1=-\left\{\frac{1}{u h}\right\}
\end{aligned}
$$

if $1 /(u h)$ is not an integer; if it is, subtract $1 / 2$. Thus, using the formula for convolution, we find that

$$
\frac{b_{h, k}}{2 \pi}=\int_{0}^{\infty}\left\{\frac{1}{h x}\right\}\left\{\frac{1}{k x}\right\} d x
$$

Remarkably, Vasyunin [Vas95] found a beautiful exact formula for the right side here. Note, first of all, that

$$
b_{h, k}=\frac{b_{H, K}}{(h, k)}
$$

where $h=(h, k) H$ and $k=(h, k) K$. Thus, it suffices to evaluate $b_{h, k}$ when $(h, k)=1$. So, assuming that $h$ and $k$ are relatively prime, and letting

$$
V(h, k):=\sum_{a=1}^{k-1}\left\{\frac{a h}{k}\right\} \cot \frac{\pi a}{k},
$$

then Vasyunin's formula implies that

$$
\frac{b_{h, k}}{2 \pi}=\frac{\log 2 \pi-\gamma}{2}\left(\frac{1}{h}+\frac{1}{k}\right)+\frac{k-h}{2 h k} \log \frac{h}{k}-\frac{\pi}{2 h k}(V(h, k)+V(k, h)) .
$$

The following estimate is easy to prove:

$$
c_{0}(1, k)=\frac{k}{\pi}\left(\log \frac{k}{2 \pi}+\gamma\right)+\frac{1}{\pi}+O(1 / k) .
$$

More challenging is the reciprocity formula, see [BC13] and [BC13b],
Theorem 6. There exists a function $g(z)$ analytic on $\mathbb{C}^{\dagger}$ which is the complex plane with the negative real axis removed such that for any $k>0$ and $(h, k)=1$,

$$
\frac{h}{k} c_{0}(h, k)+c_{0}(k, h)-\frac{1}{\pi k}=g(h / k) .
$$

One would hope that this formula could be useful in analyzing $d_{n}$.
As mentioned earlier it is believed that

$$
d_{n}^{2} \sim \frac{2+\gamma-\log 4 \pi}{\log n}
$$

as $n \rightarrow \infty$. In [BCF12] the following is proven.

Theorem 7. Let

$$
V_{N}(s)=\sum_{n \leq N} \frac{\mu(n) \frac{\log \frac{N}{n}}{\log N}}{n^{s}}
$$

If the Riemann hypothesis is true and if

$$
\sum_{|\Im(\rho)| \leq T} \frac{1}{\left|\zeta^{\prime}(\rho)\right|^{2}} \ll T^{\frac{3}{2}-\delta}
$$

for some $\delta>0$, then

$$
\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left|1-\zeta V_{N}(1 / 2+i t)\right|^{2} \frac{d t}{\frac{1}{4}+t^{2}} \sim \frac{2+\gamma-\log 4 \pi}{\log N}
$$

The condition implicitly assumes that the zeros of the Riemann zeta function are all simple. Moreover, this upper bound is "mild" in the sense that a conjecture, due to Gonek and recovered by a different heuristic method of Hughes, Keating, and O'Connell [HKO00], predicts that

$$
\sum_{|\rho| \leq T} \frac{1}{\left|\zeta^{\prime}(\rho)\right|^{2}} \sim \frac{6}{\pi^{3}} T
$$

Thus, $V_{N}$ gives an asymptotically optimal choice.

## 27. Eigenvalues of Redheffer's matrix

The Redheffer matrix $A(n)$ is an $n \times n$ matrix of 0 's and 1's defined by $A(i, j)=1$ if $j=1$ or if $i$ divides $j$, and $A(i, j)=0$ otherwise. It has the property that

$$
\operatorname{det} A(n)=M(n)
$$

the summatory function of the Möbius function. Thus, RH is true if and only $\operatorname{det} A(n) \ll$ $n^{1 / 2+\epsilon}$ and so it is of interest to study the eigenvalues; see [BFP89].

It is known that $A(n)$ has $n-[\log n / \log 2]-1$ eigenvalues equal to 1 . One way to see this is to interpret the matrix minus the identity as the incidence matrix of a graph. The coefficients of the characteristic polynomial are easily described by counting "cycles" of divisors: such as $1 \rightarrow 2 \rightarrow 4 \rightarrow 12 \rightarrow 60$ would be a cycle of length 5 that occurs in any graph with $n \geq 60$. Letting $S_{k}(n)$ be the number of such cycles of length $k$ in the graph with $n$ vertices, the desired characteristic polynomial, but with all of the eigenvalues equal to 1 removed, is of degree $N=[\log n / \log 2]+1$ and is given by

$$
P_{n}(\lambda)=(\lambda-1)^{N}-\sum_{k=1}^{N-1} S_{k}(n)(\lambda-1)^{N-1-k}
$$

Another way to think of this is to let $D_{k}(m)$ be the number of ways to factor $m$ into a product of $k$ integers each greater than 1 , taking order into account. So, $D_{k}(m)$ is like $d_{k}(m)$
except that the factor 1 is not allowed. Then

$$
S_{k}(n)=\sum_{m=2}^{n} D_{k}(m)
$$

It is not difficult to show that $A$ has two 'large' eigenvalues, one a real positive eigenvalue which is approximately $\sqrt{n}$, and the other a negative eigenvalue which is approximately $-\sqrt{n}$. It's easy to believe this because $S_{1}(n)=n-1$ so that the leading terms of $P_{n}$ are

$$
P_{n}(\lambda)=(\lambda-1)^{N}-(n-1)(\lambda-1)^{N-2}-\ldots .
$$

and the remaining eigenvalues are small by comparison.
Then RH if and only if $\operatorname{det}(A)=O\left(n^{1 / 2+\epsilon}\right)$ for every $\epsilon>0$. It has been suggested that all of these remaining eigenvalues are inside the unit circle. If so, and if some positive proportion of them have absolute value smaller than 0.9 say, then a quasi-Riemann Hypothesis would follow, i.e. there would be a vertical line strictly to the left of the half-line with no zeros to its right.


Vaughan, [Vau93]and [Vau96], has given very precise estimates for the 'large' eigenvalues and their product, upper bounds for the magnitude of any 'non-trivial' eigenvalue, and upper and lower estimates for eigenvalue close to 1 . Curiously, the closeness of eigenvalues to 1 depend on rational approximations to

$$
\alpha:=(\log 2) /(\log 3 / 2)=1.709511 \ldots ;
$$

for example, his theorems imply that the $n$ for which $\left\{\log \left(n / 2^{N-1}\right) / \log (3 / 2)\right\}>2-\alpha$ have eigenvalues markedly closer to 1 than $n$ for which the reverse inequality holds.

Here is a slightly different approach which is implicit in Vaughan's work.

$$
\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}}=\frac{1}{\zeta(s)}=\frac{1}{1+(\zeta(s)-1)}=\sum_{k=0}^{\infty}(-1)^{k}(\zeta(s)-1)^{k}=\sum_{k=0}^{\infty}(-1)^{k} \sum_{n=1}^{\infty} \frac{D_{k}(n)}{n^{s}}
$$

so that

$$
\mu(n)=\sum_{k=0}^{\infty}(-1)^{k} D_{k}(n)
$$

and

$$
M(n)=\sum_{m=1}^{n} \mu(m)=\sum_{k=0}^{\infty}(-1)^{k} S_{k}(n)
$$

Now, if we perturb this argument (due to Vaughan) slightly we can nearly recover Redeheffer's characteristic polynomial. Let

$$
\sum_{m=1}^{\infty} \frac{\mu_{w}(m)}{m^{s}}:=\frac{1}{1+w(\zeta(s)-1)}=\sum_{k=0}^{\infty}(-1)^{k} w^{k}(\zeta(s)-1)^{k}=\sum_{k=0}^{\infty}(-1)^{k} w^{k} \sum_{m=1}^{\infty} \frac{D_{k}(m)}{m^{s}}
$$

so that

$$
\mu_{w}(m)=\sum_{k=0}^{\infty}(-1)^{k} w^{k} D_{k}(m)
$$

and, after summing $m$ from 1 to $n$, we have

$$
M_{w}(n):=\sum_{m=1}^{n} \mu_{w}(m)=\sum_{k=0}^{\infty}(-1)^{k} w^{k} S_{k}(n)=\sum_{k=0}^{N}(-1)^{k} w^{k} S_{k}(n)
$$

Then

$$
(1-\lambda)^{-N} M_{1-\lambda}(n) \approx P_{n}(\lambda)
$$



## 28. Bombieri's Theorem

Bombieri [Bom00] has proven that one of the following assertions is true:

- The Riemann Hypothesis
- There are infinitely many zeros of $\zeta(s)$ to the right of the critical line.
- There are coefficients $c_{\rho}$ such that $\sum_{\rho}\left|c_{\rho}\right|^{2}=1$ and $\sum_{\Re \rho \neq 1 / 2}\left|c_{\rho}\right|^{2} \geq 1 / 2$ for which the linear combination

$$
\sum_{\rho} \frac{c_{\rho}}{\rho(1-\rho)} x^{-\rho}+A+B / x
$$

for some constants $A$ and $B$ vanishes identically in some interval $1 \leq x \leq M_{0}$ where $M_{0}>1$ is an explicitly computable constant.
The idea is to rule out the third possibility here so that one can say that if RH is false, then infinitely many zeros are off the line. If one had a quantitative version of such an assertion, it might contradict, for example, the density estimate for $N(\sigma, T)$ cited above.

Bombieri's analysis is interesting. He begins with Weil's criterion but applied to a Hilbert space of functions supported on a finite closed interval $\left[M^{-1}, M\right]$. He looks for functions $g$ for which $\|g\|=1$ and $\sum_{\rho} \tilde{g}(\rho) \tilde{\bar{g}}(1-\rho)$ is minimal. The inner product on the Hilbert space is defined by

$$
\langle f, g\rangle=\int_{M^{-1}}^{M} f(x) \bar{g}(x) d x
$$

Using Weil's explicit formula he finds a convenient expression for the Euler-Lagrange equation. He can essentially solve this as a linear combination as mentioned here.

Here are some further details. Let $\mathcal{T}(f)$ be as in Weil's explicit formula. Let $t_{a}[f]$ be the function defined by $t_{a}[f](x)=f(a x)$. Let $\mathcal{L}[f]$ be the function defined at a number $x$ by $\mathcal{L}[f](x)=\mathcal{T}\left(t_{x}[f]\right)$. The convolution of $f$ and $g$, denoted by $f * g$ is defined, as usual, to be the function whose value at $x$ is

$$
(f * g)(x)=\int_{0}^{\infty} f(u) g(x / u) \frac{d u}{u}
$$

Recall that $f^{*}(x)=f(1 / x) / x$. Observe that

$$
\left(f * g^{*}\right)(x)=\int_{0}^{\infty} f(u x) g(u) d u=\left\langle t_{x}[f], g\right\rangle
$$

To prove RH we have to show that

$$
\mathcal{T}\left(f * \overline{f^{*}}\right)>0
$$

for all suitable $f$.
The idea is to use the calculus of variations to find the minimal $f$, say on the interval $\left(M^{-1}, M\right)$ subject to something like $\langle f, f\rangle=1$. So, suppose that $f$ is a minimal function and consider $f+\epsilon \phi$ where $\phi\left(M^{-1}\right)=\phi(M)=0$. Then, $\left.\frac{d}{d \epsilon} I(f+\epsilon \phi)\right|_{\epsilon=0}=0$ where

$$
I(f)=\frac{\mathcal{T}\left(f * \overline{f^{*}}\right)}{\langle f, f\rangle}
$$

We have

$$
0=\frac{d}{d \epsilon} \Phi(f+\epsilon \phi)=\frac{\langle f, f\rangle \frac{d}{d \epsilon} \mathcal{T}\left((f+\epsilon \phi) *\left(\overline{f^{*}+\epsilon \phi^{*}}\right)\right)-\mathcal{T}\left(f * \overline{f^{*}}\right) \frac{d}{d \epsilon}\langle f+\epsilon \phi, f+\epsilon \phi\rangle}{\langle f, f\rangle^{2}}
$$

so that

$$
\langle f, f\rangle \mathcal{T}\left(f * \overline{\phi^{*}}\right)=\mathcal{T}\left(f * \overline{f^{*}}\right)\langle f, \phi\rangle
$$

for all $\phi$. Now

$$
\mathcal{T}\left(\left(f * \overline{\phi^{*}}\right)(x)\right)=\mathcal{T}\left(\left\langle t_{x}[f], \phi\right\rangle\right)=\left\langle\mathcal{T}\left(t_{x}[f]\right), \phi\right\rangle=\langle\mathcal{L}[f](x), \phi\rangle .
$$

Thus, the above becomes

$$
\langle\mathcal{L}[f]-\lambda f, \phi\rangle=0
$$

for all suitable $\phi$ where

$$
\lambda=\frac{\mathcal{T}\left(f * \overline{f^{*}}\right)}{\langle f, f\rangle}
$$

We conclude that a minimal such $f$ must satisfy $\mathcal{L}[f]=\lambda f$, i.e.

$$
\langle f, f\rangle \mathcal{L}[f]=\mathcal{T}\left(f * \overline{f^{*}}\right) f
$$

Recall that Weil's explicit formula says that

$$
\mathcal{T}(f)=\sum_{\rho} \tilde{f}(\rho)
$$

Thus,

$$
\left.\mathcal{L}[f](x)=\mathcal{T}\left(t_{x}[f]\right)=\sum_{\rho} t_{x} \tilde{[ } f\right](\rho)=\sum_{\rho} \tilde{f}(\rho) x^{-\rho}
$$

so that

$$
\sum_{\rho} \tilde{f}(\rho) x^{-\rho}=\lambda f(x)
$$

Now let $\phi(x)$ be the characteristic function of the interval $\left[M^{-1}, M\right]$ and consider functions $f$ given by

$$
f(x)=\sum_{\rho} X_{\rho} \phi(x) x^{-\rho}
$$

Then

$$
\tilde{f}(s)=\sum_{\rho} X_{\rho} \tilde{\phi}(s-\rho)
$$

So, if $X_{\rho}=\tilde{f}(\rho) / \lambda$, then $f$ is formally a solution. This leads to the eigenvalue problem

$$
\lambda X_{\rho}=\sum_{\rho^{\prime}} \tilde{\phi}\left(\rho-\rho^{\prime}\right) X_{\rho^{\prime}}
$$

We calculate that

$$
\tilde{\phi}(s)=\int_{M^{-1}}^{M} x^{s} \frac{d x}{x}=\frac{M^{s}-M^{-s}}{s}
$$

Now we introduce some new notation. Let $\rho=1 / 2+i \gamma($ with $\gamma \in \mathbf{C}) ; M=e^{t}$ with $t>0$; $\Lambda=1 / \lambda ; z_{\gamma}=X_{\rho} ; w_{\gamma}=\left(\frac{1}{4}+\gamma^{2}\right) z_{\gamma} ; K(x)=\frac{\sin x}{x}$ and

$$
H(x, y, t)=\frac{2 t K(t(x-y))}{\frac{1}{4}+y^{2}}
$$

Then the eigenvalue problem can be rewritten as

$$
w_{\gamma}=\Lambda \sum_{\gamma^{\prime}} H\left(\gamma, \gamma^{\prime}, t\right) w_{\gamma^{\prime}}
$$

Next, let

$$
\mathcal{H}(\Gamma, t)=\left[H\left(\gamma, \gamma^{\prime}, t\right)\right]_{\gamma, \gamma^{\prime} \in \Gamma}
$$

and the resolvent determinant

$$
D(\Lambda, t)=\operatorname{det} \mathcal{H}(I-\Lambda \Gamma ; t)
$$

Bombieri proves that $D(\Lambda, t)$ is an entire function in $\Lambda$ of order at most 1 , and that

$$
D(\Lambda, t)=1+\sum_{n=1}^{\infty}(-1)^{n} \Delta_{n}(t) \frac{\Lambda^{n}}{n!}
$$

where

$$
\Delta_{n}(t)=\sum_{\gamma_{1}, \ldots, \gamma_{n} \in \Gamma} \operatorname{det}\left[H\left(\gamma_{j}, \gamma_{k}, t\right)\right]_{j, k=1, \ldots, n}
$$

Moreover, the truncations

$$
D_{N}(\Lambda, t)=\left.\operatorname{det}\left[\delta_{\gamma, \gamma^{\prime}}-\Lambda H\left(\gamma, \gamma^{\prime}, t\right)\right]\right|_{\gamma, \gamma^{\prime} \in \Gamma_{N}}
$$

where $\Gamma_{N}$ is the set of $\gamma \in \Gamma$ with $|\gamma| \leq N$, converge to $D(\Lambda, t)$ uniformly on compact sets, as $N \rightarrow \infty$.

The zeros of the resolvent give the solutions to our linear system. Let $\Lambda_{0}$ be a zero of $D(\Lambda, t)$ and define

$$
D\left(\gamma, \gamma_{0} ; \Lambda, t\right)=\Lambda \sum_{n=0}^{\infty}(-1)^{n} \Delta_{n}\left(\gamma, \gamma_{0} ; t\right) \frac{\Lambda^{n}}{n!}
$$

where $\left.\Delta_{0}\left(\gamma, \gamma_{0} ; t\right)=H(\gamma, \gamma), t\right)$ and

$$
\Delta_{n}\left(\gamma, \gamma_{0} ; t\right)=\sum_{\gamma_{1}, \ldots, \gamma_{n} \in \Gamma} \operatorname{det}\left[\begin{array}{ll}
H\left(\gamma, \gamma_{0}, t\right) & H\left(\gamma, \gamma_{k}, t\right) \\
H\left(\gamma_{j}, \gamma_{0}, t\right) & H\left(\gamma_{j}, \gamma_{k}, t\right)
\end{array}\right]_{j, k=1, \ldots, n}
$$

Then $D\left(\gamma, \gamma_{0} ; \Lambda, t\right)$ is an entire function of $\Lambda$ of order at most 1 , and there exists a $\gamma_{0}$ such that

$$
w_{\gamma}=D\left(\gamma, \gamma_{0} ; \Lambda_{0}, t\right)
$$

is a solution, not identically 0 , to the system above. Moreover,

$$
f(x)=\sum_{\rho} X_{\rho} \phi(x) x^{-\rho}
$$

with $X_{\rho}=w_{\gamma}$ and $M=e^{t}$ satisfies

$$
f(x)=\Lambda_{0} \mathcal{L}[f](x) \quad x \in(1 / M, M)
$$

and $f(1 / M)=f(M)=0$.
So, we (i.e. Bombieri) have "constructed" an extremal function for Weil's "quadratic functional."

The next step is to investigate finite approximations to this problem. In other words, consider what the situation is for finite sets $\Lambda$. Bombieri considers a general situation where the numbers in $\Gamma$ are arbitrary but have the same symmetries as do zeta-zeros. He shows that all of the eigenvalues $\Lambda$ are real, and that if all of the $\gamma \in \Gamma$ are real then all of the eigenvalues $\Lambda$ are positive. In fact, the number of non-real pairs of conjugate $\gamma$ is exactly equal to the number of negative eigenvalues.

## 29. The Selberg Class of Dirichlet series

Selberg [Sel89] has introduced a class of Dirichlet series and made certain conjectures about them. We let $\mathcal{S}$ denote the class of functions in question. A Dirichlet series

$$
F(s)=\sum \frac{a(n)}{n^{s}}
$$

is in $\mathcal{S}$ provided that it satisfies the following hypotheses:

- Analyticity: $(s-1)^{m} F(s)$ is an entire function of finite order for some non-negative integer $m$
- Ramanujan Hypothesis: $a_{n} \ll n^{\epsilon}$ for any fixed $\epsilon>0$
- Functional equation: there must be a function $\gamma_{F}(s)$ of the form

$$
\gamma_{F}(s)=\epsilon Q^{s} \prod_{i=1}^{k} \Gamma\left(\lambda_{i} s+\mu_{i}\right)
$$

where $|\epsilon|=1, Q>0, \lambda_{i}>0$, and $\Re \mu_{i} \geq 0$ such that

$$
\Phi(s)=\gamma_{F}(s) F(s)
$$

satisfies

$$
\Phi(s)=\overline{\Phi(1-\bar{s})}
$$

- Euler product: $a_{1}=1$, and

$$
\log F(s)=\sum \frac{b_{n}}{n^{s}}
$$

where $b_{n}=0$ unless $n$ is a positive power of a prime and $b_{n} \ll n^{\theta}$ for some $\theta<1 / 2$.
Selberg conjectures that any Dirichlet series in the Selberg class satisfies a Riemann Hypothesis, that all of its non-trivial zeros have real part equal to $1 / 2$.

Here are two examples of functions in the Selberg class:

$$
L_{4}(s)=1-\frac{1}{3^{s}}+\frac{1}{5^{s}}-\frac{1}{7^{s}}+\cdots=\prod_{p \geq 3}\left(1-\frac{\chi_{-4}(p)}{p^{s}}\right)^{-1}
$$

satisfies the functional equation

$$
\left(\frac{4}{\pi}\right)^{s / 2} \Gamma\left(\frac{s}{2}+\frac{1}{2}\right) L_{4}(s)=\xi_{4}(s)=\xi_{4}(1-s)
$$

and, letting $x \prod_{n=1}^{\infty}\left(1-x^{n}\right)^{24}=\sum_{n=1}^{\infty} \tau(n) x^{n}$ define Ramanujan's tau-function,

$$
L_{\tau}(s)=\sum_{n=1}^{\infty} \frac{\tau(n) / n^{11 / 2}}{n^{s}}=\prod_{p}\left(1-\frac{\tau(p) / p^{11 / 2}}{p^{s}}+\frac{1}{p^{2 s}}\right)^{-1}
$$

satisfies

$$
(2 \pi)^{-s} \Gamma(s+11 / 2) L_{\tau}(s)=\xi_{\tau}(s) .
$$

One needs Mordell's theorem $\tau(m) \tau(n)=\sum_{d \mid n} d^{11} \tau\left(m n / d^{2}\right)$ and Deligne's theorem $|\tau(p)| \leq$ $2 p^{11 / 2}$, both conjectured by Ramanujan, to verify that $L_{\tau} \in \mathcal{S}$.

## 30. Real zeros of quadratic L-Functions

A collection of problems which may be relevant to solving RH is as follows. Let $L_{p}(s)=$ $\sum \chi_{p}(n) n^{-s}$ where $\chi_{p}(n)=\left(\frac{n}{p}\right)$ is the Legendre symbol which is equal to 0 if $n$ is a multiple of $p$ and otherwise is 1 if $n$ is a square $\bmod p$ and -1 if $n$ is not a square modulo the prime number $p$. The series for $L_{p}(s)$ converges if $\Re s>0$. The problem is to prove that $L_{p}(\sigma)>0$ for $0<\sigma<1$.

Depending on your point of view, this may seem like an easier problem than RH or a harder one. But I believe that it is exactly the same difficulty. It is certainly easier to state than RH! The reason I like this problem is that it removes the temptation to use analysis to prove RH. On the surface RH looks like a problem in analysis, and so analysts try all sorts of tricks. However, if one believes that the problem is essentially number theoretic, then this version puts one on more realistic turf. Also, perhaps the real difficulty with proving RH comes into view. That is that the Dirichlet series

$$
\Lambda(s)=\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^{s}}
$$

really does have a zero at $s=1$ ! It has an Euler product, as does $L_{p}(s)$. It also has a functional equation, though of a slightly different character, in that the Gamma-function appears in the denominator rather than the numerator of the factor relating this function at $s$ and at $1-s$. Moreover, can find primes $p$ such that the two Dirichlet series agree for an arbitrarily long initial stretch. Thus, there is a tendency for certain $p$ for $L_{p}(s)$ to want to have a zero at, or near $s=1$. It is this tension that, I believe, causes the intrinsic difficulty with proving RH. Somehow, one has to understand and explain the essential difference between these two objects.

Of course, a significant difference is that the $\chi_{p}(n)$ are periodic functions of $n$ (with period $p)$ whereas $\lambda(n)$ is not. This leads to the deduction that the $L_{p}(s)$ are entire functions, whereas $\Lambda(s)$ is not entire. But how can one make use of this fact?

One attempt actually does take us back into analysis. That is through the representation

$$
\begin{aligned}
\Gamma(s) L_{p}(s) & =\int_{0}^{\infty} e^{-x} x^{s} \frac{d x}{x} \sum_{n=1}^{\infty} \frac{\chi_{p}(n)}{n^{s}}=\int_{0}^{\infty} \sum_{n=1}^{\infty} \chi_{p}(n) e^{-n x} x^{s} \frac{d x}{x} \\
& =\int_{0}^{\infty} \sum_{n=1}^{p-1} \chi_{p}(n) \frac{e^{-n x}}{1-e^{-p x}} \frac{d x}{x}=\int_{0}^{\infty} F_{p}\left(e^{-x}\right) \frac{d x}{\left(1-e^{-p x}\right) x} \\
& =\int_{0}^{1} \frac{F_{p}(u) d u}{\left(1-u^{p}\right) u \log \frac{1}{u}}
\end{aligned}
$$

where

$$
F_{p}(u)=\sum_{n=1}^{p-1} \chi_{p}(n) u^{n}
$$

is the Fekete polynomial. Note that if $F_{p}(u)>0$ for $0<u<1$ then it follows that $L_{p}(\sigma)>0$ for $0<\sigma<1$. For small $p$ this idea works reasonably well. For example, $F_{5}(u)=$ $u-u^{2}-u^{3}+u^{4}>0$. In fact, of the primes up to 100 , only 43 and 67 have Fekete polynomials that have zeros in $[0,1]$. However, if $\chi_{p}(2)=\chi_{p}(3)=\chi_{p}(5)=\chi_{p}(7)=\chi_{p}(11)=-1$ then $F_{p}(0.7)<0$ as shown by Pólya. In fact, it is not known whether $F_{p}(\sigma)>0$ for all $0<\sigma<1$ and infinitely many $p$. On the other hand, it has been shown by Conrey and Soundararajan [CS02] that for infinitely many $p$, (in fact, a positive proportion of $p$ ) the inequality $L_{p}(\sigma)>0$ holds for all $\sigma>0$. Watkins [W] has shown that these L-functions with odd characters do not vanish for $p<3 \times 10^{8}$.

In the spirit of this section we present the following inequality which implies RH .
Let $q>0$ be squarefree with $q \equiv 3 \bmod 4$ and let $h(q)$ be the class number of the imaginary quadratic field $K=Q(\sqrt{-q})$. Let $\chi_{q}$ be the Jacobi symbol modulo $q$ so that $\chi_{q}$ is the quadratic character associated with $K$. Suppose that

$$
S_{q}(N):=\sum_{n=1}^{N} \chi_{q}(n)\left(1-\frac{n}{N}\right) \leqslant h(q)=S_{q}(q / 2)
$$

for all $q$ as described above and all $N<\frac{q}{4}$. Then all complex zeros of the Riemann zetafunction have real part equal to $1 / 2$.

This inequality has been checked for $q<5000$.

## 31. An orthogonal family

The book of Iwaniec and Kowalski [IK04], Section 3.8, is a good reference for the material in this section, as is [GZ80], from which much of this material is taken. See also [RVZ93] and [CSn13]. Let

$$
\eta=\frac{1+\sqrt{-7}}{2}
$$

and for integers $a$ and $b$ let

$$
N(a+b \eta):=(a+b \eta)(a+b \bar{\eta})=a^{2}+a b+2 b^{2} .
$$

We let

$$
\zeta_{K}(s)=\frac{1}{2} \sum_{(a, b) \neq(0,0)} \frac{1}{N(a+b \eta)^{s}}=\zeta(s) L\left(s, \chi_{-7}\right)
$$

where $\chi_{-7}(n)=\left(\frac{n}{7}\right)$ is the Legendre symbol (i.e. it is an arithmetic function which is periodic modulo 7 and for which $\chi_{-7}(1)=\chi_{-7}(2)=\chi_{-7}(4)=1$ and $\chi_{-7}(3)=\chi_{-7}(5)=\chi_{-7}(6)=-1$
and $\chi_{-7}(7)=0$. Note that

$$
\frac{1}{2} \sum_{(a, b) \neq(0,0)} q^{a^{2}+a b+2 b^{2}}=1+\sum_{n=1}^{\infty} a_{n} q^{n}=1+q+2 q^{2}+3 q^{4}+q^{7}+4 q^{8}+2 q^{11}+2 q^{14}+\ldots
$$

where

$$
\sum_{n=1}^{\infty} \frac{a_{n}}{n^{s}}=\zeta_{K}(s)=\left(1+\frac{1}{2^{s}}+\frac{1}{3^{s}}+\ldots\right)\left(1+\frac{1}{2^{s}}-\frac{1}{3^{s}}+\frac{1}{4^{s}}-\frac{1}{5^{s}}-\frac{1}{6^{s}}+\frac{1}{8^{s}}+\ldots\right)
$$

We define a Hecke character by

$$
\chi(a+b \eta)= \begin{cases}\epsilon_{a, b}(a+b \eta) & \text { if }(N(a+b \eta), 7)=1 \\ 0 & \text { otherwise }\end{cases}
$$

Here the choice of $\epsilon_{a, b}= \pm 1$ is determined by

$$
(a+b \eta)^{3} \equiv \epsilon_{a, b} \bmod \sqrt{-7}
$$

This can be simplified to

$$
\epsilon_{a, b}=\left(\frac{2 a+b}{7}\right) .
$$

The Hecke $L$-function is

$$
L(s, \chi)=\frac{1}{2} \sum_{(a, b) \neq(0,0)} \frac{\chi((a+b \eta))}{\left(a^{2}+a b+2 b^{2}\right)^{s+1 / 2}}
$$

which can be more simply written as

$$
L(s, \chi)=\frac{1}{2} \sum_{(a, b) \neq(0,0)} \frac{(a+b \eta)\left(\frac{2 a+b}{7}\right)}{\left(a^{2}+a b+2 b^{2}\right)^{s+1 / 2}} .
$$

This is the $L$-function of a cusp form of level 49 and weight 2 and is the $L$-function of the elliptic curve $y^{2}+x y=x^{3}-x^{2}-2 x-1$, a rank 0 CM elliptic curve of conductor 49. The $L$ function $L_{E}(s)=L(s, \chi)$ satisfies the functional equation

$$
\left(\frac{7}{2 \pi}\right)^{s} \Gamma(s+1 / 2) L(s, \chi)=\Phi(s)=\Phi(1-s)
$$

We are interested in the primitive parts of the $L$-functions of the symmetric powers of $L(s, \chi)$. This amounts to looking at a sequence of Hecke Grössencharacters, denoted by $\chi^{2 n-1}, n=1,2, \ldots$ The series for $L\left(s, \chi^{2 n-1}\right)$ is

$$
L\left(s, \chi^{2 n-1}\right)=\frac{1}{2} \sum_{(a, b) \neq(0,0)} \frac{(a+b \eta)^{2 n-1}\left(\frac{2 a+b}{7}\right)}{\left(a^{2}+a b+2 b^{2}\right)^{s+n-1 / 2}}
$$

(Note that $L\left(s, \chi^{2 n}\right)$ is identically zero.) The Euler product for $L\left(s, \chi^{2 n-1}\right)$ is

$$
L\left(s, \chi^{2 n-1}\right)=\prod_{p=a^{2}+a b+2 b^{2}}\left(1-\frac{\epsilon_{a, b}(a+b \eta)^{2 n-1}}{p^{s+n+1 / 2}}\right)^{-1}\left(1-\frac{\epsilon_{a, b}(a+b \bar{\eta})^{2 n-1}}{p^{s+n+1 / 2}}\right)^{-1}
$$

In general, if

$$
L(s)=\prod_{p}\left(1-\frac{\alpha_{p}}{p^{s}}\right)^{-1}\left(1-\frac{\overline{\alpha_{p}}}{p^{s}}\right)^{-1}
$$

with $\left|\alpha_{p}\right|=1$, then the symmetric $k$ th power is (up to some bad factors)

$$
L\left(s, \mathrm{sym}^{k}\right)=\prod_{p}\left(1-\frac{\alpha_{p}^{k}}{p^{s}}\right)^{-1}\left(1-\frac{\alpha_{p}^{k-2}}{p^{s}}\right)^{-1} \ldots\left(1-\frac{\overline{\alpha_{p}^{k-2}}}{p^{s}}\right)^{-1}\left(1-\frac{\overline{\alpha_{p}^{k}}}{p^{s}}\right)^{-1} .
$$

Thus we see in our situation for the symmetric powers of the L-function of a CM elliptic curve that

$$
L\left(s, \chi, \operatorname{sym}^{2 n-1}\right)=L\left(s, \chi^{2 n-1}\right) L\left(s, \chi^{2 n-3}\right) L\left(s, \chi^{2 n-5}\right) \ldots L(s, \chi)
$$

It is convenient to define the function $\chi^{(2 n-1)}$ at positive rational integers $m$ by

$$
\chi^{(2 n-1)}(m)=\frac{1}{2} \sum_{a^{2}+a b+2 b^{2}=m} \chi^{2 n-1}((a+b \eta)) .
$$

Then

$$
L\left(s, \chi^{2 n-1}\right)=\sum_{m=1}^{\infty} \frac{\chi^{(2 n-1)}(m)}{m^{s+n-1 / 2}}
$$

The functional equation for $L\left(s, \chi^{2 n-1}\right)$ is

$$
\left(\frac{7}{2 \pi}\right)^{s} \Gamma(s+n-1 / 2) L\left(s, \chi^{2 n-1}\right)=\Phi_{2 n-1}(s)=(-1)^{n-1} \Phi_{2 n-1}(1-s)
$$

and in asymmetric form

$$
L\left(s, \chi^{2 n-1}\right)=(-1)^{n-1} X_{2 n-1}(s) L\left(1-s, \chi^{2 n-1}\right)
$$

where

$$
X_{2 n-1}(s)=\left(\frac{7}{2 \pi}\right)^{1-2 s} \frac{\Gamma(1-s+n-1 / 2)}{\Gamma(s+n-1 / 2)}
$$

Here the center of the critical strip is at $s=1 / 2$.
Rodriguez-Villegas and Zagier [RVZ93] have proven a formula, conjectured by Gross and Zagier [GZ80], for the central value of the $L\left(s, \chi^{2 n-1}\right)$, namely

$$
L\left(1 / 2, \chi^{2 n-1}\right)=2 \frac{(2 \pi / \sqrt{7})^{n} \Omega^{2 n-1} A(n)}{(n-1)!}
$$

where

$$
\Omega=\frac{\Gamma(1 / 7) \Gamma(2 / 7) \Gamma(4 / 7)}{4 \pi^{2}}=0.81408739831 \ldots
$$

By the functional equation $A(n)=0$ whenever $n$ is even. For odd $n$ Gross and Zagier [GZ80] conjectured that $A(n)$ is a square and gave the following table (in the notation of Rodriguez-Villegas and Zagier):

| $n$ | $A(n)$ | $L\left(1 / 2, \chi^{2 n-1}\right)$ |
| :--- | :--- | :--- |
| 1 | $1 / 4$ | 0.9666 |
| 3 | 1 | 4.7890 |
| 5 | 1 | 0.9885 |
| 7 | $3^{2}$ | 0.7346 |
| 9 | $7^{2}$ | 0.1769 |
| 11 | $\left(3^{2} \cdot 5 \cdot 7\right)^{2}$ | 9.8609 |
| 13 | $(3 \cdot 7 \cdot 29)^{2}$ | 0.6916 |
| 15 | $(3 \cdot 7 \cdot 103)^{2}$ | 0.1187 |
| 17 | $(3 \cdot 5 \cdot 7 \cdot 607)^{2}$ | 1.0642 |
| 19 | $\left(3^{3} \cdot 7 \cdot 4793\right)^{2}$ | 1.7403 |
| 21 | $\left(3^{2} \cdot 5 \cdot 7 \cdot 29 \cdot 2399\right)^{2}$ | 6.6396 |
| 23 | $\left(3^{3} \cdot 5 \cdot 7^{2} \cdot 10091\right)^{2}$ | 0.3302 |
| 25 | $\left(3^{2} \cdot 7^{2} \cdot 29 \cdot 61717\right)^{2}$ | 0.2072 |
| 27 | $\left(3^{2} \cdot 5^{2} \cdot 7^{2} \cdot 13 \cdot 53^{2} \cdot 79\right)^{2}$ | 1.2823 |
| 29 | $\left(3^{4} \cdot 5^{2} \cdot 7^{2} \cdot 113 \cdot 127033\right)^{2}$ | 8.4268 |
| 31 | $\left(3^{5} \cdot 5 \cdot 7^{2} \cdot 71 \cdot 1690651\right)^{2}$ | 0.6039 |
| 33 | $\left(3^{4} \cdot 5 \cdot 7^{2} \cdot 1291 \cdot 1747169\right)^{2}$ | 0.0591 |

Rodriguez-Villegas and Zagier [RVZ93] proved that $A(n)=B(n)^{2}$ where $B(1)=1 / 2$ and $B(n)$ is an integer for $n>1$. In fact they prove a remarkable recursion formula:

Define a sequence of polynomials $b_{k}(x)$ by the recursion

$$
21 b_{k+1}(x)=\left((32 k x-56 k+42)-(x-7)(64 x-7) \frac{d}{d x}\right) b_{k}(x)-2 k(2 k-1)(11 x+7) b_{k-1}(x)
$$

with initial conditions $b_{0}(x)=1 / 2$, and $b_{1}(x)=1$. Then, with $A$ and $B$

$$
B(2 n+1)=b_{n}(0)
$$

Moreover, equation (6) of [RVZ93] states that for odd $n$

$$
B(n) \equiv-n \bmod 4,
$$

a result that in one fell swoop proves the non-vanishing of $L\left(1 / 2, \chi^{2 n-1}\right)$ for all odd $n$.
It would be interesting to use these recursion formulae to try to understand a discretization of the values of this family of L-functions, from which one might profitably apply a random matrix model to infer more detailed statistical behavior of these values. The integers $B(n)$ that appear in the formula of Villegas-Zagier are growing quickly, presumably to counteract, by virtue of the expected Lindelöf Hypothesis, the $C^{n}(n-1)$ ! growth in the denominator. The question of how just how small these $L$-values can be is an interesting one.

One thing is that for infinitely many $n$

$$
b_{n}(0)^{2} \geq \frac{7^{n} 2^{6 n} \pi^{6 n}(2 n)!}{(\Gamma(1 / 7) \Gamma(2 / 7) \Gamma(4 / 7))^{4 n}}
$$

whereas the Riemann Hypothesis for a suitable set of L-functions implies that

$$
b_{n}(0)^{2} \ll \frac{7^{n} 2^{6 n} \pi^{6 n}(2 n)!}{(\Gamma(1 / 7) \Gamma(2 / 7) \Gamma(4 / 7))^{4 n}} \exp \left(\frac{\log n}{\log \log n}\right) .
$$

Consider the generating function

$$
B(x, y):=\sum_{n=0}^{\infty} b_{n}(x) y^{n} .
$$

Note that

$$
B_{x}(x, y)=\sum_{n=0}^{\infty} b_{n}^{\prime}(x) y^{n}
$$

also,

$$
B_{y}(x, y)=\sum_{n=1}^{\infty} n b_{n}(x) y^{n-1}=\sum_{n=2}^{\infty}(n-1) b_{n-1}(x) y^{n-2}
$$

and

$$
B_{y y}(x, y)=\sum_{n=1}^{\infty} n(n-1) b_{n}(x) y^{n-1}=\sum_{n=2}^{\infty}(n-1)(n-2) b_{n-1}(x) y^{n-2} .
$$

Using these we derive the partial differential equation for $B(x, y)$ :

$$
\begin{aligned}
& (88 x+56) y^{2} B_{y y}+(220 x y+140 y-64 x+112) y B_{y} \\
& \quad+\left(128 x^{2}-910 x+98\right) B_{x}+(44 x y+28 y-42) B+21=0
\end{aligned}
$$

The growth of the coefficients in the power series solution of this equation remarkably encodes the Lindelöf Hypothesis for this family of L-functions.

## 32. Positivity

The issue of possible real zeros arises especially in connection with Dirichlet L-functions for real characters. It is conjectured that $L\left(1 / 2, \chi_{d}\right) \geq 0$; this inequality would imply that there are no Landau-Siegel zeros. Remarkably there are instances of families of L-functions where one does know the non-negativity - or even positivity - of the central value. Such is true for example for primitive L-functions of degree 2 with no character, by work of Waldspurger [Wal81] and Kohnen and Zagier [KZ81], and Katok and Sarnak [KatS93]. For example, Kohnen and Zagier prove that if $L_{f}(s)$ is the L-function of a weight $2 k$ newform for the full
modular group then the central value of the L-function twisted by the real character $\chi_{D}$, where $D$ is a fundamental discriminant with $(-1)^{k} D>0$, is given by

$$
L_{f}\left(k / 2, \chi_{D}\right)=\frac{\omega_{f} c(|D|)^{2}}{|D|^{k-1 / 2}}
$$

where $c(|D|)$ is an integer (the coefficient of a half-integral weight form related to $f$ by the Shimura lift) and where $c_{f}>0$ is an explicit constant.

## 33. Epstein zeta-functions; Haseo Ki's Theorem

Let $Q(u, v)$ be a positive definite quadratic form. The Epstein zeta-function is

$$
\zeta_{Q}(s)=\frac{1}{2} \sum_{(m, n) \neq(0,0)} \frac{1}{Q(m, n)^{s}}
$$

where the sum is over all pairs of integers except $(0,0)$. This has a functional equation

$$
\left(\frac{\sqrt{d}}{2 \pi}\right)^{s} \Gamma(s) \zeta_{Q}(s)=\xi_{Q}(s)=\xi_{Q}(1-s)
$$

It has an Euler product in certain situations, nine to be exact, namely

$$
Q_{1}(u, v)=\frac{1}{4}\left(u^{2}+v^{2}\right), \ldots Q_{9}(u, v)=u^{2}+u v+41 v^{2}
$$

This has a Fourier expansion (the Chowla-Selberg formula) in the variable $x$ where $z=x+i y$ and $z$ is a root of $Q(u, v)=0$ (so $z=x+i y=-b /(2 a)+i \sqrt{d} /(2 a))$. This Fourier expansion screams out RH, though in fact RH is probably true only in those nine cases. Haseo Ki [Ki05], building on the work of many previous authors, has shown that each finite truncation of this Fourier series has all but finitely many of its zeros on the half-line. Let $K_{s}(x)$ denote the K-Bessel function (defined below). Then the Fourier expansion, can be written as

$$
\xi_{Q}(s)=y^{s} \eta(2 s)+y^{1-s} \eta(2-2 s)+4 y^{1 / 2} \sum_{n=1}^{\infty} n^{s-1 / 2} \sigma_{2 s-1}(n) \cos (2 \pi n x) K_{s-1 / 2}(2 \pi n y)
$$

where $\eta(s):=\pi^{-s / 2} \Gamma(s / 2) \zeta(s)=\eta(1-s)$. Each term of the Fourier expansion (with the constant term being $y^{s} \eta(2 s)+y^{1-s} \eta(2-2 s)$ ) is invariant under $s \rightarrow 1-s$. Also each term has all of its zeros on the $1 / 2$-line. Ki showed that, for each $N$, all but finitely many zeros of

$$
y^{s} \eta(2 s)+y^{1-s} \eta(2-2 s)+4 y^{1 / 2} \sum_{n=1}^{N} n^{s-1 / 2} \sigma_{2 s-1}(n) \cos (2 \pi n x) K_{s-1 / 2}(2 \pi n y)
$$

are on the 1/2-line.
As mentioned earlier, there are only 9 values of $z$ for which we expect that RH is true. For all other values it is almost certainly false. Consequently, we don't hold much hope for this approach, unless the Euler product can be worked into the picture.

The K-Bessel function is defined in various (equivalent) ways here. As a Fourier integral,

$$
\int_{-\infty}^{\infty}\left(t^{2}+y^{2}\right)^{-s} e(-n t) d t=2 \pi^{s} \Gamma(s)^{-1}|n|^{s-1 / 2} y^{1 / 2-s} K_{s-1 / 2}(2 \pi|n| y)
$$

As a solution of a differential equation, $y=K_{\nu}(z)$ satisfies

$$
z^{2} y^{\prime \prime}+z y^{\prime}-\left(z^{2}+\nu^{2}\right) y=0
$$

As a Mellin transform,

$$
\int_{0}^{\infty} u^{-\nu} e^{-x(u+1 / u)} \frac{d u}{u}=2 K_{\nu}(2 x),
$$

and as an inverse Mellin transform, If $c>\max \{0,-2 \nu\}$,

$$
4 x^{\nu} K_{\nu}(2 x)=\frac{1}{2 \pi i} \int_{(c)} \Gamma(s / 2) \Gamma(\nu+s / 2) x^{-s} d s
$$

It follows from a theorem of Pólya that for any $x>0$ all of the zeros of

$$
k(s):=K_{s-1 / 2}(x)
$$

are on the line $\Re s=1 / 2$.

## 34. Some other equivalences of interest

- Equidistribution of Farey sequence: Let $r_{v}$ be the elements of the Farey sequence of order $N, v=1,2, \ldots \Phi(N)$ where $\Phi(N)=\sum_{n=1}^{N} \phi(n)$. Let $\delta_{v}=r_{v}-v / \Phi(N)$. Then RH if and only if

$$
\sum_{v=1}^{\Phi(N)} \delta_{v}^{2} \ll N^{-1+\epsilon}
$$

Also, RH if and only if

$$
\sum_{v=1}^{\Phi(N)}\left|\delta_{v}\right| \ll N^{1 / 2+\epsilon} .
$$

- Lagarias theorem: RH if and only if

$$
\sigma(n) \leq H_{n}+\exp \left(H_{n}\right) \log H_{n}
$$

for every $n$ where $H_{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$

- (Hinkannen, Complex Variables 4, 1997) RH if and only if

$$
\Re \frac{\xi^{\prime}(s)}{\xi(s)}>0
$$

for $\Re s>1 / 2$.
This is easy to show. Basically

$$
\Re \frac{\xi^{\prime}}{\xi}(s)=\sum_{\rho} \Re \frac{1}{s-\rho}=\sum_{\rho} \frac{\sigma-\beta}{|s-\rho|^{2}}
$$

If RH holds, then all $\beta=1 / 2$ and so $\sigma>1 / 2$ implies each term of the sum is positive. If there is a $\rho=\beta+i \gamma$ with $\beta>1 / 2$, then by choosing $s=\sigma+i \gamma$, the term in the sum corresponding to $\rho$ is $1 /(\sigma-\beta)$ which for $\sigma$ very close to $\beta$ but smaller than it will cause the entire sum to be negative.

- V. V. Volchkov has shown that RH is equivalent to the equality

$$
\int_{0}^{\infty} \int_{1 / 2}^{\infty} \frac{1-12 y^{2}}{\left(1+4 y^{2}\right)^{3}} \log (|\zeta(x+i y)|) d x d y=\pi \frac{3-\gamma}{32}
$$

- Convergence of Carey's series: RH if and only if

$$
\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\left|\sum_{k=0}^{n} \frac{c_{2 n+1,2 k+1}}{2 k+2} \log \left(\frac{2 k+1}{2 k+2} \frac{(-1)^{k} B_{2 k+2}(2 \pi)^{2 k+2}}{2(2 k+2)!}\right)\right|^{2}<\infty
$$

where $c_{m, r}$ denotes the coefficient of $x^{r}$ in the Legendre polynomial of degree $m$ and $B_{k}$ is the $k$ th Bernoulli number. Specifically,

$$
c_{2 n+1,2 k+1}=\frac{(-1)^{n-k}(2 n+2 k+2)!}{2^{2 n+1}(n-k)!(n+k+1)!(2 k+1)!}
$$

and

$$
\zeta(2 k)=\frac{(-1)^{k+1}(2 \pi)^{2 k}}{2(2 k)!} B_{2 k}
$$

## 35. Zeros on the line

In this section we briefly describe the methods that have show that many zeros are on the line.
35.1. Simplest method. The simplest way to conclude that infinitely many zeros are on the $1 / 2$-line seems to be to contrast the behaviors of

$$
\left|\int_{0}^{T} Z(t) d t\right|
$$

and

$$
\int_{0}^{T}|Z(t)| d t
$$

We can show that these behave differently asymptotically as $T \rightarrow \infty$ which implies that there are infinitely many zeros on the line. We easily have

$$
\int_{0}^{T}|Z(t)| d t \geq\left|\int_{0}^{T} \zeta(1 / 2+i t) d t\right| \sim T
$$

by moving the path of integration in the latter integral to the right of 1 and integrating term by term. On the other hand

$$
\int_{0}^{T} Z(t) d t
$$

has substantial cancelation because of the oscillations in $\chi(1-s)^{-1 / 2} \sim \exp (i t / 2 \log t)$ and can be bounded from above in various ways by $T^{3 / 4}$ for example.

Incidentally, this assertion has not been proven for degree 3 L -functions.
35.2. Hardy and Littlewood's method. The above can be strengthened to show that there are $\gg T$ zeros on the line up to a height $T$. Basically one compares

$$
\int_{0}^{T}\left|\int_{t}^{t+H} Z(u) d u\right| d t
$$

with

$$
\int_{0}^{T}\left(\int_{t}^{t+H}|Z(u)| d u\right) d t
$$

35.3. Siegel's method. Siegel employed a formula found in Riemann's notes

$$
\pi^{-s / 2} \Gamma(s / 2) \zeta(s)=\pi^{-s / 2} \Gamma(s / 2) f(s)+\pi^{-\frac{1-s}{2}} \Gamma((1-s) / 2) \bar{f}(1-s)
$$

where

$$
f(s)=\int_{\mathcal{L}} \frac{x^{-s} e^{\pi i x^{2}}}{e^{\pi i x}-e^{-\pi i x}} d x
$$

(here $\mathcal{L}$ is a line of slope -1 that passes through $1 / 2$ ) to assert that $\zeta(1 / 2+i t)=0$ whenever the argument of

$$
\pi^{-s / 2} \Gamma(s / 2) f(s)
$$

is congruent to $\pi / 2$ modulo $\pi$. After applying the argument principle, the problem boils down to showing that the entire function $f(s)$ has many zeros to the left of the $1 / 2$-line. Most of the zeros seem to be near the $1 / 2$-line which makes this proposition seem daunting. But through some quite sophisticated analysis Siegel is able to show that the entire $f(s)$ actually has $\gg T$ zeros to the left of the 0 -line at a height smaller than $T$. In this way he deduces at least $T$ zeros on the line for $\zeta(s)$.
35.4. Selberg's method. This is like the Hardy-Littlewood method except that in place of $Z(t)$ one uses $Z(t)|\eta(t)|^{2}$ where $\eta(t)$ is an approximation to $\zeta(1 / 2+i t)^{-1 / 2}$; thus $|\eta(t)|^{2}$ acts as a "mollifier" for $\zeta(1 / 2+i t)$ to mitigate the loss of a $\log T$ that occurs when Cauchy's inequality is invoked in the Hardy-Littlewood method.
35.5. Levinson's method. This is like Siegel's method except that $f(s)$ is basically replaced by $\zeta^{\prime}(1-s)$ and a mollifier is used to avoid losing a logarithm and Littlewood's lemma is invoked to give an upper bound for the number of zeros of $\zeta^{\prime}(1-s)$ with real part larger than $1 / 2$. Now the analysis is delicate; it's clear from the start that one will obtain a lower bound of the right order of magnitude but the sign of that magnitude is in question. At the end of the calculation one might conclude that at least $-10 \%$ of the zeros are on the critical line. Fortunately, Levinson gets $1 / 3$ of the zeros on the line; a very respectable result.


Figure 5. Zeros of the Riemann-Siegel function $f(s)$
35.6. Improvements in Levinson. Levinson's proportion has been improved to $40 \%$ then $41 \%$ and the current record is $41.2 \%$ due to Feng. These improvements have come about by using longer and more elaborate mollifiers, and also the calculus of variations to help choose optimal weights in the mollifier function. A sample result is

Theorem 8. Let

$$
B(s, P)=\sum_{n \leq y} \frac{\mu(n) P\left(\frac{\log \frac{y}{n}}{\log y}\right)}{n^{s}}
$$

where $P$ is a polynomial with $P(0)=0 ; V(s, Q)=Q\left(\frac{-1}{L} \frac{d}{d s}\right) \zeta(s)$; and let $\sigma_{0}=1 / 2-R / L$ where $L=\log T$. Then

$$
\frac{1}{T} \int_{1}^{T}\left|V\left(\sigma_{0}+i t, Q\right) B\left(\sigma_{0}+i t, P\right)\right|^{2} d t \sim(c(R, P, Q)
$$

for $y=T^{\theta}$ with $\theta<4 / 7$ where

$$
c(R, P, Q)=|P(1) Q(0)|^{2}+\left.\frac{1}{\theta} \int_{0}^{1} \int_{0}^{1}\left|\frac{d}{d u}\left(e^{R(y+\theta u)} Q(y+\theta u) P(x+u)\right)\right|_{u=0}\right|^{2} d x d y
$$

The length $\theta$ of the mollifier is critical here; Farmer [Far93] has conjectured that the above asymptotic formula holds true for arbitrary fixed $\theta$; this is called the "long mollifiers" conjecture. It would imply that $100 \%$ of the zeros of $\zeta(s)$ are on the critical line. In fact, Bettin (unpublished) has an argument that the long mollifiers conjecture implies RH.

A qualitative improvement in Levinson's method is that the proportion of zeros on the critical line of $\xi^{(n)}(s)$ the $n$-th derivative of the Riemann xi-function approaches 1 as $n \rightarrow \infty$. So, in some ways Levinson's method is very satisfying. Also, Levinson's method can be arranged to produce simple zeros, a fact observed independently by Selberg (unpublished) and Heath-Brown [H-B79]. It is known [BCY11] that at least $40.5 \%$ of the zeros of $\zeta(s)$ are simple and on the critical line.

## 36. Critical zeros of other L-Functions

Hafner has used Selberg's method to prove that a positive proportion of the zeros of degree 2 L-functions are on the critical line. Levinson's method doesn't quite work for degree 2 Lfunctions but it does give bounds for the multiplicity of zeros, [Far94].

Conrey, Iwaniec and Soundararajan [CIS13] have shown that at least $60 \%$ of the collective zeros of all Dirichlet L-functions are on the critical line; similarly at least $36 \%$ of all the twists by Dirichlet characters of a fixed degree 2 L-function are on the critical line; and at least $0.5 \%$ of all the twists by Dirichlet characters of a fixed degree 3 L -function are on the critical line.

Work on real zeros of Dirichlet L-functions is extremely interesting in this context. In some ways this study is a microcosm of work on the Riemann Hypothesis.

Soundararajan [Sou00] has shown that at least $7 / 8$ of these L-functions don't vanish at the center.

Conrey and Soundararajan [CS02] have shown that at least $20 \%$ of such L-functions have no real zeros.

There is a simple approach using Fekete polynomials

$$
F_{\chi_{d}}(t)=\sum_{n=1}^{|d|-1} \chi_{d}(n) t^{n} ;
$$

if $F_{\chi_{d}}(t)>0$ for $0<t<1$ then $L\left(s, \chi_{d}\right)$ has no positive real zeros. But this idea fails quickly, in particular for $d=-163$. In fact Montgomery and Baker proved that $L\left(s, \chi_{d}\right)$ oscillates quite a lot on the real interval from 0 to 1 , likely as many as $\log \log |d|$ times. Chowla and Selberg [ChSe67] considered the Dedekind zeta function $\zeta(s) L\left(s, \chi_{d}\right)$ for the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ (with $d<0$ ) and identified it as an Epstein zeta-function for which they found an explicit Fourier expansion which was used to prove that the $L\left(s, \chi_{-163}\right)$ has no positive real zeros. Bateman and Grosswald [BG64] identified the Bessel functions in the formula to give the following: Suppose that $d=b^{2}-4 a c<0$ and let

$$
Z(s)=\frac{1}{2} \sum_{(m, n) \neq(0,0)} \frac{1}{\left.a m^{2}+b m n+c n^{2}\right)^{s}} .
$$

Then

$$
a^{s} Z(s)=\zeta(2 s)+\pi^{1 / 2} k^{1-2 s} \zeta(2 s-1) \frac{\Gamma(s-1 / 2)}{\Gamma(s)}+\frac{\pi^{2}}{\Gamma(s)} k^{-s+1 / 2} H(s)
$$

where

$$
H(s)=4 \sum_{n=1}^{\infty} n^{s-1 / 2} \sigma_{1-2 s}(n) \cos (\pi n b / a) K_{s-1 / 2}(2 \pi k n) ;
$$

here $\left.\sigma_{r}(n)=\sum_{d \mid n} d^{r}\right)$. Low [Low68] used this formula to prove that no $L\left(\sigma, \chi_{d}\right) \neq 0$ for $0<\sigma<1$ and real odd characters with $|d|<800000$ Mark Watkin's extended this to $|d|<3 \times 10^{8}$ when $\chi_{d}$ is an odd character. Watkins mentions that for real even characters $\chi_{d}$ it was proven by Rosser (unpublished) that $L\left(\sigma, \chi_{d}\right) \neq 0$ for $0<\sigma \leq 1$ and $|d| \leq 986$ . Recently David Platt [Pla13] has verified the Riemann Hypothesis for each Dirichlet Lfunction with a character of modulus $q$ smaller than 400,000 up to a height at least $10^{8} / q$, so this includes the real even characters and surpasses Rosser's result.

Also, there is extensive work by Iwaniec and others about what one can prove if LandauSiegel zeros (i.e. real zeros very near to $s=1$ of Dirichet L-functions for real characters) exist: infinitely many twin primes; zeros of zeta lie in an arithmetic progression; the pair correlation function is periodic; $100 \%$ of the zeros of $\zeta(s)$ are on the critical line in certain intervals; precisely $50 \%$ of L-functions associated with cusp forms on $\Gamma_{0}(N)$ vanish at the central point; and $x^{2}+y^{6}$ is prime infinitely often. The study of Landau-Siegel zeros is extremely instructive and may well offer substantial clues about the Riemann Hypothesis.

## 37. Random Matrix Theory

There have been remarkable developments in the statistical theory of L-functions based on Random Matrix theory, see [Meh04] for a general reference. These have their beginnings in Montgomery's pair correlation conjecture and the ensuing Montgomery - Odlyzko Law and the ensuing work by Katz and Sarnak [?] on symmetry types of families and in the Keating - Snaith [KS00] work on conjectures for moments of families of L-functions. Now we have a detailed (conjectural) picture of averages of products and ratios of products of L-functions that can be used to precisely describe the statistical behavior of values and zeros of L-functions in families. While these are not directly related to the Riemann Hypothesis they give a glimpse of the depth of complexity that these functions are capable of. The Keating and Snaith conjecture for moments of zeta asserts that

$$
\int_{0}^{T}|\zeta(1 / 2+i t)|^{2 k} d t \sim g_{k} a_{k} T \frac{(\log T)^{k^{2}}}{k^{2}!}
$$

where

$$
a_{k}=\prod_{p}\left(1-\frac{1}{p}\right)^{(k-1)^{2}} \sum_{\ell=0}^{k-1} \frac{\binom{k-1}{\ell}^{2}}{p^{\ell}}
$$

and

$$
g_{k}=\frac{k^{2}!}{1^{1} \cdot 2^{2} \cdots \cdots k^{k} \cdot(k+1)^{k-1} \cdots \cdots(2 k-1)^{1}}
$$

In particular

$$
\begin{gathered}
\int_{0}^{T}|\zeta(1 / 2+i t)|^{2} d t T \log T \\
\int_{0}^{T}|\zeta(1 / 2+i t)|^{4} d t \sim 2 \prod_{p}\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\right) \frac{\log ^{4} T}{4!}
\end{gathered}
$$

which were proven by Hardy and Littlewood and Ingham, and

$$
\int_{0}^{T}|\zeta(1 / 2+i t)|^{6} d t \sim 42 \prod_{p}\left(1-\frac{1}{p}\right)^{4}\left(1+\frac{4}{p}+\frac{1}{p^{2}}\right) \frac{\log ^{9} T}{9!}
$$

conjectured in [CoGh98], and

$$
\int_{0}^{T}|\zeta(1 / 2+i t)|^{8} d t \sim 24024 \prod_{p}\left(1-\frac{1}{p}\right)^{9}\left(1+\frac{9}{p}+\frac{9}{p^{2}}+\frac{1}{p^{3}}\right) \frac{\log ^{16} T}{16!}
$$

conjectured in [CoGo01]. Heath-Brown [HB79] proved a formula for the 4th power moment with a power of $T$ savings; the mainterm was of the shape $T P_{4}(\log T)$ where $P_{4}$ is a 4 th degree polynomial. Motohashi [Mot97] found an exact formula for a weighted 4th power moment. The general conjectures have been elaborated to predict all lower order main terms with a power of $T$ savings in the error term. More precisely we have the following conjecture [CFKRS05]
Conjecture 2. Let $A$ and $B$ be sets of complex numbers ("shifts") each smaller than $1 / 10$ in absolute value. Let

$$
Z_{\zeta}(A ; B)=\prod_{\alpha \in A, \beta \in B} \zeta(1+\alpha+\beta)
$$

and

$$
\begin{aligned}
\mathcal{A}(A ; B) & =\prod_{p} \prod_{\alpha \in A, \beta \in B}\left(1-\frac{1}{p^{1+\alpha+\beta}}\right) \\
& \times \int_{0}^{1} \prod_{\alpha \in A} z_{p, \theta}(1 / 2+\alpha) \prod_{\beta \in B} z_{p,-\theta}(1 / 2+\beta) d \theta
\end{aligned}
$$

where $z_{p, \theta}(x)=1 /\left(1-e(\theta) / p^{x}\right)$. Then for some $\delta>0$,

$$
\begin{aligned}
& \int_{0}^{T} \prod_{\alpha \in A} \zeta(1 / 2+i \tau+\alpha) \prod_{\beta \in B} \zeta(1 / 2-i \tau+\beta) d \tau \\
&=\int_{0}^{T} \sum_{\substack{S S A \\
T \subset B \\
|S|=|T|}} e^{-\ell\left(\sum s+\sum t\right)} \mathcal{A} Z_{\zeta}(\bar{S} \cup(-T) ; \bar{T} \cup(-S)) d \tau \\
& \quad+O\left(T^{1-\delta}\right) .
\end{aligned}
$$

The conjecture above has been systematically tested; all theoretical and numerical results are in accordance. Diaconu, Goldfeld and Hoffstein [DGH03] have an approach to these conjectures through multiple Dirichlet series. Also Bump and Beineke, see [BB04] and [BB04b], have constructed an Eisenstein series on $G L(2 k)$ whose L-function is a product of shifted zetas and whose constant term has the same structure as the main term in the predicted average of that L-function, but without the arithmetic factors.

As an application we conjecture that

$$
\int_{0}^{T}|\zeta(1 / 2+i t)|^{6} d t=\int_{0}^{T} P_{3}\left(\log \frac{t}{2 \pi}\right) d t+O\left(T^{1-\delta}\right)
$$

where

$$
\begin{aligned}
P_{3}(x)= & 0.000005708527034652788398376841445252313 x^{9} \\
& +0.00040502133088411440331215332025984 x^{8} \\
& +0.011072455215246998350410400826667 x^{7} \\
& +0.14840073080150272680851401518774 x^{6} \\
& +1.0459251779054883439385323798059 x^{5} \\
& +3.984385094823534724747964073429 x^{4} \\
& +8.60731914578120675614834763629 x^{3} \\
& +10.274330830703446134183009522 x^{2} \\
& +6.59391302064975810465713392 x \\
& +0.9165155076378930590178543 .
\end{aligned}
$$

Numerically we have

$$
\int_{0}^{2350000}|\zeta(1 / 2+i t)|^{6} d t=3317496016044.9
$$

whereas

$$
\int_{0}^{2350000} P 3\left(\log \frac{t}{2 \pi}\right) d t=3317437762612.4
$$

Perhaps the best confirmation of our conjecture is the theorem of [CIS12].
Theorem 9. Let $A=\left\{\alpha_{1}+i y, \alpha_{2}+i y, \alpha_{3}+i y\right\}$ and $B=\left\{\beta_{1}-i y, \beta_{2}-i y, \beta_{3}-i y\right\}$ with $\alpha_{j}, \beta_{j} \ll 1 / \log Q$, and $\Psi$ smooth on $[1,2], \Phi$ Schwarz on $\mathbf{R}$. Then

$$
\begin{aligned}
& \sum_{q} \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Phi(y) \sum_{\chi}{ }^{b} \prod_{\alpha \in A} L(1 / 2+\alpha, \chi) \prod_{\beta \in B} L(1 / 2+\beta, \bar{\chi}) d y \\
& =\sum_{q} \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Phi(y) \sum_{\chi}{ }^{b} \mathcal{S}_{A ; B}(q) d y+O\left(Q^{19 / 10+\epsilon}\right)
\end{aligned}
$$

where $\mathcal{S}$ is the prediction from the recipe.

Regarding the precise distribution of zeros there is the Ratios Conjecture of Conrey, Farmer, and Zirnbauer [CFZ08]. A special case is

Conjecture 3. Let $\Re \gamma, \Re \delta>0$ and $\Im \alpha, \beta, \gamma, \delta \ll T^{1-\epsilon}$. Let $s=1 / 2+$ it and

$$
R_{\zeta}(\alpha, \beta, \gamma, \delta)=\int_{0}^{T} \frac{\zeta(s+\alpha) \zeta(1-s+\beta)}{\zeta(s+\gamma) \zeta(1-s+\delta)} d t
$$

Then

$$
\begin{aligned}
R_{\zeta}=\int_{0}^{T} & \left(\frac{\zeta(1+\alpha+\beta) \zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta) \zeta(1+\beta+\gamma)} A_{\zeta}(\alpha, \beta, \gamma, \delta)\right. \\
& \left.+\left(\frac{t}{2 \pi}\right)^{-\alpha-\beta} \frac{\zeta(1-\alpha-\beta) \zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta) \zeta(1-\alpha+\gamma)} A_{\zeta}(-\beta,-\alpha, \gamma, \delta)\right) d t \\
& +O\left(T^{1-\delta}\right)
\end{aligned}
$$

The Euler product $A$ is given by

$$
A_{\zeta}(\alpha, \beta, \gamma, \delta)=\prod_{p} \frac{\left(1-\frac{1}{p^{1+\gamma+\delta}}\right)\left(1-\frac{1}{p^{1+\beta+\gamma}}-\frac{1}{p^{1+\alpha+\delta}}+\frac{1}{p^{1+\gamma+\delta}}\right)}{\left(1-\frac{1}{p^{1+\beta+\gamma}}\right)\left(1-\frac{1}{p^{1+\alpha+\delta}}\right)}
$$

As a consequence of this conjecture we can obtain lower order terms in Montgomery's pair correlation (see [CSn07]). This formula was obtained earlier by Bogomolny and Keating.

Theorem 10. Assuming the ratios conjecture,

$$
\begin{aligned}
\sum_{\gamma, \gamma^{\prime} \leq T} f(\gamma & \left.-\gamma^{\prime}\right)=\frac{1}{(2 \pi)^{2}} \int_{0}^{T}\left(2 \pi f(0) \log \frac{t}{2 \pi}+\int_{-T}^{T} f(r)\left(\log ^{2} \frac{t}{2 \pi}+2\left(\left(\frac{\zeta^{\prime}}{\zeta}\right)^{\prime}(1+i r)\right.\right.\right. \\
& \left.\left.\left.+\left(\frac{t}{2 \pi}\right)^{-i r} \zeta(1-i r) \zeta(1+i r) A(i r)-B(i r)\right)\right) d r\right) d t+O\left(T^{1 / 2+\epsilon}\right)
\end{aligned}
$$

here the integral is to be regarded as a principal value near $r=0$,

$$
A(\eta)=\prod_{p} \frac{\left(1-\frac{1}{p^{1+\eta}}\right)\left(1-\frac{2}{p}+\frac{1}{p^{1+\eta}}\right)}{\left(1-\frac{1}{p}\right)^{2}}
$$

and

$$
B(\eta)=\sum_{p}\left(\frac{\log p}{\left(p^{1+\eta}-1\right)}\right)^{2}
$$

We believe that this formula is very accurate, indeed, down to a power savings in $T$. It includes all of the lower order terms that arise from arithmetical considerations and should include all of the fluctuations found in any of the extensive numerical experiments that have been done. We have not scaled any of the terms here so that terms of different scales are shown all at once.

In [CSn07] we prove an analogue for the one-level density of zeros of Dirichlet L-functions with real charactera.

Theorem 11. Assuming the ratios conjecture for the family of quadratic L-functions, we have

$$
\begin{aligned}
& \sum_{d \leq X} \sum_{\gamma_{d}} f\left(\gamma_{d}\right)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} f(t) \sum_{d \leq X}\left(\log \frac{d}{\pi}+\frac{1}{2} \frac{\Gamma^{\prime}}{\Gamma}(1 / 4+i t / 2)+\frac{1}{2} \frac{\Gamma^{\prime}}{\Gamma}(1 / 4-i t / 2)+\right. \\
& \left.2\left(\frac{\zeta^{\prime}(1+2 i t)}{\zeta(1+2 i t)}+A_{D}^{\prime}(i t ; i t)-\left(\frac{d}{\pi}\right)^{-i t} \frac{\Gamma(1 / 4-i t / 2)}{\Gamma(1 / 4+i t / 2)} \zeta(1-2 i t) A_{D}(-i t ; i t)\right)\right) d t \\
& \quad+O\left(X^{1 / 2+\epsilon}\right)
\end{aligned}
$$

where

$$
A_{D}(-r ; r)=\prod_{p}\left(1-\frac{1}{(p+1) p^{1-2 r}}-\frac{1}{p+1}\right)\left(1-\frac{1}{p}\right)^{-1}
$$

and

$$
A_{D}^{\prime}(r ; r)=\sum_{p} \frac{\log p}{(p+1)\left(p^{1+2 r}-1\right)}
$$

The picture below made by Mike Rubinstein shows shadows at vertical heights that are approximately one-half of the heights of the zeros of $\zeta(s)$. This is not surprisng given the formula above.

## 38. Concluding remarks

We don't have a good clear approach to the Riemann Hypothesis. It is remarkable that it has so many unclear approaches! Nevertheless, these approaches are leading to interesting mathematics.

Some observations: the simplest most natural things we try don't get very far. The conclusion is that $\zeta(s)$ is a more subtle and complicated beast than any previous experience prepares us for.

Somehow the Fourier theory should have worked. However, the discrete nature of RH and the examples of 8 in a continuous family (in the counterexample section) warn us away from the analytic approaches that can't pick out this discrete set of examples.

There are interesting sets of necessary and sufficient conditions for RH based on the coefficients in the expansion of $\xi$ : the Grommer inequalities, the Karlin-Nuttall inequalities, the inequalities that follow from one of Jensen's conditions, and the Li coefficients. It might be interesting to understand the connections between all of these. An idea of Li was to try to give an interpretation to his coefficients; for example if they are related to counting something then they would be non-negative. Also, can any of these conditions be applied in the function field setting?


Figure 6. Zeros of $L\left(s, \chi_{d}\right)$

We do need the Euler product. The straight conjecture about almost periodicity and general Dirichlet series, while intriguing, is probably really hard.

The Rodriguez-Villegas and Zagier recursion formula is extremely tantalizing! And there are connections with this approach and with a continuous family of p-adic Eisenstein series that is worth investigation.

We have mentioned but not discussed the Stepanov - Bombieri proof of Weil's theorem. It is very interesting and potentially brings a new theory and techniques from transcendental number theory into play. In general one should possibly try to find classical analytic approaches that work to prove Weil's theorem.

We have not mentioned the Langlands' program for automorphic representations. A sample conjecture is that the L-function attached to any symmetric power of another L-function is itself meromorphic and has a functional equation. Unfortunately, even if we knew such a powerful statement to be true we still wouldn't know how to use it to deduce RH.

Also we haven't written about Iwaniec's ideas to use families of elliptic curves which have rank $>1$ and root numbers which capture the Möbius function. The idea of using the existing landscape of L-functions with multiple zeros to say something about the Riemann zeta -function is very attractive. For example:


The above is the plot of the $Z$ function for the first elliptic curve of rank 4. It has huge negative spikes essentially at the zeros of the Riemann zeta-function! See Rubinstein [Rub13] for an explanation of this phenomenon.

Somehow we may not be using the functional equation in a good enough way. The most obvious way to prove that an analytic $a(z)$ function has only real zeros is to express it as $a(z)=b(z)+c(z)$ where something like $|b(z)|<|c(z)|$ when $\Im z>0$ and where $|b(z)|>|c(z)|$ when $\Im z<0$. This is similar to the de Branges approach. To go back to basics, consider the example

$$
2 \cos z=e^{i z}+e^{-i z}
$$

The first function is larger when $y<0$ and the first function is larger when $y>0$. So the only place it can vanish is on the real axis. Here, the approximate functional equation of $\zeta(s)$ is very suggestive. It gives us

$$
\chi(1-s)^{-1 / 2} \zeta(s)=\chi(1-s)^{-1 / 2} f(s)+\chi(s)^{-1 / 2} f(1-s)
$$

which essentially expresses $\zeta(s)$ as a sum of complex conjugates on the $1 / 2$-line. Unfortunately it is not the case that one of these functions dominates in half of the plane. In fact the entire function $f(s)$ seems to have infinitely many zeros on each side of the $1 / 2$-line. Nevertheless Siegel used the above as a starting point for a proof that $\zeta(s)$ has $\gg T$ zeros on the critical line up to a height $T$. Levinson's starting point is also a decomposition of this form. Differentiating $H(s) \zeta(s)=H(1-s) \zeta(1-s)$ (where $H(s)=\pi^{-s / 2} \Gamma(s / 2)$ ) we obtain

$$
H(s)\left(\frac{H^{\prime}}{H}(s)+\frac{H^{\prime}}{H}(1-s)\right) \zeta(s)=-H(s) \zeta^{\prime}(s)-H^{\prime}(1-s) \zeta^{\prime}(1-s)
$$

which effectively gives $\zeta(s)$ as a sum of complex conjugates in the critical line. This time, by Speiser's theorem (as proven by Montgomery and Vaughan) one of these functions does dominate. However, it is difficult to see how to prove this. Nevertheless Levinson uses this as a starting point in his proof that at least one-third of the zeros are on the critical line.

The decomposition of the Fourier integral by the functional equation is another example. Perhaps it could be more useful. Perhaps start with the $L_{\tau}(s)$ so as to avoid the pole at $s=1$. Also, perhaps the infinite product for $\Delta$ could be useful now. Maybe use a circle method/saddle point method to analyze the ensuing functions near rational points with small denominator; then prove the desired inequality. However, again the counterexample of the eight warns us away.

Another possibility is to try to use the " 2 " that is everywhere, especially in approaches to the Landau-Siegel zero. One could try to do long mollifiers, with optimal coefficients that involve a smoothed Möbius function; then turn the tail of the sum into a sum over zeros; manipulate that somehow by a transform; then turn it back into a sum like one started with. In this way estimate a long mollification. This "2" arises in Selberg's work when he bounds $S(T)$ point-wise by a Dirichlet polynomial (he bounds the sum over zeros by a Dirichlet polynomial); see also Soundararajan's proof of the upper bound for moments of $\zeta(s)$ (contingent on RH). Perhaps it is the same two that appears in Brun-Titchmarsh upper bounds for primes in arithmetic progressions.

In the Nyman-Beurling approach or the long mollifier approach one should probably try to make use of the reciprocity formula for the cotangent sum.

A new powerful tool might be the theory of stable polynomials (see the references to Branden and Borcea). There are examples of how these get used in the proof of the Leeyang theorem and in the recent proof of the Kadison-Singer theorem. They involve functions of several variables but apply to problems in one variable.

Finally I want to mention Random Matrix Theory. Over the last 15 years we have developed incredibly precise descriptions of statistics of families of L-functions; these include things like moments in families with power savings and statistics of zeros, like pair correlations, again with power savings. The conjectures in the end are very symmetric and easy to describe though they are admittedly combinatorially complicated. These conjectures lead the way sometimes to proofs. For example in the work [CIS12] on the sixth moment of Dirichlet L-functions. The rigorous proofs involve spectacular combinatorial maneuvers. Knowing where we are headed is the thing that keeps us going. One thing we have learned is that
there is always a miracle at the end! A coming together in unexpected ways of apparently dissonant terms. We see this in Levinson's original paper (the cancelation of 24 terms); in Soundararajan's paper [Sou00] on the cubic moment of quadratic L-functions; in the work of Kowalski, Michel, and Vanderkam [KMV00] on orthogonal moments; and certainly it always comes up in the asymptotic large sieve. Also in lower order term of moments and in the combining of constants in shifted convolution problems. There is always some term that cannot readily be computed; but then a functional equation enters and the impossible term gets combined with another problematic term to give a complete sum. So, it is always the functional equation that saves the day. This makes me think that we may not be utilizing the functional equation correctly in our attempts.

A point is that we are now seeing the mechanics of moments on a new ultra-fine scale with the help of the magnification of RMT. One especially amazing thing is that there are often two completely different mechanisms at work in the same problem. This is because on the one hand we have to identify polar terms (i.e. Gamma-factor terms or terms arising from the infinite-prime) in one way, generally a way that is consistent with RMT behavior. Then we have to do the same with the prime parts which are analytically simpler, i.e. no singular behavior to keep track of, but which involve elaborate combinatorial tricks to nail down. It is surprising that here are the two sides to this, that it always works, and that we cannot see the commonality in what we are doing so as to be able to take advantage of something unseen going on.

As we proceed we need to bear in mind is that the function we are dealing with is capable of unruly behavior:


There is much work still to be done analyzing the complexities of $\zeta$. Ultimately the light bulb will switch on!

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