The confluent hypergeometric function

The confluent hypergeometric function is given by

$$ {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\,;\, z\right) = \lim_{b\to\infty} {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix}\,;\, \frac{z}{b}\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}, \quad c \notin \{0, -1, -2, \ldots\}. \tag{1} $$

This is a solution of the confluent hypergeometric differential equation

$$ zy''(z) + (c - z)y'(z) - ay(z) = 0. \tag{2} $$

For $c \notin \mathbb{Z}$ the general solution of the confluent hypergeometric differential equation (2) can be written as

$$ y(z) = A\, {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\,;\, z\right) + B\, z^{1-c}\, {}_1F_1\left(\begin{matrix} a+1-c \\ 2-c \end{matrix}\,;\, z\right) $$

with A and B arbitrary constants.

Based on Euler's integral representation for the ${}_2F_1$ hypergeometric function, one might expect that the confluent hypergeometric function satisfies

$$ {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\,;\, z\right) = \lim_{b\to\infty} {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix}\,;\, \frac{z}{b}\right) = \lim_{b\to\infty} \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}\left(1 - \frac{zt}{b}\right)^{-b} dt. $$

Now we have

$$ \lim_{b\to\infty} \left(1 - \frac{zt}{b}\right)^{-b} = e^{zt}, $$

which leads to

Theorem 1. *For* $\mathrm{Re}\, c > \mathrm{Re}\, a > 0$ *we have*

$$ {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\,;\, z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt}\, dt. \tag{3} $$

Proof. Note that we have

$$ \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt}\, dt = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^1 t^{n+a-1}(1-t)^{c-a-1}\, dt $$

and for $\mathrm{Re}\, a > 0$ and $\mathrm{Re}\,(c-a) > 0$

$$ \int_0^1 t^{n+a-1}(1-t)^{c-a-1}\, dt = B(n+a, c-a) = \frac{\Gamma(n+a)\Gamma(c-a)}{\Gamma(n+c)} = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} \frac{(a)_n}{(c)_n} $$

for $n = 0, 1, 2, \ldots$. This implies that

$$ \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt}\, dt = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} = {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\,;\, z\right). $$

This integral representation can be used to prove Kummer's transformation formula:

Theorem 2.

$$ {}_1F_1\left({a \atop c};z\right) = e^z\,{}_1F_1\left({c-a \atop c};-z\right). \tag{4} $$

Proof. We use the substitution $t = 1-u$ to obtain

$$ \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{zt}\,dt = \int_0^1 (1-u)^{a-1}u^{c-a-1}e^{z(1-u)}\,du = e^z\int_0^1 u^{c-a-1}(1-u)^{a-1}e^{-zu}\,du. $$

This implies that

$$ {}_1F_1\left({a \atop c};z\right) = e^z\,{}_1F_1\left({c-a \atop c};-z\right). $$

Note that this also follows from Pfaff's transformation formula for the ${}_2F_1$:

$$ {}_2F_1\left({a,\,b \atop c};z\right) = (1-z)^{-b}\,{}_2F_1\left({b,\,c-a \atop c};\frac{z}{z-1}\right), $$

by replacing z by z/b and taking the limit $b\to\infty$.

We also have a Barnes-type integral representation for the confluent hypergeometric function. In order to find this representation we compute its Mellin transform. By using Kummer's transformation formula (4) we obtain

$$ \begin{aligned} \int_0^\infty z^{s-1}{}_1F_1\left({a \atop c};-z\right)dz &= \int_0^\infty z^{s-1}e^{-z}{}_1F_1\left({c-a \atop c};z\right)dz \\ &= \sum_{n=0}^\infty \frac{(c-a)_n}{(c)_n\,n!}\int_0^\infty e^{-z}z^{s+n-1}\,dz = \sum_{n=0}^\infty \frac{(c-a)_n}{(c)_n\,n!}\,\Gamma(s+n). \end{aligned} $$

Now we have $\Gamma(s+n) = \Gamma(s)\,(s)_n$ and by using Gauss's summation formula

$$ \sum_{n=0}^\infty \frac{(c-a)_n}{(c)_n\,n!}\,\Gamma(s+n) = \Gamma(s)\,{}_2F_1\left({c-a,\,s \atop c};1\right) = \frac{\Gamma(c)}{\Gamma(a)}\,\frac{\Gamma(s)\Gamma(a-s)}{\Gamma(c-s)}. $$

This leads to

Theorem 3.

$$ \frac{\Gamma(a)}{\Gamma(c)}\,{}_1F_1\left({a \atop c};z\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)}{\Gamma(c+s)}\Gamma(-s)(-z)^s\,ds, \quad |\arg(-z)| < \pi/2 \tag{5} $$

where the path of integration is curved, if necessary, to separate the poles $s = -a-n$ from the poles $s = n$ with $n \in \{0,1,2,\ldots\}$.

Proof. The proof is similar to the proof of Barnes' integral representation for the ${}_2F_1$ hypergeometric function. Application of Cauchy's residue theorem then gives that the integral equals the sum of residues

$$ \sum_{n=0}^\infty \frac{\Gamma(a+n)}{\Gamma(c+n)}\,\frac{(-1)^n}{n!}\,(-z)^n = \frac{\Gamma(a)}{\Gamma(c)}\sum_{n=0}^\infty \frac{(a)_n}{(c)_n}\,\frac{z^n}{n!} = \frac{\Gamma(a)}{\Gamma(c)}\,{}_1F_1\left({a \atop c};z\right). $$