Global Existence and Uniqueness
of the Non-stationary 3D-Navier-Stokes
Initial-boundary Value Problem

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Backup data from the related homepage
https://www.navier-stokes-equations.com

The root cause of the 3D-NSE Millennium problem: inappropriate mechanical H(1) energy norm estimates

"In fact, to date, 3D regular flows are known to exist either for all times but for data of "small size", or for data of "arbitrary size" but for a finite interval of time only, ... it is not known whether, in the 3D case, the associated initial-boundary value problem is well-posed in the sense of Hadamard, ...

"The prescription of the pressure p as a solution of the Neumann problem at the boundary walls or at the initial time is independently of the velocity v and, therefore, could be incompatible with the initial-boundary value NSE problem, which could render the problem ill-posed."
G. P. Galdi, The NSE: A Mathematical Analysis

Note: The below problem statement is based on standard physical "mechanical energy" H(1)-norm estimates, defined by the underlying Dirichlet integral based H(1)-inner product. The proposed solution concept is concerned with an additional (additive) "dynamical energy" norm concept (complementary to H(1) with respect to the H(1/2) norm) as introduced in www.unified-field-theory.de

https://www.fuchs-braun.com/

The 3D-NSE problem

The Navier-Stokes Equations (NSE) describe a flow of incompressible, viscous fluid. The three central questions of every PDE is about existence, uniqueness and smooth dependency on initial data can develop singularities in finite time, and what these might mean. For the NSE satisfactory answers to those questions are available in two dimensions, i.e. 2D-NSE with smooth initial data possesses unique solutions which stay smooth forever. In three dimensions, those questions are still open. Only local existence and uniqueness results are known. Global existence of strong solutions has been proven only, when initial and external forces data are sufficiently smooth. Uniqueness and regularity of non-local Hopf solutions are still open problems. The question of global existence of smooth solutions vs. finite time blow up (ODE problem) is one of the Clay Institute millennium problems.

Cannone M., Harmonic Analysis Tools for Solving the Incompressible Navier-Stokes Equations.pdf (597.36KB)

Galdi G., An Introduction to the Navier-Stokes Initial-Boundary Value Problem.pdf (416.2KB)

Giga Y., Weak and strong solutions of the Navier-Stokes initial value problem.pdf (2.06MB)

Serrin J., Mathematical Principles of Classical Fluid Mechanics.pdf (364.25KB)

We especially refer to (GiY) lemma 3.2 concerning an estimate of P(ugrad)v.

The Navier-Stokes equations describe the motion of fluids. The Navier–Stokes existence and smoothness problem for the three-dimensional NSE, given some initial conditions, is to prove that smooth solutions always exist, or that if they do exist, they have bounded energy per unit mass.

The Serrin gap occurs in case of space dimension n=3 as a consequence of the Sobolev embedding theorem with respect to the energy Hilbert space H(1) with the Dirichlet integral as its inner product.

The d'Alembert "paradox" is not about a real paradox but it is about the failure of the Euler equation (the model of an ideal incompressible fluid) as a model for fluid-solid interaction. The difficulty with ideal fluids and the source of the d'Alembert paradox is that in incompressible fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The d Alembert paradox.pdf (182.01KB)

The following details are basically taken from the paper / book of M. Cannone (CaM) / M. Shinbrot (ShM), whereby all *not itallic* marked text is cited from those references.

Given a smooth datum at time zero, will the solution of the NSE continue to be smooth and unique for all time?

There is no uniqueness proof except for over small time intervals:
the existence of weak solutions can be provided, essentially by the energy inequality. If solutions would be classical ones, it is possible to prove their uniqueness. On the other side for existing weak solutions it is not clear that the derivatives appearing in the inequalities have any meaning.

It has been questioned whether the NSE really describes general flows.

The difficulty with ideal fluids, and the source of the d'Alembert paradox, is that in such fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbering on each other, provided they are separated by a streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The uniqueness question is among the most important unsolved problems in fluid mechanics: “instant fame awaits the person who answers it. (Especially if the answer is negative!)” uniqueness of the solutions of the equations of motion is the cornerstone of classical determinism.

Moreover, as for the solutions of the Euler equations of ideal fluids, or the Boltzmann equation of rarefied gases, or the Enskog equation of dense gases either.

The question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the NSE blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost.

In the twentieth century, instead of explicit formulas in particular cases, the problems were studied in all their generality. This leads to the concept of weak solutions. The prize to pay is that only the existence of the solutions can be ensured. In fact the construction of weak solutions as the limit of subsequence of approximations leaves open the possibility that there is more than one distinct limit, even for the same sequence of approximations.

For the stationary NSE the existence of a solution has been proven for all space dimensions. The uniqueness has been proven for n<5 under certain conditions to the data. The underlying functions space is a reflexive, separable Banach space X, which is compact embedded into its dual space X(*).

For the full non-linear NSE case (non-stationary, non-linear) the classical solution definition are provided in (CaM). Basically the existence of solutions is proven only for “large” Banach spaces. The uniqueness is proven only in “small” Banach spaces.

Related literature

M. Shinbrot, Lectures on fluid mechanics, Dover Publications, New York, 2012

The 3D-NSE Millennium solution

A global unique H(1/2) based dynamical energy inner product based weak solution of the 3D-Navier-Stokes equations

Note: The proposed solution concept is concerned with an additional (additive) "dynamical energy" norm concept (complementary to the standard Dirichlet integral based H(1) energy norm with respect to the extended H(1/2) Hilbert space norm) as introduced in www.unified-field-theory.de

https://www.fuchs-braun.com/

Physically speaking, ...

... in the current purely H(1) based mechanical energy based modelling framework the crucial 3D-NSE non-linear term governs a mechanical particle flow accompanied by an additional pressure variable governing the interactions between those mechanical particles and the interaction between mechanical flow and mechanical boundary particles enabling Kolmogorov's statistical L(2)=H(0) based turbulence (disturbance) theory

... in the proposed extended H(1/2) mechanical + dynamical energy based modelling framework the mechanical particle flow may be interpreted as physical disturbance (mathematically modelled by a compact operator) of a corresponding overall dynamical quanta energy field system, where the classical 3D-NSE initial & boundary conditions become only relevant for the „mechanical world" and Kolmogorov's statistical L(2)=H(0) resp. energetical H(1)-based mechanical turbulence theory may still be applied.

The 3D-NSE solution concept

We provide a global unique (weak, generalized Hopf) H(1/2)-solution of the generalized 3D Navier-Stokes initial value problem. The global boundedness of a generalized energy inequality with respect to the energy Hilbert space H(1/2) is a consequence of the Sobolevskii estimate of the non-linear term (1959). The extended (energy) Hilbert space is in line with the proposed Krein space based quanta potential energy Hilbert space concept in unified-field-theory.de. It enables an alternative mathematical model for Mie's concept of an electric pressure enhancing the Maxwell equations. The second unknown function in the NSE is the pressure p; the pressure function p can be represented as Riesz operator transforms of (u x u), while the gradient (force) operator applied to the unknown pressure function p becomes the Calderón-Zygmund integrodifferential operator applied to the (velocity) NSE-solution function u (EsG) p. 44.

Braun K., Global existence and uniqueness of 3D Navier-Stokes equations_1.pdf (483.9KB)

Some earlier related papers are

J. A. Nitsche footprints related to NSE problems.pdf (794.84KB)

Braun K., The Prandtl (hypersingular) integral equation with double layer potential and the exterior Neumann problem.pdf (315.97KB)

Braun K., Unusual Hilbert and Hoelder space frames for the (elementary particles) transport (Vlasov) equation.pdf (1.11MB)

Braun K., Generalized wavelet theory and non-linear, non-periodic boundary value problems.pdf (439.35KB)

Braun K., Some remarkable pseudo-differential operators of order -1, 0, 1.pdf (664.7KB)

Braun K., A new ground state energy model.pdf (1.07MB)

June 1986, Interior error estimates of the Ritz method for Pseudo-Differential equations (3).pdf (2.44MB) which is about a well posed variation model of the NSE in a H(-1/2) framework. This goes along with a related energy measure of the "rotating" fluids which corresponds to the H(1/2) Hilbert space norm and a related classical (averaging/aggregating) energy measure which corresponds to the L(2)(H(1/2)) norm.

In the following we give some challenges and hurtles of current situation in the "standard" H(0)=L(2) framework with related energy Hilbert space H(1):

In case of n=3 whether a *strong solution* of the non-linear, non-stationary Navier-Stokes equations exists on whole given *time-interval* without any smallness condition is a fundamental open mathematical problem. Answers exists, e.g. for initial value function with a non-smallness regularity assumption, which is basically about the 1/2 scale of Sobolev spaces as domain of the initial value functions (Fujija-Kato/Sohr).

The pressure of the NSE requires neither initial value nor boundary conditions, but the NSE problem statement is about well posedness of the NSE system. The special role of the Hilbert space H(1/2) is related to the Laplace boundary layer problem, accompanied by J. Plemelj's extended Green formulas with respect to reduced regularity assumptions to the underlying domains.

Solutions of the divergence problem with homogenous Dirichlet data are based on Calderon-Zygmund theory or rely on the Stokes equation with inhomogenous data. Corresponding solutions with Sobolev space domains are built on the Bogovskii solution operator, which can be extended continously to an operator acting from W(s,p) to W(s+1,p) for s>-2+1/p (Geissert M., Heck H., Hieber M.)

The questions concerning the existence of weak solutions of the non-linear, non-stationary Navier-Stokes equations have been basically answered. Corresponding "extrapolation" to related strong solutions by density arguments are different per problem category (linear/non-linear, stationary/non-stationary, space dimension), basically due to the structure of the Stokes operator (and its related global/local "incompatibility" between pressure and velocity), the Serrin gap (related to the non-linear term) according to Sobolev embedding theorems and the logarithm convexity of the Sobolev spaces.

R*egularity and uniqueness of weak solutions for n=3* are still pending (Giga/Sohr), basically due to the Serrin gap challenge, which is a consequence of the Sobolev embedding theorem. An existence and uniqueness result of solutions in L(r) of the Navier-Stokes initial value problem is given in Giga/Miyakawa (as well, that the solutions are smooth up to the boundary if the external force is smooth). The essential step in Giga/Miyakawa is an *estimate of the nonlinear term (u,grad)u*, based on the resolvent of fractional powers of the Stokes operator, applying the calculus of pseudodifferential operators (Giga Y., "*Weak and strong solutions of the Navier-Stokes initial value problem*", Lemma 3.2). We emphasis that the fractional powers of the Stokes operator generates corresponding Hilbert scale, which enables e.g. (weak) a variational representation of the NSE initial value problem with respect to negative Hilbert scale inner products (e.g. the inner product of the Hilbert space H(-1/2)). The same is supported by lemma 2.5.2 (Sohr H., p. 152), which is about the boundedness (with corresponding appropriate definition of domains) of negative fractional Stokes operators applied to the Helmholtz projection to define extended, bounded Helmholtz projection operators with respect to related (negative) Hilbert space inner products (Sohr H., p. 264 ff.)

Basically all existence proofs of weak solutions of the Navier-Stokes equations are given as limit (in the corresponding weak topology) of existing approximation solutions built on finite dimensional approximation spaces. The approximations are basically built by the Galerkin-Ritz method, whereby the approximation spaces are e.g. built on eigenfunctions of the Stokes operator or generalized Fourier series approximations. Thereby the quality of the approximations spaces itself seems to be not relevant, e.g. special "quasi-optimal" approximations properties of finite element approximation spaces weren’t applied for those solutions.

The "power 3" challenge with its corresponding (time variable related) blow-up effect is a given due to the Sobolev embedding theorem and corresponding Serrin scale conditions. This is basically a consequence of an underlying ordinary differential (Riccati) problem. A reduced Hilbert scale factor of the (energy) Hilbert space is proposed to reduce the problematic "power of 3" term down to "power of 2" (still a challenging Riccati ODE) term, as one element of a solution concept. The consequences to the physical model parameters would be to define a pressure force based on a non-harmonic pressure function at the boundary layers.

From a Galerkin approximation method perspective addressing the non-regular behavior of the N-S-E solution(s) at zero and blow-up time go in line with local convergence analysis for non-linear parabolic equations with reduced regularity of the initial value function and/or not fulfilled compatibility conditions. The transformation of the (one dimensional) free boundary Stefan problem to a (non-linear) fixed boundary value equation provides the simplest model problem for such situations, which still has open questions, as well.

From a functional analysis perspective the above goes in line with (hyper) singular integral operators, which correspond to single / double layer potential operator, the tangential derivative ofthe single layer operator and the normal derivative of the double layer operator.

The idea: an appropriately defined distributional Hilbert space framework

In an appropriately defined distributional Hilbert space framework the incompressible (!) N-S-E are well posed. The shift on the Hilbert scale to the left “closes” the Serrin gap in case of n=3 and"time-weighted" norms "control" the singular velocity and pressure behavior for t --> 0. At the same time the Hilbert scale shift jeopardies the application of the Sobolev embedding theorem to make conclusions about classical solutions. By standard functional analysis this should enable the building of a counterexample (following the idea of J. Heywood) that existing weak solutions in "standard" scaled Sobolev spaces lead to "strong solution".

Just from a common sense feeling the same argument should not be valid in case for compressible "fluids". This would be in line with J. Plemelj's concept of a mass element, replacing an only mass density, which requires less regularity assumptions to enable the Hilbert scale shift to the left, while keeping the option to apply the Sobolev embedding theorem. This would also be in line with

[ArA] Arthurs A. M., "Complementary variational principles", where one example (§ 4.8) is about nonlinear variation method applied to compressible fluid flow.

Remark: It's suggested to apply J. Plemelj's theory also to the still not solved "radiation problem" (R. Courant, D. Hilbert, "*Methods of mathematical physics*, II", 1937, VI, §5, section 6, VI §10, section 3).

J. Plemelj proposed an alternative definition of a normal derivative, based on Stieltjes integral. It requires less regularity assumptions than standard definition; the "achieved" "regularity reduction"is in the same size as a reduction from a C(1) to a C(0) regularity. Plemelj's concept is proposed to be applied to ensure physical model requirements modeled by normal derivatives within a distributional Hilbert space framework.

The concept of Pseudo-Differential operators is about the study of the differential and integral operators within the same algebra of operators. The concept of Hilbert scale is about an appropriate Hilbert space with respect to the eigenpairs of self-adjoint, positive definite operators with corresponding order (respectively coercive operators in combination with the Garding inequality). The study of uniqueness and existence of weak solutions of the Navier-Stokes equation is about appropriate weak variation representation of the NSE and related approximation solutions, which converge to the NSE solution(s).

For the quasi-optimal approximation estimates of Ritz-Galerkin method in Hilbert scales (which, of course, also includes collocation methods) we refer to

[BrK1] Braun K., "*Interior Error Estimates of the Ritz methods for Pseudo-Differential Equations*".

Defining the right Hilbert space framework goes along with defining the proper Pseudo-Differential operators with their related domains (!). In this context we refer to the model (singular integral) Pseudo-Differential operators in

[BrK2] Braun K., "*An alternative quantizationof H=xp*"
[BrK3] Braun K., "*A new ground state energymodel"*
[LiI] Lifanov, I. K., Nenashev A. S., "*Generalized functions on Hilbert spaces, singular integral equations, and problems of aerodynamics and electrodynamics*".

For the linkage to the normal derivative concept of Plemelj (especially to the double layer potential, as well as its normal derivative, a hyper singular operator of Calderon-Zygmund type, which is essentially a once differentiation operator) we refer to

[AmS] Amini S., "*On Boundary Integral Operators for the Laplace and the Helmholtz Equations and Their Discretization*":

- the operator defined by the normal derivative of the single layer potential is the dual operator of the double layer potential
- the operator defined by the tangential derivative of the single layer potential is the Hilbert transform (S(0)). It is discontinuous on the surfaces with a jump according to the Plemelj formula
- the operator of the normal derivative of the double layer potential is a hyper-singular operator, which behaves as the derivative of the Hilbert transform operator (S(1)). It is continuous (!) on the surface.

The terms "vortex density" and "rotation of a fluid element" are sometimes used synonymy. The rotation of a vector field v describes the vortex field of v. The corresponding vortex force at a point x is defined as the product of the constant normal vector of all (infinitesimal small) areas F containing x with the rot(v). The vortex density in combination with the concept of"circulation" (L. Prandtl) is interpreted as the root cause of the "rotation" of a fluid element. The definition of"circulation" is integral part of the definition of "vortex force". We propose the use Plemelj's concept/definition of a mass element (alternatively to a mass density) to define a "vortex" of a fluid element, which would require less regularity assumptions to the vector field v than H(1), but being still consistent with the definitions of fluid density, vortex force and rotation in case of corresponding H(1)-regularity assumptions.

We note that
- moving from the "convection form" to the"rotation form" goes along with a movement from "kinematic pressure" to "Bernoulli pressure"
- the Euler equation is nonlocal, i.e. one cannot compute the time derivative of the solution u at (x,t) only from the knowledge of the function u in the neighborhood of x at the time t. This gets proven by taking the divergence of the NSE, which gives the Laplacian of the pressure p. Therefore the pressure p is determined by local information, but not the gradient of p (P. Constantin, "*On the Euler equations of incompressible fluids*").

Comments to the following three related papers of J. A. Nitsche

[NiJ1] Nitsche J. A., "L(infinitety) boundedness of the FE Galerkin operator for parabolic problems"
[NiJ2] Nitsche J. A., contains an extended list of references in the context of "Finite element approximations and non-linear parabolic differential equation"
[NiJ3] Nitsche J. A., "Direct proofs of some unusual shift theorems".

ad [NiJ1] : The paper gives an ("optimal", i.e. shift on Hilbert scale by the factor"2") shift theorem for the heat equation based on Fourier transforms estimates. The "trick" (which is about changing the order of integration with respect to the time-variable) to prove an optimal Sobolev scale shift by scale factor "-2" (analogue to the elliptic Laplace equation) is proposed to be applied for appropriately defined time-depending norm estimates for the Oseen kernel in the context of harmonic analysis for solving the incompressible Navier-Stokes equations.

The link of [NiJ1], [NiJ2] to the N-S-E is given by the Leray-Hopf operator and its Oseen kernels. The Fourier transformations of the Oseen kernel (see Lerner N., "A note on the Oseen kernels") are required to apply the "heat equation shift theorem trick", in [NiJ1].

ad [NiJ3] : For the Stokes equation an unusual shift theorem (negative norm estimates) has been proven by solving auxiliary problems, building on the Cauchy-Riemann differential equations. The effect is that the Stokes problem is decoupled into two elliptic problems. The definition of the auxiliary functions (w,z) for given C-R- relationships for (u,v) can be interpreted as defining curl(u) and div(v) as representations of the C-R equations (n=2). For n=3 the curl operator (which is for n=2 one of the two C-R equations) is linked to the Leray-Hopf projector P (and therelated Riesz operators) by the following properties (M. Lerner):

- the commutator (P,curl) vanishes
- P(curl) = curl
- (curl(u),curl(v)) =(gradP(u),gradP(v)) = D(P(u),P(v)).

We note that curl(u) describes the mean rotation of a fluid and 1/2*rot(u) gives the mean value of the angular velocity.

Replacing the gradient operator by the Calderon-Zygmund operator enables the definition of a "curl"-operator with distributional Hilbert space domain. This follows the same idea concerning alternative "energy inner product in distributional Hilbert spaces" ([BrK1]) to overcome current issues with different domain regularity/scope of momentum and location operator.

The conjecture is that with the above shift theorems for appropriately defined norms can be proven for both, velocity and pressure, anticipating the divergent behavior for t --> 0.

We emphasis that the L(2)-norm of the (Oseen) tensor kernel of the solutions of the incompressible N-S-E can be estimated by t*exp(-a) with a=1/4 ([CaM]), which is the same (space dimension n independent!) divergence behavior as for the one-dimensional Stefan problem with non-regular initial value function [NiJ4] .

As a consequence for the space dimension n=3 the „natural" energy norm H(1) is being replaced by an appropriate alternative with reduced regularity requirements. The concept of J. Plemelj still keeps the physical model relevance and, at the same time, strengths the capabilities of a potential. For the non-stationary case this norm is modified by timely-weight (integral-dt) norms. For such a Hilbert space there is a quasi-optimal shift theorem for the NSE. The initial value function for the pressure is basically of Dirac function type. We note, that the embedding properties of the Dirac function into the continuous function space are depending from the space dimension. The same is valid for the Serrin gap. The correspondingly required reduced regularity assumptions to the normal derivative are "delivered" by the concept of J. Plemelj ([PlJ]), respectively the corresponding Pseudo-Differential operator.

The „pressure“ artefact to model "pressure force" per unit volume (Navier)

The "pressure" (especially the continuous pressure in the context of airfoil uplift theory in an ideal fluid, which is vortex-free per definition) is a mathematical arte-fact, while the "gradient of pressure" is a force per unit volume. We propose to represent the "pressure force" by the components of a corresponding Plemelj-Stieltjes potential, which requires less regularity assumptions than current standard model.

We claim, that the root cause of still unsolved open questions of the NSE is not related to the specifics of non-linear and/or the non-stationary obstacles, it is due to the fact, that by design it can not be assigned to one of the elliptic, parabolic or hyperbolic PDE clusters. This is due to the fact, that the model puts velocity and pressure into a certain relationship, which leads to mathematical incompatibilities. Those become visible, when it's possible to decouple the NSE equations into separate PDEs for velocity and pressure.The proposed model for a Mie theory accompanied by the concept of an electric pressure overcomes this issue.

From G. Galdi we quote:

"*the field p(x,t) can be formally obtained - by operating with "div" on both sides of the NSE - as a solution of a Neumann problem. From this it is clear that to describe the values of the pressure at the bounding walls or at the Initial time independently of v, could incompatible with the NSE and, therefore, could render the problem ill-posed*".

The Gauss integral theorem states, that the integral of the normal derivative of a harmonic function vanishes. This is intuitively obvious if one interprets u as a velocity potential. Then the Gauss integral theorem means that for incompressible, irrotational flow with no source, the net mass flow through a prescribed boundary is zero.

Proposition
The solution of boundary integral operator equations (especially the hyper singular operator in the context of the Laplace equation (S. Amini), which is the normal derivative of the double layer potential) is proposed to be applied alternatively to define a non-harmonic pressure function p. With respect to the Gauss theorem this means, that the pre-requisite of an irrotational flow is omitted.

In other words, it is basically about an alternative representation/ definition of the pressure function, not as a solution of the Neumann problem with corresponding (too high) regularity requirements, but as solution of a boundary integral operator equation, which "allows" the "rotation" of fluids.

With respect to the definition of the (singular) integral operator we refer to Kevorkian J., Partial Differential Equations, (2.12.2), p. 98)

With respect to the Fourier transform of the uniform distribution of unit mass over the unit sphere with center at the origin for n=2, which is the Bessel function of first kind J(0)(x) we refer to

B. E. Petersen, "*Introduction to the Fourier Transform and Pseudo-Differential Operators*", example 2.5.

Kevorkian J., Partial Differential Equations, Analytical Solution Techniques

The Airfoil uplift force modeled as "pressure force" per unit volume (Navier)

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Kevorkian J., Partial Differential Equations, Analytical Solution Techniques

Space-scale turbulence:
coherent (Kolmogorov) & incoherent (Heisenberg)
turbulent flow decomposition

"*Fourier transform would be the appropriate tool to analyze the intrinsic structure of a turbulent flow if and only if the turbulent flow field is a superposition of waves. Only in this case are wave numbers well defined and the Fourier energy spectrum is meaningful for describing and modeling turbulence. If, on the contrary, turbulence were a superposition of point vortices then the Fourier spectrum in this case would be meaningless. The problem we still face in turbulence theory is that we have not yet identified the typical "object" that composes a turbulent field.*
M. Farge

In (BrK2) we provide an alternative (*quantitative* model to Kolmogorov's purely *qualitative* statistical turbulence model. It takes into account the quantitative fluid behavior as its described by the Euler or the Navier-Stokes equations. In order to highlight and focus on the new conceptual elements (and to avoid technical difficulties) we restrict ourself to the one-dimensional Constantin-Lax-Majda (CLM) vorticity equation with a viscosity term.

In the following we provide further information related to (BrK2).

The first building element is the CLM-vorticity equation with viscosity term considered in a H(-1/2) physical (Hilbert) space framework (in sync with the H(1/2) solution concept of our 3D-NSE solution). The Hilbert scale framework is enabled by the well known Generalized Fourier Transform (GFT) concept. The second building element is about the Continuous Wavelet Transform (CWT) (as proposed by (FaM)) with its natural Hilbert space framework H(-1/2) (due to its admissibility condition). As a consequence of those two building elements the physical and the wavelet framework is identical. The third building element combines the CWT with the Hilbert transform concept to enable "space-scale decomposition" of the H(-1/2) Hilbert space leading to "localized" Heisenberg uncertainty inequality restricted to the (complementary to the H(0)-test space) closed subspace of H(-1/2). This means that the "Heisenberg uncertainty" which is about the interaction of "fluid elements" and the "fluid momentum", is only valid in the closed H(-1/2)-H(0) subspace of H(-1/2).

Working titles of the following early thoughts

" A turbulent flow field as superposition of fractional Hilbert wavelets modelling coherent structures (vortices, shocklets) and incoherent noise" (FaM1)

"Space-scale decomposition of turbulent fields into localized wavelet oscillations of finite energy" (FaM1)

"A Dawson (wavelet) function based turbulence model"

"A simple one-dimensional turbulent flow (weak H(-1/2) variation equation) model based on a revisited CLM model for the vorticity equation with viscosity term" (MaA) 5.2

"A single answer to the two "Heisenberg (relativity-turbulence) questions?"

Slogans

"when H(0) conservation of law (w/o contributions from the non-linear term) meets H(-1/2) conservation law (with contribution of the non-linear term reflecting how divergence affects the velocity)"
"when today's H(0) low- and high-pass filtering (turbulence) concept meets H(-1/2) space-scale decomposition wavelet (turbulence) concept, (FaM)"
"when H(-1/2) (physical & wavelet) space meets non-stationary random functions with finite variance and related H(1/2) spectrum (FrU) 4.5"

((FaM) 5.1): "*The definition of the appropriate "object" that composes a turbulent field is still missing. It would enable the study how turbulent dynamics transports these space-scale "atoms", distorts them, and exchanges their energy during the flow evolution. If the appropriate "object" has been defined that composes a turbulent field it would enable the study how turbulent dynamics transports these space-scale "atoms", distorts them, and exchanges their energy during the flow evolution*"

The story line

1. The trilinear form of the non-linear NSE is antisymmetric. Therefore the energy inequality of the NSE with respect to the physical H(0) space does not take into account any contribution from the non-linear term. At the same time the regularity of the non-linear term cannot be smoother than the linear term. In (BrK2) an alternative physical space H(-1/2) is proposed to overcome those drawbacks providing the adequate variation equation framework to guarantee a unique 3D-NSE solution in H(1/2).
2. (FaM1): the turbulent regime develops when the non-linear term of the NSE strongly dominates the linear term. Superposition principle holds no more for non-linear phenomena. Therefore turbulent flows cannot be decomposed as a sum of independent subsystems that can be separately studied. A wavelet representation allows analyzing the dynamics in both space and scale, retaining those degrees of freedom which are essential to compute the flow evolution.
3. Methods based on wavelet (Galerkin) expansions in L(2) framework face the issue that in Galerkin methods the degrees of freedom are the expansion coefficients of a set of basis functions and these expansion coefficients are not in physical space (means in wavelet space). First map wavelet space to physical space, compute non-linear term in physical space and then back to wavelet space, is not very practical (MeM).
4. The admissibility condition is basically the norm of the H(-1/2) Hilbert space. Therefore ct(x) is a candidate for a wavelet as element of H(-1/2)-L(2). The Hilbert transform is an isomorphism on any Hilbert scale H(b), b real. Therefore the Hilbert transformed ct(x) distributional H(-1/2)-"function" is a wavelet, as well ((WeJ).
5. (FaM1): the turbulent H(-1/2)-signal can be split into two contributions: coherent bursts, corresponding to that part of the signal which can be compressed in a L(2)-wavelet basis, plus incoherent noise, corresponding to that part of the signal which cannot be compressed in a L(2)-wavelet basis, but in the H(-1/2)-wavelet basis. For the n=1 periodic case the later one corresponds to the alternative zero-state energy model of the harmonic quantum oscillator.

A revisited (H(-1/2-) CLM vorticity turbulence model with viscous term would enable non-stationary random functions with finite variance and related spectrum ((FrU) (4.54)) with respect to the corresponding H(1/2) energy norm/spectrum space.

If the solution of the Euler equation is smooth then the solution to the slightly viscous NSE with same initial data is also smooth. Adding diffusion to the CLM model makes the solution less regular (MuA). As a consequence of this the CLM model lost most of the interest in the context of NSE analysis. In (MuA) a nonlocal diffusion term is proposed removing this drawback.

The (MuA)-(DeS) proposal results in unbalanced energy norm terms. This drawback can be removed applying the Symm operator A also to the non-linear term (resulting in with same H(1/2) energy norm). This is nothing else then changing from the weak variation original equation (MuA) (10) in a H(0) framework to the weaker H(-1/2) representation (BrK). The spectral coefficient analysis and the proof of the required properties of the nonlinear operator F(w):=w*H[w] follow analog to (SaT) with reduced Hilbert scale number by -1/2.

The today's generalized revisited CLM equations (e.g. (WuM)) affect only the "energy" relevant change to the linear (dissipative) term while the non-linear terms are not adapted correspondingly to conserve the energy balance. In case the nonlinear term governs the linear term (which is the case for the turbulent flow phenomenon) this "modification" to the physical model is at least worth to be challenged.

The building concept for a turbulent flows model is about an alternatively proposed physical H(-1/2)Hilbert space which enables
- a Dawson function based hydrodynamics (including gases dynamics, (HoE)) statistics and

- a space-scale decomposition of turbulent fields into wavelet oscillations of finite energy (FaM1).

The Dawson function replaces the Gaussian function. The L(2) subspace of H(-1/2) governs the coherent flow structures while the closed subspace H(-1/2)-L(2) governs the incoherent ("noise") structures (eigendifferentials, wave packages).

The alternatively proposed Hilbert space might also enable a common model for statistics of gases and highly turbulent fluid flows (HoE). The concept of (ChF) might provide the appropriate (Nitsche-based domain decomposition) method for the solution of the underlying hypersingular integral equations.

The building of a physical (weak) H(-1/2) NSE representation can go along with an integral representation of the NSE. By this approach the successfully (RH solution & zero point energy modelling approach) applied (hypergeometric) Kummer functions enter the stage from the other (fluid ensemble statistics and Hilbert transformed probability (Gaussian) function) side. For the related Fourier analysis we refer to (PeR). For relaxed "vanishing at infinity" conditions in case of unbounded turbulent flow applying generalized harmonic analysis we refer to ((MoA), (LuJ).

“A single answer to the two “Heisenberg (relativity-turbulence) questions?”

The "transform" tool set is about the Generalized Fourier Transform (GFT), the Windowed FT (WFT), the Continuous Wavelet Transform (CWT) and the Hilbert Transform (HT). GFT, WFT and CWT do not allow localization in the phase space. All of them can be interpreted as phase-space representations. The WFT and the CWT (resp. its inverse transforms) were known in mathematics since quite a while as a "derivative" of Calderon's reproducing formula. From a group theory perspective WFT and CWT are the same, as both are built in the same way:

- WFT is built on Weyl-Heisenberg group
- CWT is built on affine-linear groups (---> affine connexions).

Both are facing the same consequences from the Heisenberg uncertainty inequality. The Heisenberg principle states that the product of the variances of localization and momentum (of a quantum or a fluid) is always greater than a positive constant. Therefore localization & momentum cannot be measured with arbitrarily exactness at the same point in time. The function which has minimal uncertainty in the neighborhood of the expectation values of localization and momentum is the Gaussian function. This impacts the "quality" of the WFT.

The advantage of the CWT against the WFT is, that the product of the variances does not "a priori" depends from the wavelet function itself, but "only" from the "zoom" parameter "a" governing the so-called "zoom effect" property of the CWT.

The answer to Heisenberg's question above is related to the "quantum gravity" problem. This is about the interaction of quanta (in our case the interaction of fluids) and the corresponding momentum of the continuum (in our case the fluid).

There is no way out from the mathematical constraint of the Heisenberg uncertainty inequality. However our solution concept take advantage of properties of the Hilbert transform (e.g. the constant Fourier coefficient of a Hilbert transform vanishes; the Hilbert tranform of a wavelet is a wavelet) which go along with the admissibility condition of a wavelet. The Hilbert transform of the ("WFT-Heisenberg-optimal") Gaussian function (which is not a wavelet) conserve the properties of the Gaussian (in a L(2)-sense), while it becomes a wavelet. The "natural" Hilbert space of (Hilbert transformed) wavelets is H(-1/2) (due to the admissibility condition). By this there is a Hilbert space framework given with identical physical and wavelet space enabling Galerkin-wavelet representation and supporting the turbulence model solution concept of (FaM).

We emphasis that the regularity of the Dirac function requires a fractional Hilbert space with scale factor -n/2+e (e := "epsilon" >0), i.e. the regularity of the Dirac function depends from the space-dimension and the regularity of the Dirac function in case of n=1 is "nearly" a H(-1/2) "function", while for n>1 an H(-1/2) "alternative Dirac" function is more regular than its origin. The "H(-1/2) physical space" concept can also be applied to the QED, e.g. for an alternative model of a quantizied Dirac particle in a given Maxwell field overcoming today's underlying vaccum state modelling challenge (absorption and emission operators).

The extensions to 3D NSE and Reynolds number

In the following we recall corresponding central supporting statements and concepts which allow to extent the concept above to space dimension n > 1:

- Hilbert transform --> Riesz transforms
- Hermite polynomials --> Hilbert transformed Hermite polynomials
- wavelet transform --> time-frequency analysis of functions with symmetry properties

(FaM1): "*The notion of "local spectrum" is antinomic and paradoxical when we consider the spectrum as decomposition in terms of wave numbers for as they cannot be defined locally. Therefore a "local Fourier spectrum" is nonsensical because, either it is non-Fourier, or it is nonlocal. There is no paradox if instead we think in terms of scales rather than wave numbers. Using wavelet transform then there can be a space-scale energy be defined ((FaM) (51)-(58)) with a correspondingly defined scale decomposition in the vicinity of location x and a correspondingly defined local wavelet energy spectrum. By integration this defines a local energy density and a global wavelet energy spectrum. The global wavelet spectrum can be expressed in terms of Fourier energy spectrum. It shows that the global wavelet energy spectrum corresponds to the Fourier spectrum smoothed by the wavelet spectrum at each scale.*

The concept of (FaM) enables the definition of a space-scale Reynolds number, where the average velocity is being replaced by the characteristics root mean square (rms) velocity Re(l,x)) at scale l and location x. At large scale (i.e. l ~ L) Re(L) coincides with the usual large-scale Reynolds number, where Re(L) is defined as the integral of Re(l,x) over all x of the R(n) space (FaM) (60)-(61)."

Note 1: for a given geometrical shape of the boundaries the Reynolds number is the only control parameter of the flow. If friction forces are small compared to inertia forces this corresponds to large Reynolds numbers (ScH). In (FaM) an alternative scale "l" and location "x" dependent definition of a "Reynolds" control data is proposed. It is an enabler for a turbulence model building on fluid "H(-1/2)-objects" and "H(1/2)-fluid flow" properties overcoming variance & power-law spectrum divergence challenges of current (basically K41, nonsensical "local Fourier spectrum") turbulence theory. In (FaM) it is proposed "*to study the energy spectrum of turbulent flows (which are statistically stationary (in time) and homogeneous (in space) using a wavelet (energy spectrum) representation, alternatively to the energy spectrum given by the modulus of the Fourier transform of the velocity auto-correlation (FaM1). The wavelet representation allows analyzing the dynamics in both space and scale. The wavelet analysis and synthesis can be performed locally, in contrast to the Fourier transform where the local nature of the trigonometric functions does not allow performing a local analysis. Since wavelet transforms conserve the energy and preserve locality in physical space, one can extend the concept of energy spectrum and define a local energy spectrum (FaM1) (4). Although the wavelet transform analyzes the flow using localized functions rather than complex exponentials, one can show that the global wavelet spectrum converges towards the Fourier energy spectrum provided the analyzing wavelet has enough vanishing moments. The global wavelet spectrum, defined by integrating local energy spectrum over all positions gives the correct exponent for a power-law Fourier energy spectrum of order (-b) if the analyzing wavelet has at least M > (b-1)/2 vanishing moments.*"

Note 2: The Gaussian (normal distribution) function is no wavelet. The Mexican hat wavelet is built out of the Gaussian function as its 2nd derivative, i.e. the Mexican hat is a wavelet of order 2. Also the first derivative of the Gaussian function is a wavelet. The Gaussian function itself does not fulfill the admissibility and the zero-average conditions (vanishing constant Fourier term) The later (also missing) property is the baseline for the alternatively proposed Zeta function theory to prove the Riemann Hypothesis (BrK), using the fact that the Hilbert transform has always a vanishing constant Fourier term. Both functions, the Gaussian function and its Hilbert transform (the Dawson function) are L(2)-norm equivalent due to the related property of the Hilbert transformation operator. The commutator of the

Hilbert transform operator applied to odd function vanishes, as well, i.e. especially for the Dawson function it holds, (xH-Hx)(F)(x)=0.

From (WeJ) we quote:

"*... the Hilbert transform is the unique, bounded linear operator in L(2) that commutes with translation, dilation and reflection. Therefore the HT is the unique operator that preserves the L(2) solution space of the one-dimensional, linear translation-dilation equations. The higher dimensional scaling relations involve translations, dilations and rotations of the argument vector. The appropriate bounded linear operators on L(2)(R(n)) that maps solutions into solutions would be the Riesz operators. We suggest that the Hilbert wavelets may have several useful numerical applications. These include exterior boundary value problems and the inversion of the Radon transform. The inverse Radon transform requires evaluation of derivatives and Hilbert transforms. A Galerkin approximation could be a natural application of the Hilbert wavelets.*"

The Galerkin method based on wavelet expansion requires (ongoing) mappings between wavelet and physical space (during computing process) which is not practical if both spaces are different which is the case for most of current (weak) variation PDE representations in a L(2)-Hilbert space framework (MeM). The proposed H(-1/2) (weak) physical space concept enables identical wavelet and physical Hilbert spaces, while at the same time enabling the full power of Galerkin method computing non-linear terms in this (newly) physical space. This points back to the "solution" section of this page with the weak H(1/2)-NSE solution of the corresponding weak NSE representation in the H(-1/2) Hilbert space and the (H(0)=L(2)-based) physical principles of quantum theory (HeW).

Note 3 (GrK): "*Time-frequency analysis is a modern branch of harmonic analysis that uses the structure of translations and modulations (or time-frequency shifts) for the analysis of functions and operators. It is a form of local Fourier analysis that treats time and frequncy simultaneously and symmetrically. ... It originates in the early development of quantum mechanics by H. Weyl, E. Wigner, and J. von Neumann around 1930 ....*
The ideal time-frequency representation of f would provide the occurring frequency spectrum at each instant x. The main obstruction to this ideal is the uncertainty principle:

the uncertainty principle makes the concept of an instantaneous frequency impossible

... To find the frequency spectrum of a signal f at a time x, one localizes f to a neighborhood of x and takes its Fourier transform. This leads to the short-time Fourier transform (STFT). THe localization procedure is parametrized by a window function g. ... Pseudodifferential operators can be represented as superpositions of translations and modulation operators. ....
Gaussians, its time-frequency shift properties, its Fourier transform and Plancherel's theorem play a very special role in time-frequency analysis. ... conceptually one can think of f and its Fourier transform as two different, equivalent representations of the same object f. Each of these representations contains the same information, but each one makes visible and accessible rather different features of f. ... In time-frequency analysis we search for representations that combine the features of both f and its Fourier transform into a single function, a so-called time-frequency representation. ... It is clear that neither f nor its FT alone can accomplish the tasks of of a time-frequency representation. ...
The musical score is a useful analogy for illustrating several .. concepts in time-frequncy analysis.

... in raw, qualitative form, the uncertainty principle states that

***a function f and its Fourier transform cannot be supported on arbitrarily small sets.** ....*

... the uncertainly principle is often loosley formulated as follows:

***A realizable signal occupies a region of area at least one in the time-frequency plane**.*

... The analogy between "time-frequency" and "location-momentum" leads to many similarities between signal analysis and quantum mechanics. ...

... in order to obtain information about local properties of f, in particular about some "local frequency spectrum", we restrict f to an interval and take the Fourier transform of this restriction. Since a sharp cut-off introduces artificial discontinuities and can create unwanted problems, we choose a smooth cut-off function as a "window".

Note 4: The Riesz operators are the natural generalization of the Hilbert transform. They are singular integral operators which are not continuous at the origin, because of the singularity at the origin. They are related to the Leray-Hopf (Helmholtz-Weyl) operator. They are idempotent, commute with translations and homothesis and they are "rotation" invariant (StE). Therefore for the Riesz transformed Gaussian the same properties are valid as for the Gaussian (time-frequency shift, density in L(2), Plancherel theorem, uncertainty principle) while ensuring rotation invariance and vanishing constant Fourier terms. This enables the application of harmonic analysis techniques of radial functions (commutative hypergroups) with its relationship to the continuous wavelet transform (RaH). The related Hankel transform (Fourier-Bessel transform which is essentially the Fourier transform of a radial function) provides the tool set to define corresponding Hilbert scales. The radial wavelet transform coincides (up to normalization) with the wavelet transform of the Bessel-Kingman hypergroup of index (n-2)/2 with corresponding Haar measure ((RaH) 1.4). For commutative hypergroups it is possible to develop harmonic analysis analogous to the one on locally compact Abelian groups. One prominent example is the group of homogenous isotropic random field generated by the parallel shifts of rotation and reflection. A stationary phase distribution is called in ergodic theory an invariant measure. In (HoE) the average of a phase function with respect to a given phase distribution function is defined by a corresponding Stieltjes integral.

In (ShA) the group of isometric transformations on R(n) (generated by the parallel shifts, the rotations and the reflections) with the homogeneous isotropic random field (and its special case of a stationary process) is considered. For homogeneous isotropic fields the covariance function is defined by Bessel functions with index (n-2)/2 combining with a non-negative random measure which provides the link back to the above.

Note 5 (MoA) p.18: In (HoE) the use of the techniques of "characteristic functionals" of a turbulent velocity field in an incompressible field is proposed. These characteristic functionals define uniquely e.g. a probability distribution P(dw) in the phase space of a turbulent flow, and hence finding them would give a complete solution of the problem of turbulence. The conceptual elements are (stationary) phase distribution and an average of a phase function F(u) with respect to a given phase distribution defined by a Lebesgue-Stieltjes integral over F(u)dP(du) where P(du) denotes the "differential" of the probability distribution P. In case the phase distribution is stationary, then the average of any phase function stays constant in time.

The proposed Hopf equation is linear, although the dynamics of fluid is nonlinear, the fundamental problem of statistical fluid mechanics. The Stieltjes integral approach is in line with the Plemelj concept. Hopf's equation is formally similar to the Schwinger equations of quantum field theory, which are equations in functional derivatives for the Greens function of interacting quantum fields. In this context we again refer to the proposed alternative zero point energy model and the related Hilbert space framework (isomorph to H(-1/2)) with inner product in the form ((du,v)).

Note 5: The "Energy Gradient Theory" of H.-S. Dou is based on an alternative laminar flow NSE representation in terms of the gradient of the total mechanical energy. It applies the concept of the magnitude and the direction of the gradient of the total mechanical energy for unit volumetric fluids. It enables time averaged NSE for turbulent flows. It is related to the concept of a "local Reynolds number K", which represents the direction of the gradient of the total mechanical energy for pressure driven flows controlling the stability of a flow under certain disturbance.

The magnitude and the direction of the gradient of the total mechanical energy are singular (boundary layer) integral operators with to-be-appropriately-defined domains, which provide the linkage to the Calderon-Zygmund theory and the approximation theory in Hilbert scales.

This means that Dou's energy gradient theory has its weak representation in appropriate Sobolev space H(a) (energy) Hilbert space framework enabling (complementary: principle of Thomson) variation resp. energy operator minimization (Trefftz, Prager and Synge, Friedrich) methods. The singular integral operator representing of this laminar flow NSE indicates a less energy Hilbert scale factor than 1 (e.g. in sync with the regularity requirements of the Plemelj normal derivative.

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The „line force“ arte-fact to model a "displacement current force" (Maxwell)

The Biot-Savart law describes the magnetic field force (Maxwell), calculated from a electric current density (or the flow/current field (Navier), calculated from a vortex field). The current density (not the origin current force) is "generated" by:

- a "truly" electric current
- a mathematically required "change" of the electric field.

The "route cause" of electric current in the Biot-Savart law is captured by the magnetic constant.

The later one is a (purely mathematically required) arte-fact, which got the (a posteriori) physical interpretation of a Maxwell (displacement) current; this is similar to the a posteriori "defined" (boundary layer created) vortex in an a priori vortex-free "ideal" fluid framework with related (vortex-free) potential streams). The mathematical tool is mainly about the Stokes theorem, which anticipates the "standard" normal derivative and its related regularity conditions.

The Stokes theorem in its integral form is most directly linked to the comments of the tab "J. Plemelj and ..." with respect to a a more general concept of a "mass element" and a "potential". Following those conceptual ideas/proposals, linking it with the proposed new energy inner product/ norm, it leads to a varaitonal representation of the Maxwell equations in a "less regular" (distributional) Hilbert space framework with corresponding less regular energy Hilbert space than H(1).

The „manifold“ arte-fact to model "multiple extended quantities" (Riemann)

The terminology of *"multiple extended quantities*" was introduced by B. Riemann, synonymly to a "continuous manifold". It is conceptually based on two essential attributes: "continuity" and "multiple extension".

The history of manifolds is the attempt to build a mathematical structure to model the whole (the continuum) and the particular (the part) to put its combination them into relationship to describe motion, action etc. (Helmholtz, Riemann, Poincare, Lie and others).

From the paper from E. Scholz below we recall the two conceptual design strategies for a "continuum":

Strategy I:
Design of an "atomistic" theory of the continuum (which to H. Weyl's opinion contradicts to the essence of the continuum by itself)

Strategy II: develop a mathematical framework which symbolically explores the "relationship between the part and the whole" for the case of the continuum.

The later one leads to the concept of affine connexion, based on the concept of a manifold, which were developed during a time period of about 100 years.

The concept of manifolds leads to the concept of co-variant derivatives, affine connexion and Lie algebra to enable analysis and differential geometry, but (according to H. Weyl, Scholz 1) ...a .." *truly infinitesimal geometry ... should know a transfer principle for length measurements between infinitely close points only*."

Our alternative definition of the energy (inner product) Dirichlet integral (see Braun K., "*A new ground state energy model*"), which is rotation invariant with respect to infinitely close points, is proposed to build a truly infinitesimal geometry, which then would lead to a "principle of general contra-variance".

The „string" arte-fact to model "matter components" (Veneziano)

The string "theory" concept is about replacing the 0-dimensional "particle" "entity/ object" (which is not the same as today's quantum object model (!)) by a 1-dimensional "string" "entity/ object". It aims to model different kinds of quantum vibrations, which then "generate" "energy", respectively different kinds of "quantum energy states"...

... unfortunately, the simple fact is, that compared to today's particle-field model dualism/paradox this approach generates no additional value for any experimental setup or interpretation of experimental data, nor any value added to existing conceptual challenges, e.g. concerning the "particle-field paradox" inherent "contact point / body" interaction issue.

Every conceptual challenge with respect to synchronize/ integrate GRT and quantum field theory remains unsolved/ not answered, respectively generates even more challenges, as now e.g. the"contact body" problem needs to be solved for a one-dimensional object, than "only" for a "zero" dimensional "object" (...:)) . Different kinds of (open and closed) "strings" add additonal complexity (finite (!) length, still requiring "end points /particles" for all kinds of those "manifolds", e.g. open intervals, as well as for closed intervals).

The standard model of a quantum x in current quantum mechanics is an element x of a Hilbert space. If one would built a feasible analogue model of a "string-quantum" in current quamtum mechanics framework (a Kind of all-inclusive package for the quantum itself, its location and its vibration/ energy) basically one needs to define something simliar to the already existing wave packages. In current quantum mechanics this leads to the Heissenberg uncertainty relation (respectively "uneigentliche" eigenvectors) , which is just annother view on the particle-field dualism.

In "Braun K., An alternative quantization of H=xp" an alternative distributional Hilbert space framework in combination with Pseudo-Differential Operators are proposed to overcome the above challenge.

Background Information: String "Theory" and "Loop Quantum Gravity"

The superstring theory is based on the idea to replace the "particle" (no extension, no interaction between particles) by a string with extension into one dimension. It enables vibrations and different kind of vibration states.

S. Lie developed the concept of (continuous) contact transformations to provide a mathematical framework (gauge theories) to "solve" the contact body problem for "manifolds" (as model of "multiple extended quantities"). Basically it adds to each manifold (of each multiple extended quantities model per nature force) an appropriate group to enabled "geometric model" properties. Those groups must be "orthogonal" to each others, Therefore the required space dimension increases per added "force model" (and no interaction between those forces are possible):

The Standard (field) Model of Elementary Particles (SMEP) is given by SU(3) x SU(2) x U(1). Its components are the following interaction dynamics fields:

1. Electromagnetic Interaction Dynamics (EID): U(1)
2. Weak Interaction Dynamics (WID): SU(2) x U(1)
3. Strong Interaction Dynamics (SID): SU(3).

A symmetry group to link the superstring field concept with SMEP is still missing:

4. "String Interaction Dynamics" (SUSY: Super Symmetry?)?

à A SUSY field model requires an inflation of space dimension n>10 with no validation against physical standards like the Huygens principle, but also with respect to results concerning the characterization of the space-time dimension n=4. There is no mathematical model existing.

In the context of overcoming the long list of constraints, which jeopardize any solution to build a mathematical model, we quote from

Kaku M., "*Introduction to Superstrings and N-Theory*", Springer Verlag, NewYork, 1999
"*Because general relativity and quantummechanics can be derived from a small set of postulates, one or more of these postulates must be wrong. The key must be to drop one or more of these assumptions about Nature on which we have constructed general relativity and quantum mechanics. Over the years several proposals have been made to drop some of our common sense notions about the universe: continuity, causality,unitarity, locality, point particles.*"

We note that the "Loop Quantum Gravity (LQG)", as alternative to the super string theory, is also built on a Hilbertspace K(diff), modeling 3D diffeomorphism invariance and transformation properties of spin network states under diffeomorphism ((RoC) 6.4). TheHamiltonian for the fields in a standard analysis framework is defined by ((RoC) 6.4.2)

H := H(Einstein)+H(Yang-Mills) + H(Dirac) + H(Higgs) ,((RoC) 7.3.

(RoC) 1.2.1: "*The LQG is characterized by the choice of a different algebra of basisfield functions, as in Quantum Field Theory (QFT). In conventional QFT this isgenerally the canonical algebra formed by the positive and negative frequencycomponents of the filed modes. The quantization of this algebra leads to thecreation and annihilation operators a and a(+). The characterization of thepositive and negative frequencies requires a background space-time. In contrast to this, what characterizes LQG is the choice of a different algebra of basisfield functions: a non-canonical algebra based on the holonomies of the gravitational connection. The holonomy (or "Wilson loop") is the matrix of the* ***parallel transport along a closed curve***."

This means, that also the LQG is built on the same handicaps, as H. Weyl's affine geometry. For an alternative approach for a quantum gravity model (with respect to the concepts of"continuity" and "particles") we refer to (RoC).

(RoC) Rovelli C., „*Quantum Gravity*", Cambridge University Press, 2004

History of string theory

The string theory and the early days of dual models is going back to the "Regge trajectory" and Veneziano amplitude and duality. From (GrM) below we recall:

1.2.1 Duality and the Graviton
One of the difficulties of dual theories of strong interactions, apart from the failure to accommodate the parton properties, was that these models always predicted a variety of massless particles, none of which are present in the hadronic world. These massless particles show up, for instance, in the poles that appear at s=0 and t=0 in the Veneziano amplitude, once one set a(0)=1 in order to eliminate ghosts. Dual models turn out to have massless particles of various spins. In particular, the closed-string sector of dual models turns out to have massless spin two particle. Investigation reveals - as one might suspect on general grounds - that its couplings are similar to those of general relativity. Might we interpret this particle as the graviton?

From a mathematical conceptual point of view it "just" "adds (while keeping orthogonality relationships with the prize to be paid, ending up with a high number of dimensions)" the 4 current arte-facts from gravitation and QED, QCE, QFE field theory, which are

- gluon (QCD) --> color charge
- photon (QED) --> electrical charge
- "Higgs-(…on)" (QFD) --> flavor charge, weak iso spin
- "graviton"--> mass.

The "existence" of the first two "particles" are "proven", which (just!) means the energy/spin/scattering etc. effects were measured and the corresponding mathematical model built on the arte-facts explains the observed energy/spin/scattering values; *to conclude by this to the existence of those particles looks (to the author's opinion) a little bit like self fulfilling prophecy*. The third arte-fact seems recently to be "proven" by same methodology, the last one most probably won't be possible to "prove" by same methodology, just because of the scale (exp(-40)) relationship compared to the other "forces"!!!

It is basically about the Newtonian law formulation in a Hamiltonian formalism. Newton's law considers the force on a body to produce a definite motion, that is, a definite effect is always associated with a certain ***cause***. The Hamilton's principle considers the motion of a body to result from the attempt of Nature to achieve a certain ***purpose***, namely to minimize the time integral of the difference between the kinetic and potential energies.

The definition of the "Hamiltonian purpose" might be interpreted as "sense", in the way, as it is defined by M. Heidegger (§32, Verstehen und Auslegen, p. 151) as "*das, worin sich die Verständlichkeit von etwas hält*" ("that, wherein the understandability of something keeps").
M. Heidegger, "Sein und Zeit", Max Niemeyer Verlag Tübingen, 2001.

D. E. Neuenschwander, "Emmy Noether's Wonderful Theorem", 3.1, 3.4

Newton´s law (equations of motion) <---> action principle

Quote: "o*nly when the kinetic (K) and potential (U) energies and the calculus of variations came together could it be realized how F=m*a can be subsumed into the latter, through the Lagrangian K-U.*"

Kaku M., (KaM):
"*This equivalence breaks down at the quantum level. The equations of motion is only an approximation to the actual quantum behavior of matter. Thus the action principle is the only acceptable framework for quantum mechanics.*"

The Lagrange formalism (building on the physical concepts of "force" and "work") and the Hamiltonian formalism (building on "energy" and "action") are equivalent. The transformation is given by the Legendre transformation. To the author's opinion a GUT theory needs to be built into a mathematical framework, where both formalism are no longer equivalent, i.e. the Legendre transformation is not possible resp. the regularity requirements are not fulfilled. This is, where J. Plemelj could come into the game (currently the "force" is calculated/represented as gradient of a corresponding potential in the usual sense). In this new framework, "force" and "work" are not existing in the sense of Newton (but in a less regular sense, following Plemelj's alternative definition of a potential), but the "energy" and "action" definitions keep valid (energy norm and corresponding minimization problem). If "force" is not required to be defined in the standard way, there is no need to prove the existence of corresponding "test" particle, to which the "force" is acting to (the massless bosons fill anyway 99% of the "volume" of a nucleon in the "presence" of force). From a mathematical point of view the force gradient is currently seen and modeled as a continuous field, while the related energy is accepted and modeled by discrete quantum leaps. This is at least "remarkable" and indicates also a switch to a Stieltjes integral representation of the underlying force potential.

In a nutshell: to the author´s opinion the current conceptual problem to unify the QED, QCD, QFD and gravitation theory is a mathematical problem (defining the adequate framework/Hilbert space environment) and not a problem of physics. It´s already proven, that there are neither particles nor fields in the way the current mathematical framework requires them to describe the existing physical models. The reflections of Hermann Weyl (see below (WeH), introduction) concerning space, time, matter and its "strong", "weak" or "anything else" "interactions" seems to be still valid. H.Weyl ((BlD), (WeH1)) introduced the concepts of gauge transformation and gauge invariance to compute the field strength (or curvature) of gauge potential (or connection). In case G=U(1) (or equivalent, the group of rotations in the plane), the gauge potential is essentially the 4-vector potential of electromagnetism and the field strength is the electromagnetic field (BlD).

Note: the group SU(2) (or equivalently, the unit quaternions) corresponds to the gauges prescribing a point-dependent choice of isotopic spin axis (introduced by Yang and Mills).

In order to show that the Veneziano amplitude obeys duality the analytical behavior of the Euler beta function is applied ((GrM), 1.1.1). The beta function especially describes very well the high-energy behavior of the Veneziano model. A string is a given state of oscillation that has a mass m ((GrM), 2.1). The mass squared can be determined in terms of internal modes of oscillation of the string. Both representations, for open and closed strings, correspond to a Laurent series with vanishing "zero" Fourier term. The Gamma and the Beta function play a key role in the Riemann Hypothesis. The approach given in

www.riemann-hypothesis.de ,

is basically about the following: Transformation of the constant, non-vanishing Fourier term of the Theta function into a vanishing Fourier term (by applying the Hilbert Transformation to the Theta function). The prize to be paid is change of the framework from analytical function to L(2)-Hilbert space environment. A corresponding approach might lead to an appropriate Hilbert space definition (H(-1/2)?), which, as well, might fit to the proposed Hilbert space environment for the Stefan problem.

We note that the Dirac delta function is an element of H(-s/2) for s>n/2, whereby n denotes the dimension of the field.

Following the road to surface minimization formulation would create a loop back to the Poincare conjecture;

see e.g. Michael T. Anderson, "Geometrization of 3-manifolds via the Ricci flow":

The QCD (Quantum Cromodynamics) is the gauge field theory that describes the strong inter**actions** of colored partons (quarks and gluons). It is the SU(3) component of the SU(3)xSU(2)xU(1) standard model of particle physics. The mathematical formalism is given by the Lagrangian of QCD. The parton distributions are crucial ingredients in the theoretical predictions of scattering cross-sections at hadron colliders. **J. Plemelj'**s concept enables the definition of a parton induced potential, based on partons' (infinitely small) mass, not based on a parton distribution!

The QED field theory describes the weak inter**actions** (the term "force" is no longer needed!) of electromagnetic fields.

A mathematical concept to handle non-linear differential or integral equations is given by K. O. Friedrich's dual (complementary) extremal problem formalism. For example in electrostatic field theory the dual principle to the "*principle of minimal potential energy*" is the "*principle of Thomsen*". The "method of Noble" leads to a saddle point problem, described by a system of adjoint operator equations S, T, where S, T are represented as Gateaux derivatives with respect to u resp. v of a Hamiltonian (energy density) function W(u,v).

The author would expect that a similar duality relationship holds true for two QCD complementary "energy" principles for the fermions (e.g. exclusion principle of Pauli and/or the only two energy states of Fermions?) and (truly) bosons (the "zero until infinity" energy states of bosons with its corresponding statistical distribution of the number of particles). The spin-statistics theorem and proper definitions of the zero point and vacuum fluctuation energy should be an outcome of this duality relation. A corresponding duality relation in the appropriate Hilbert space framework is expected between Planck's oscillator (anticipating the zero point energy, but leading to divergent norm integrals) and Fermi's oscillator.

E.Schroedinger, "Statistical Thermodynamics", Cambridge University Press, 1952
R. P. Feynman, A. R. Hibbs, Quantum mechanics and path integrals, Dover Publications, Inc., Mineola, New York, 2010, pp. 240, 244, 268, 294.

References related to complementary (dual) variational problems:

L. B. Rall, On complementary variational principles, J. Math. Anal. Appl. 14 (1966) 174-184,
A. M. Arthurs, Complementary variational principles, Oxford, 1970, *chapter 4*.

The general concept to define a valid physical model is to define mathematical field model (by (physical) Lagrange principle with corresponding mathematical variational principles), which requires per definition the existence of a (mathematical arte-fact) test particle ("Aufpunkt"). From this physical model e.g. corresponding energy (quantum) values are calculated, which then are compared to experimental measurement results. If the measured values fits to the theoretically calculated "forecast" the (physical!!) existence of the required test particle is assumed to be proven.

Applying the dual pair principles of "energy principles" and its "dual principles" for e.g. Fermions and Bosons in combination with the opportunities provided by "J. Plemelj" "to model potentials just built on infinitely small particles might open alternative way to define quantum fields.

Following our proposal from
www.riemann-hypothesis.de

for an alternative zero point energy model the appropriate Hilbert space framework modeling the dual (variational) equations should not be H(0), but H(-a), a>0.
We emphasize that our model, representing a quantum as an element of e.g. H(-1/2) Hilbert space, models such a spontaneous breakdown just by applying the projection operator from H(-1/2) into H(0) Hilbert space, resulting into the fact, that a quantum dual operator (which defines the corresponding H(-1/2)-energy (-norm)), "breaks down spontaneously" to an asymmetric operator within the smaller (today´s standard) H(0) Hilbert space.

From the famous paper of P. Higgs we recall in this context:

. ..."the idea that the apparently approximate nature of the internal symmetries of elementary-particle physics is the result of asymmetries in the stable solution of exactly symmetric dynamical equations .... is an attractive one. .... Within the framework of quantum field theory such a "spontaneous" breakdown of symmetry occurs if a Lagrangian, fully invariant under the internal symmetry group, has a structure that the physical vacuum is a member of a set of (physically equivalent) states which transform according the a nontrivial representation of the group. .... That vacuum expectation values of scalar fields, ... might play such a role in the breaking of symmetries.... in a theory of this type the breakdown of symmetry occurs already at the level of classical field theory...."We emphasize, that our model fits to this statement, while being valid at the same time for the Maxwell equations without any further requirements for additional space-time dimensions to keep consistency between the models.

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(YaC) Yang C. N., Mills R. L., Conservation of isotopic spin and isotopic gauge invariance, Phys. Rev. 96, 191-195, 1954
(BlD) Bleecker D., Gauge Theory and Variational Principles, Dover Publications, Inc., 1981

The wave radiation model: a hyperbolic initial value problem

"the objective world has only been constructed at the price of taking the self, that is, mind, out of it, remaking it; obviously, therefore, it can neither act on it nor be acted on by any of ist parts. ... If this problem of the action of mind on matter cannot be solved within the framework of our scientific representatiion of the objective world, where and how can it be solved?" - E. Schrödinger, Mind and Matter

... with a radiation operator definition at t=0
... and the Ritz-Galerkin method for the wave equation

There is a conceptual difference between elliptic and parabolic problem versus hyperbolic problems.

The Maxwell equations build a symmetric hyperbolic system. The decoupled Maxwell equations consist of two linear wave equations. The telegraph equation is a generalized wave equation, where different choises of the underlying "main parameter a) leads to hyperbolic, elliptic and parabolic differential equations.

The Yang-Mills equations are a nonlinear generalization of the Maxwell equations which are semi-linear.

The Euler and Einstein equations are quasi-linear, but not semi-linear hyperbolic equations. Even the simplest linear hyperbolic equation (the wave equation) shows different ('not optimal') properties of shift theorems in standard Sobolev spaces.

At the same time, there is the Courant conjecture, which makes the 2D and 4D space-time worlds unique compared to all other dimensions n (Courant-Hilbert, Methods of Mathematical Physics) p. 763: "*Families of spherical waves for arbitrary time-like lines exist only in the case of two or four variables, and then only if the differential equation is equivalent to the wave equation*".
In this context we also note that in current "quantum radiation theories" the Huygens principles is neglected.

With respect to hyperbolic problems the adequate energy Hilbert space cannot be more regular than the proposed H(1/2) (energy) Hilbert space for the NSE solution. We provide a counter example that it is most likely that any H(a) for any a>0 is not sufficient. An alternative ("exponential decay") inner product is proposed with 'optimal' shift theorem behavior of the d'Alembert operator. The prize to be paid is an extension of current Hilbert scale inner products with polynomial decay coefficients to an inner product with exponential decay.

We present an 'optimal order' shift theorem for the wave equation with respect to this alternative ("hyperbolic") energy norm. At the same time we emphasis that there is only one quantum potential energy "measure" (as proposed in http://www.fuchs-braun.com/), which is the quantum potential energy norm.

We note the elegant role of the proposed NSE-H(1/2) energy norm in universal Teichmüller theory.

The wave radiation problem is described by the corresponding radiation operator equation [CoR] VI §10.3). The radiation operator defines the radiation condition on the t-axis for x=0 for a given function g(t). It is in sync with Plemelj's alternative definition of the normal derivative.

We provide a characterization of the 4-dimensional Minkowski space which is about the fact that the differential of the Fourier transform of the uniform distribution of unit mass over the unit sphere at the origin vanishes.

Following the same concept as for the parabolic (heat equation) case (Nitsche/Wheeler) we sketch a proof of the L(infinite) boundedness of the Ritz operator in case of finite element approximation spaces for the solution of the wave equation. The analysis restricts to the space dimension m=1 and m=3. The required assumptions to the finite element approximation spaces are linear resp. quadratic finite elements in order to enable the Nitsche duality technique.

The 1-dimensional non-linear parabolic (Stefan) model problem with not regular initial value data

The classical Stefan problem aims to describe the temperature distribution in a homogeneous medium undergoing a phase change, for example ice passing to water; Stefan problems are examples of free boundary problems; appropriate variable transformation leads to nonlinear parabolic initial-boundary value problems with a fixed domain.

We provide an 'optimal' finite element approximation error estimate for a one-dimensional non-linear parabolic model problem with non-regular initial value data. The solution concept is based on corresponding results of J. A. Nitsche for finite element approximation error estimates for the one-dimension Stefan problem ([NiJ], [NiJ1], [NiJ2]) leveraging on the NSE solution concept of this homepage. As the approach is not depending from the space dimension it can also be applied to the 3-D non-stationary, non-linear Navier-Stokes equations to improve the 'not-optimal' finite element approximation error estimates in [HeJ].

The basic idea is in line with the proposal of the "Sapce-scale turbulence" section regarding:

let ux, uxx denote the first and second derivative of the function u(x) and e the viscosity term. In (MuA) it is proposed to replace e*uxx by -e*H[ux](x). For the latter term it holds the equality -e*H[ux](x) = e*A[uxx](x) whereby A denotes the Symm operator in the framework of L(2)-integrable periodic function on R with domain H(-1/2) (BrK1) (DeS) (MuA) (OkH).

In this case the same approach is applied to the auxiliary problem for a solution v in ([NiJ], [NiJ1], [NiJ2]) i.e. the auxiliary function v:=ux is replaced by v:=-H[ux].

For an analog NSE analysis the linearized (analog heat equation related) auxiliary problem (with corresponding 'optimal' parabolic shift theorem and related 'optimal' Ritz-Galerkin approximation estimates, [NiJ5]) needs to be replaced by the corresponding analog of the NSE theory which is about "an unusual shift theorem for Stokes flow" (see related J. A. Nitsche paper) and a corresponding "L(infinity)-analysis of the Galerkin (finite element) approximation of the solution of the Stokes equation" (J. A. Nitsche, lecture notes).

Here we are

Braun K., Optimal finite element approximation estimates for non-linear parabolic problems with non regular intial value data.pdf (563.69KB)

Wavelets: a mathematical microscope tool

The wavelet theory is established in the Fourier Hilbert space framework. In order to apply the Calderón inverse formula in a Hilbert scale framework it requires the so-called admissibility condition defining a wavelet analyzing a (signal) function. "A wavelet synthesis can be performed locally as opposed to the Fourier transform which is inherently nonlocal due to the space-filling nature of the trigonometric functions" (FaM1). So the question is, how to bundle both necessary concepts into one applying it to corresponding "right" physical NSE variation representation with related energy Hilbert space.

Mathematically speaking, the L(2) based wavelets are accompanied by the distributional Hilbert space H(-1/2). The proposed quantum potential energy model H(1/2) is based on a Hilbert scale framework built on the spectrum of a considered kinematical phenomenon govenred by a corresponding self-adjoint kinetic energy operator. The "coarse grained" kinematical Hilbert space pair (H(0),H(1)) accompanied by concepts like the Shannon entropy is compactly and densely embedded into the overall Hilbert space pair (H(-1/2),H(1/2)), where its complementary closed sub-space pair defined the complementary quantum potential element and energy Hilbert space pair.

We note that L(2) functions with vanishing constant Fourier term are wavelet mother functions, and that the Hilbert transform of L(2) functions fulfill this condition.

Wavelet analysis can be used as a mathematical microscope, looking at the details that are added if one goes from a scale "a" to a scale "a+da", where " da " is infinitesimally small. We mention that an alternative model for an "a" to a scale "a+da model is the concept of the ordered field of ideal points, an extension to the ordered field of real numbers with same cardinality, but having additionally infinitesimal elements (also called non-Archimedean numbers).

The mathematical microscope wavelet tool 'unfolds' a function over the one-dimensional space R into a function over the two-dimensional half-plane of "positions" and "details". This two-dimensional parameter space may also be called the position-scale half-plane. The wavelet duality relationship provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "microscope observations" of two wavelet (optics) functions f and g can be compared with each other by the "reproducing" ("duality") formula.

A wavelet-Galerkin turbulent flow field representation with expansion coefficients in identical physical & wavelet space

"*When Fourier meets Navier*" in a physical H(1/2) energy Hilbert space there is a global unique H(1/2) solution of the corresponding weak variation equation of the non-linear, non-stationary Navier-Stokes equations ("solution").

The standard numerical approximation method for PDE is the Ritz-Galerkin method which is usually equipped with finite element approximation spaces (FEM). In case of negative scaled Hilbert spaces the Ritz-Galerkin method accompanied by finte element approximation spaces becomes a boundary element method (BEM) related to the underlying singular equation representation.

The required finite (boundary) element approximation properties of the BEM face some challenges in case of non-linearity and/or non-periodic boundary conditions. The wavelet extension method is used to represent functions that have discontinuities and sharp peaks, and for accurately deconstructing and reconstructing finite, non-periodic and/or non-stationary signals, (MeM).

Mehra M., Wavelets and Differential Equations - A short review.pdf (175.43KB)

Here we go with a collection of some useful supporting tool sets:

Braun K., Generalized wavelet theory and non-linear, non-periodic boundary value problems_1.pdf (439.35KB)

J. Plemelj and his more general definitions of
"normal derivatives", "mass elements" and "potentials"

The intention of the section is to motivate an alternative mathematical framework, which does not require “ideal” boundary layer assumptions. The mathematical requirements to define boundary layers and corresponding potentials are very much depending from the definition and regularity requirements of the normal derivative. It is perpendicular to the boundary itself and therefore requires regularity assumptions, affecting "points" outside of the domain w/o any physical meaning.

There is a similar challenge related to Einstein's field equations, when trying to describe a ground state energy in a vacuum: tensor analysis, manifolds and the affine connexion concept require differentiable manifolds, while for the physical model it would be sufficient to require continuous manifolds only.

With respect to the "mass element" and the "potential" topic in combination with the "author's paper" --> "a new ground state energy model", we trust to make the following statement:

"Plemelj's mathematical concept of a "mass element", its corresponding "potential" and alternative energy norm definition of a "mass element" in combination with the concept of Pseudo-differential operators (e.g. rotation invariant) Hilbert/Riesz (singular integral) operators and related Hilbert scale definitions) provides the mathematical missing piece to enable H. Weyl's vision of a "truly infinitesimal geometry" to overcome the incompliancies and inconsistencies of the today's paradox of the kinetic-particle and dynamical-field models. "

We note that every Hilbert space is a Banach space, and every Banach space is a metric space; in general the other way around is not true; only a Hilbert space has an inner product. The Riemann manifold is only a (Pythagorean sort of) metric space. The concept of manifolds requires the concept of exterior derivatives.

For an overview of history of current particle-field dualism with respect to philosophical and physics aspects we refer to the great work of H. Weyl:

1. H. Weyl, " *philosophy of mathematics and natural science*", Princeton University Press, 2009

2. H. Weyl, "*The continuum, a critical examination of the foundation of analysis*", Dover Publications, Inc., New York, 1987.

3. Weyl H., „*Was ist Materie*“, Berlin, Verlag von Julius Springer, 1924

We quote from 3. Weyl H.:
Kap.III Die Feldtheorie, (14): “... *Die Definition des Feldes mit Hilfe seiner ponderomotorischen Wirkung auf einen Probekörper ist nur ein Provisorium. Durch das Hereinbringen des geladenen Probekörpers stört man immer in etwa das Feld, das es eigentlich zu beobachten galt; befindet er sich einmal im Felde, so gehört er so gut wie die übrigen Konduktoren mit zu den das Feld erzeugenden Ladungen .....*“

J. Plemelj: "*Potentialtheoretische Untersuchungen*"
B. G. Teubner, Leipzig (1911)

Quote, page VII: *„hierbei war es vor allem nötig, die klassische Form des Potentiales V, ..., in der allgemeineren Gestalt des* ***Stieltjesschen Integrals*** *vorauszusetzen, wobei dm(s) das Element der Masse in einem infinitesimalen Stücke der Berandung im Punkt s vorstellt. Diese allgemeine Form geht in die klassische erst dann über, wenn die Massenverteilung in der Berandung differenzierbar ist. .....während früher jedes Potential V(p) normale Ableitungen besass, also solche selbst durchaus stetige Potentiale, die keine bestimmte normale Ableitung besitzen, von vornherein als durch ein V(p) nicht darstellbar erkannt werden könnten, ist es in der neuen Gestalt von einer ähnlichen Allgemeinheit, wie das Potential der Doppelschicht. ....Die Tendenz, der unbequemen Ableitungen am Rande aus dem Weg zu gehen, ... veranlasste mich zur Einführung des oben erwähnten Strömungsbegriffes ....Die grosse Verallgemeinerung gibt sich wieder dadurch kund, dass die Strömung beim Potential W praktisch wenig einschränkenden Bedingungen stetig verläuft, wobei von der Existenz der normalen Ableitungen noch keine Rede sein kann.“*

Quote, §8, "*Bisher war es üblich, für das Potential die Form ( ) zu nehmen. Eine solche Annahme erweist sich aber als eine derart folgenschwere Einschränkung, dass dadurch dem Potential der grösste Teil seiner Leistungsfähigkeit hinweg genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential nur in der Form ( ) verwendbar*.
("up to now it was usual to take for the potential the form (..). This assumption manifeste as such a weighty Restriktion, that in doing so the biggest piece of the capability of the potential is withdrawn. For in greater depth investigatives only the potential in the form (..) is appropriable.")

J. Plemelj: "*Potentialtheoretische Untersuchungen*"
B. G. Teubner, Leipzig (1911)

J. Plemelj Potentialtheoretische Untersuchungen 1.pdf (3.82MB)

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J. Plemelj Potentialtheoretische Untersuchungen 13.pdf (3.62MB)

J. Plemelj Potentialtheoretische Untersuchungen 14.pdf (3.37MB)

J. Plemelj's extended Green identities

Let du/dn denote the directional derivative of u in the direction of the outward pointing normal n to the surface element. An uniquely defined and continuous normal n enables a meaningful definition of du/dn, which is one central object in the Green identities.

The proofs of the Green identities are based on the assumption, that the potential together with its first derivatives is finite and continuous to the boundary, whereby a smooth continuation to the continuous boundary values is given.

The alternatively proposed definition from Plemelj of the directional derivative of u on the boundary enables an extension of the Green identities with reduced regularity assumptions to the potential function on the boundary, where the normal derivative is replaced by a differential of the conjugate potential function (in case of space dimension n=2), with the following reduced regularity assumptions:

the normal derivative of a potential function u needs no longer to be existing everywhere on the boundary, but only the conjugate function of this potential u is required to be continuous on the boundary. This compares to an extention from differentiable functions to continuous functions. The extended Green formulas are (in a nutshell) given as follows ((PlJ) p. 9-13):

let D(u,v) denote the Dirichlet integral (inner product) with domain D and let <u,v> denote the related inner product on the boundary of D. Let further **u** denote the conjugate function of a potential function u. Then it holds:

i) D(u,u)=<u,d**u**> is positive for all non constant potential functions u
ii) <v,d**u**>=<u,d**v**>

By operating with "div" on both sides of the NSE the field p(x,t) can be formally obtained as a solution of a Neumann problem. Therefore, to describe the values of the pressure at the bounding walls or at the initial time independently of v, could incompatible with the NSE and, therefore, could render the problem ill-posed. The above **extended Green formulas** provide a framework **to overcome** this Neumann "**pressure**" **incompatibility problem**.

Let ((v,w)) denote the inner product of the Hilbert space H(-1/2). Then the Plemelj extension of the Green formula enables a (weak) variation representation of the potential equation with respect this inner product, whereby the corresponding energy norm is given by the norm of the Hilbert space H(1/2).

In other words: Plemelj's extension of the Green identities enables an extended domain H(1/2) of the corresponding "energy"inner product, which is derived from the Dirichlet integral.

In the fluid theory beside the Laplacian and the Gradient operator the following operators play a key role:

1. the Trace operator div(u)
2. the Vorticity operator curl(u)
3. the Helicity operator u*curl(u).

The identity

(u,grad)u = grad(u)/2 - u*curl(u)

enables a properly defined (Plemelj) directional derivative of u(,grad)u ***on*** the boundary.

For the following we refer to the below paper from Y. Giga (GiY). Let n=3 and p=2, then it holds:

1. (GiY), *Sobolev embedding theorem*:
the Hilbert space H(1/2) is a subset of L(3)

2. (GiY) *Lemma 3.2*:
Let r=3, "delta"=3/4, "theta"="roh"=1/4 then the absolute value of the trilinear form ((P(u,grad)v,w)) is uniformly bounded with respect to the H(1/2)-norms of u,v and w.

We claim that as a consequence of the above a weak representation of the NSE with respect to the inner product of H(-1/2) is well posed, i.e. there exists an unique solution, which is continuously depending from the initial and boundary values.

The semigroup estimates ((GiY), Lemma 2.1, see also Nitsche J. A., approximation theory in Hilbert scale; extensions) indicate the to be defined appropriate exponent of the time variable in order to achieve a balance between the above H(-1/2) variation representation of the NSE and a corresponding time-weighted H(0) variation representation with "standard" regularity requirements to the potential.

Giga Y., Weak and strong solutions of the Navier-Stokes initial value problem.pdf (2.06MB)

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J. A. Nitsche and his footprints

the famous Nitsche trick was first published in

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a (nearly, w/o early 1950th papers) complete reference list is given in:

Nitsche J. A., 1926-1996.pdf (3.61MB)

A "quasi-optimal" shift theorem for linear parabolic equations with appropriately defined norms in a Hilbert scale framework is proven in

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Stefan (non-linear parabolic) problem related papers are:

Nitsche J. A., A Finite Element Method for Parabolic Free Boundary Problems I.pdf (3.94MB)

Nitsche J. A., A Finite Element Method for Parabolic Free Boundary Problems II.pdf (1.58MB)

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Some additional papers:

Nitsche J. A., Least squares approximations to first order elliptic systems.pdf (1.53MB)

Nitsche J. A., Finite Element Approximations for Solving the Elastic Problem.pdf (809.75KB)

Nitsche J. A., 1982, Schauder estimates for finite element approximations on second order elliptic boundary problems I.pdf (3.83MB)

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Nitsche J. A., lecture notes I, Approximation Theory in Hilbert Scales.pdf (2.31MB)

Nitsche J. A., lecture notes II, Extensions and Generalizations.pdf (1.26MB)

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