Addendum: A Two-Density Kummer–Digamma Extension for Additive Prime Number Theory

Supplement to "A toolbox to build a non-harmonic Fourier series based two-semicircle method"

with the kind support of ChatGPT

1. Purpose and relation to the original toolbox

The original paper develops a toolbox based on the imaginary parts of the zeros of the Kummer function, the real negative zeros of the Digamma function, and non-harmonic Fourier/Riesz-basis methods related to Kadec's 1/4 theorem, Avdonin's 1/4-in-the-mean theorem, Paley–Wiener spaces and Hilbert spaces of fractional order.

It further proposes two number-theoretical distribution functions, each of prime-counting order. The present addendum develops this two-density structure in a form designed for additive prime number theory. The central modification is to use two different Kummer/Digamma spectra together with one common density operator.

$$F^{K}(x) \sim x / \log x, \quad F^{D}(x) \sim x / \log x \quad (A1)$$

This separates the spectral difference from the choice of the density-generating function and leads naturally to a mean channel and a contrast channel.

2. The two weighted zero sequences

Let ω_n denote the normalized absolute imaginary parts of the Kummer zeros and w_n the negative Digamma zeros. Following the original construction, put

$$b_n^{(K)} := \omega_n, \quad b_n^{(D)} := 1/2 - w_n \quad (A2)$$

$$a_n^{(K)} := \frac{1}{n}\sum_{k=1}^{2n} b_k^{(K)}, \quad a_n^{(D)} := \frac{1}{n}\sum_{k=1}^{2n} b_k^{(D)} \quad (A3)$$

The localization inherited from the original construction gives

$$2n - 1 < a_n^{(K)} < 2n + 1 \quad (A4)$$
$$2n + 1 < a_n^{(D)} < 2n + 3 \quad (A5)$$

After centering and normalization, write the corresponding spectral sequences as

$$\lambda_n^K = n + \delta_n^K, \quad \lambda_n^D = n + \delta_n^D \quad (A6)$$

The Digamma side has the characteristic logarithmic displacement

$$-w_n \sim n - 1/\log n \quad (A7)$$

Using the previously established Kummer asymptotics together with the weighted construction, the working spectral contrast is

$$\Delta_n := \lambda_n^K - \lambda_n^D = 1/\log(2n) + o(1/\log n) \quad (A8)$$
$$|\Delta_n| = O(1/\log n) \quad (A9)$$

3. A common density-generating function

The original paper uses two related logarithmic functions. The present extension proposes instead to use one common density-generating function for both spectra:

$$f(x) := \log(\pi \sin(\pi x/2)) \quad (A10)$$

$$\int_0^1 f(x)dx = \log(\pi/2) \quad (A11)$$

For the normalized spectral grids set

$$x_n^{NK} := \lambda_n^K /N, \quad (x_n^{ND} := \lambda_n^D/N \quad (A12)$$

$$x_n^{NK} - x_n^{ND} = \Delta_n/N \quad (A13)$$

The logarithmic singularity at the origin is integrable. Under uniform grid approximation and the required endpoint control, the generalized Landau/Riemann-sum argument therefore yields the same limiting mean for both sequences.

$$\frac{1}{N}\sum_{n\le N} F(x_n^{NK}) \to \log(\pi/2) \quad (A14)$$

$$\frac{1}{N}\sum_{n\le N} F(x_n^{ND}) \to \log(\pi/2) \quad (A15)$$

Thus the same density operator can generate two distributions with the same leading prime-counting scale while leaving their difference attributable to the two zero sequences.

4. Mean and contrast variables

$$M := (F^K + F^D)/2, \quad D := (F^K - F^D)/2 \quad (A16)$$

$$F^K = M + D, \quad F^D = M - D \quad (A17)$$

At the level of the corresponding Stieltjes measures this gives

$$\mu^K = \mu^M + \mu\Delta, \quad \mu^D = \mu^M - \mu\Delta \quad (A18)$$

$$\mu^K * \mu^D = \mu^M * \mu^M - \mu\Delta * \mu\Delta \quad (A19)$$

The common density therefore produces the main additive term, whereas the Kummer–Digamma difference enters only quadratically. The linear cross terms cancel by commutativity of convolution.

5. Size of the Kummer–Digamma contrast

$$f'(x) = (\pi/2)\cot(\pi x/2) \quad (A20)$$

With $u_{n,}{}^N := (x_{n,}{}^{NK} + x_{n,}{}^{ND})/2$, first-order expansion gives

$$f(x_n^{NK}) - f((x_n^{ND}) \approx (\pi\Delta_n/2N)\cot(\pi u_n^N/2) \quad (A21)$$

Near the left endpoint $u_{n,}{}^N \sim n/N$, hence

$$(\pi/2N)\cot(\pi n/2N) \sim 1/n \quad (A22)$$

Together with $\Delta_n = O(1/\log n)$, this yields the model contrast scale

$$d_n \sim 1/(n \log n) \quad (A23)$$

An exact identity before linearization is λ_n^D

$$f(x_n^{NK}) - f((x_n^{ND}) = \log[\sin(\pi\lambda_n^K/2N) / \sin(\pi\lambda_n^D/2N)] \quad (A24)$$

$$\sum_{n\geq 2} n\,|d_n|^2 \sim \sum_{n\geq 2} 1/(n \log^2 n) < \infty \quad (A25)$$

Thus the unweighted density contrast reaches the critical positive Sobolev scale $H^{1/2}$ at the model level.

6. Quadratic additive correction

$$(d * d)^N := \sum_{m+n=N} d_m\, d_n \quad (A26)$$

For d_n of model size $1/(n \log n)$, endpoint contributions suggest

$$(d * d)^N = O((\log\log N)/(N \log N)) \quad (A27)$$

This estimate is a model estimate and requires uniform control of the remainders for use as a theorem. Its structural significance is that a first-order Kummer–Digamma spectral difference produces a second-order additive correction.

7. Passage to genuine prime arithmetic

The distributions F^K and F^D have the correct prime-counting scale, but this alone does not imply prime or prime-power support of their local increments. To preserve the arithmetic support exactly, the classical von Mangoldt function Λ is retained and the Kummer–Digamma information is introduced as a pair of weights.

$$\eta\Delta(n) \sim c/(n \log n) \quad (A28)$$

$$\Lambda^K(n) := \Lambda(n)[1 + \eta\Delta(n)] \quad (A29)$$

$$\Lambda^D(n) := \Lambda(n)[1 - \eta\Delta(n)] \quad (A30)$$

$$(\Lambda^K + \Lambda^D)/2 = \Lambda \quad (A31)$$

$$\Lambda\Delta(n) := (\Lambda^K(n) - \Lambda^D(n))/2 = \Lambda(n)\eta\Delta(n) \quad (A32)$$

Hence no new prime criterion is introduced. The classical von Mangoldt support remains intact; the Kummer–Digamma structure modifies only the two density channels.

8. Preservation of the Chebyshev main term

$$\Psi^K(X) := \sum_{n\leq X} \Lambda^K(n)\ , \quad \Psi^D(X) := \sum_{n\leq X} \Lambda^D(n) \quad (A33)$$

$$E\Delta(X) := \sum_{n\leq X} \Lambda(n)\eta\Delta(n) \quad (A34)$$

Under the model bound $\eta\Delta(n) = O(1/(n \log n))$, partial summation gives

$$|E\Delta(X)| \ll \sum_{n\le X}\Lambda(n)/(n\log n) = O(\log\log X) \quad (A35)$$

$$\Psi^{K}(X) = \psi(X) + O(\log\log X) \quad (A36)$$

$$\Psi^{D}(X) = \psi(X) - O(\log\log X) \quad (A37)$$

$$\Psi^{K}(X) \sim X, \quad \Psi^{D}(X) \sim X \quad (A38)$$

Thus the two channels preserve the classical Chebyshev main density while retaining a separately generated contrast.

9. Additive prime pairs

$$R^{KD}(N) := \sum_{m+n=N}\Lambda^{K}(m)\Lambda^{D}(n) \quad (A39)$$

Under the natural symmetric interpretation of the two positions in a pair, the mean/contrast decomposition gives

$$R^{KD}(N) = R\Lambda(N) - R\Delta(N) \quad (A40)$$

$$R\Lambda(N) := \sum_{m+n=N}\Lambda(m)\Lambda(n) \quad (A41)$$

$$R\Delta(N) := \sum_{m+n=N}\Lambda(m)\Lambda(n)\eta\Delta(m)\eta\Delta(n) \quad (A42)$$

For primes p and q the model contrast has

$$\Lambda(p)\eta\Delta(p) \sim c/p \quad (A43)$$

$$R\Delta(N) \approx c^2 \sum_{p+q=N;\ p,q\ \text{prime}} 1/(pq) \quad (A44)$$

Equation (A44) is a model form of the contrast term, not an established Goldbach asymptotic.

10. Modified circle-method representation

$$e(x) := \exp(2\pi i x) \quad (A45)$$

$$S^{x}(\alpha) := \sum_{n\le X}\Lambda(n)e(n\alpha) \quad (A46)$$

$$T^{x}(\alpha) := \sum_{n\le X}\Lambda(n)\eta\Delta(n)e(n\alpha) \quad (A47)$$

$$S^{K} = S + T, \quad S^{D} = S - T \quad (A48)$$

$$S^{K}(\alpha)S^{D}(\alpha) = S(\alpha)^2 - T(\alpha)^2 \quad (A49)$$

$$R^{KD}(N) = \int_0^1 [S(\alpha)^2 - T(\alpha)^2]\, e(-N\alpha)\, d\alpha \quad (A50)$$

The exact integer relation m+n=N is therefore preserved by ordinary Fourier orthogonality. The non-harmonic Kummer/Digamma systems generate the density information; they do not replace the integer frequencies in the additive condition.

11. Hilbert-space size of the contrast channel

On primes, the leading coefficient has model size 1/p, so

$$T(\alpha) \sim c \sum_{p \text{ prime}} e(p\alpha)/p \quad (A51)$$

$$\sum_{p \text{ prime}} 1/p^2 < \infty \quad (A52)$$

$$T \in H(0) = L^2 \quad (A53)$$

More precisely, for every ε>0,

$$\sum_{p \text{ prime}} p^{1-2\varepsilon} \left|\frac{1}{p}\right|^2 = \sum_{p \text{ prime}} \frac{1}{p^{1+2\varepsilon}} < \infty \quad (A54)$$

$$T \in H^{\frac{1}{2}-\varepsilon} \text{ for every } \varepsilon > 0 \quad (A55)$$

At the endpoint, however,

$$\sum_{p \text{ prime}} p|1/p|^2 = \sum_{p \text{ prime}} 1/p = \infty \quad (A56)$$

so the model contrast is generally not in $H^{-1/2}$. For the truncated series the endpoint divergence is only doubly logarithmic:

$$\|T_X\|^2_{H^{\frac{1}{2}}} \sim c^2 \log\log X \quad (A57)$$

12. A logarithmically refined critical Hilbert space

The endpoint behavior suggests refining rather than replacing the Hilbert scale used in the original paper. For β>1 define

$$\|a\|^2_{H^{1/2}_{\log,\beta}} := \sum_{n\geq 3} \frac{n}{(\log\log n)^\beta} |a_n|^2 \quad (A58)$$

For the prime-supported model $a_p \sim 1/p$,

$$\sum_{p \text{ prime}} \frac{n}{(\log\log n)^\beta} < \infty \quad \text{for } \beta > 1 \quad (A59)$$

$$T \in H^{1/2}_{\log,\beta}, \quad \beta > 1 \quad (A60)$$

$$H^{1/2} \subset H^{1/2}_{\log,\beta} \subset H^{1/2-\varepsilon} \quad (A61)$$

The coefficient-space dual has norm

$$\|b\|^2_{H^{-1/2}_{\log,-\beta}} := \sum_{n\geq 3} \frac{(\log\log n)^\beta}{n} |b_n|^2 \quad (A62)$$

This provides a logarithmically refined version of the $H^{-1/2}$ framework already present in the original toolbox.

13. Compatibility with the non-harmonic Riesz systems

$$T^K a := \Sigma_n a_n \exp(i\lambda_n{}^K t), \quad T^D a := \Sigma_n a_n \exp(i\lambda_n{}^D t) \quad (A63)$$

The logarithmic refinement can be imposed on the coefficient side of the Riesz synthesis operators. Define

$$L_\beta a := \left\{ \frac{\sqrt{n}}{(\log\log n)^{\beta/2}} a_n \right\} \quad (A64)$$

$$\|f\|^2_{\mathcal{H}^K_{1/2,\log,\beta}} := \| L_\beta (T^K)^{-1} f \|_{l^2} \quad (A65)$$

$$\|f\|^2_{\mathcal{H}^D_{1/2,\log,\beta}} := \| L_\beta (T^D)^{-1} f \|_{l^2} \quad (A66)$$

Thus the refined critical scale is compatible with the Kummer/Digamma coefficient representation without changing the underlying arithmetic convolution.

14. Exact additive convolution versus non-harmonic spectral representation

Let $f^K = T^K a$ and $f^D = T^D b$. Exact addition is kept in coefficient space:

$$(a * b)_N := \sum_{m+n=N} a_m b_n \quad (A67)$$

If T^E is the synthesis operator of the target channel, define

$$\mathcal{B}(f^K, f^D) := T^E [((T^K)^{-1} f^K) * ((T^D)^{-1} f^D)] \quad (A68)$$

This separates the analytic non-harmonic representation from the exact integer-indexed arithmetic relation.

15. Interpretation of the two densities

The term “independent densities” is used structurally, not as a claim of probabilistic independence of the actual primes. The two marginal densities are generated separately before the additive coupling is imposed.

$$\text{Kummer density} \times \text{Digamma density} \rightarrow \text{product density} \rightarrow \text{additive constraint } m + n = N \quad (A69)$$

At measure level, if μ^K and μ^D are the two marginal measures, addition is introduced by $\Sigma(x,y)=x+y$ and

$$\sum_{\#}(\mu^K \otimes \mu^D) = \mu^K * \mu^D \quad (A70)$$

16. Main structural result of the addendum

The construction can be summarized by the following sequence of transformations:

$$\text{Kummer and Digamma zero sequences}$$

$$\downarrow$$

$$\Delta_n = \lambda_n^K - \lambda_n^D = O(1/\log n)$$

$$\downarrow$$

$$\text{common log} - \text{sine density operator}$$

$$\downarrow$$

$$\eta\Delta(n) \sim 1/(n \log n)$$

$$\downarrow$$

$$\Lambda^K = \Lambda(1+\eta\Delta), \quad \Lambda^D = \Lambda(1-\eta\Delta)$$

$$\downarrow$$

$$(\Lambda^K + \Lambda^D)/2 = \Lambda$$

$$\downarrow$$

$$S^K = S + T, \quad S^D = S - T$$

$$\downarrow$$

$$S^K S^D = S^2 - T^2$$

$$\downarrow$$

$$R^{KD}(N) = R\Lambda(N) - R\Delta(N)$$

The classical Mangoldt channel is preserved exactly. All new information is concentrated in the thinner contrast channel T.

17. Status of the extension and remaining proof obligations

17.1 Inherited from the original paper

- the Kummer and Digamma zero sequences and their localization;
- the weighted sequences constructed from these zeros;
- the Kadec/Avdonin and Paley–Wiener/Riesz-basis framework;
- the generalized Landau-type framework;
- two number-theoretical distribution functions of asymptotic order x/log x;
- the $H^{-1/2}$ Hilbert-space framework and the intended connection to additive prime-number problems.

17.2 New deductions and constructions in this addendum

- use of one common density-generating function for the two zero sequences;
- the mean/contrast decomposition and the identity $\mu^K*\mu^D = \mu^M*\mu^M - \mu\Delta*\mu\Delta$;
- preservation of exact integer convolution while using non-harmonic spectral representations;
- the Mangoldt-weighted pair $\Lambda^K=\Lambda(1+\eta\Delta)$, $\Lambda^D=\Lambda(1-\eta\Delta)$;
- the exact preservation of the classical mean channel;
- the circle identity $S^KS^D=S^2-T^2$;
- the logarithmically refined critical Hilbert scale.

17.3 Remaining proof tasks

- uniform control of the remainder in the common-function Kummer/Digamma density comparison;
- a rigorous transfer from $\Delta_n=O(1/\log n)$ to a precise arithmetic contrast $\eta\Delta(n)$;
- prime-weighted estimates for $\sum_{n\le X}\Lambda(n)\eta\Delta(n)$;
- mapping properties of the quadratic operator $Q(T)=T^2$ in the logarithmically refined Hilbert scale;
- sufficiently strong estimates of the Fourier coefficients $R\Delta(N)=\widehat{T^2}(N)$ for the intended additive application.

No proof of the binary Goldbach conjecture or of the Riemann hypothesis is claimed at this stage. The addendum instead reduces the proposed two-density method to a more specific analytic problem.

18. Conclusion

The original toolbox suggests that Kummer and Digamma zero sequences can generate two number-theoretical density structures of prime-counting order. The present addendum reorganizes this idea so that the classical arithmetic structure is retained as the exact mean channel.

$$\Lambda^K = \Lambda + \Lambda\Delta, \quad \Lambda^D = \Lambda - \Lambda\Delta \quad \text{(A71)}$$

$$\Lambda^K * \Lambda^D = \Lambda * \Lambda - \Lambda\Delta * \Lambda\Delta \quad \text{(A72)}$$

$$(S+T)(S-T) = S^2 - T^2 \quad \text{(A73)}$$

The characteristic logarithmic spectral displacement is converted by the common log-sine density operator into a much thinner contrast. After Mangoldt weighting, the leading prime-supported coefficients have model size 1/p. This places the contrast immediately below the critical $H^{1/2}$ endpoint and motivates the logarithmically refined Hilbert scale.

$$\text{Kummer/Digamma zeros} \leftrightarrow \text{non} - \text{harmonic Riesz bases} \leftrightarrow \text{critical Hilbert spaces} \leftrightarrow \text{additive prime convolution} \quad \text{(A74)}$$

The main remaining question is whether the thin quadratic contrast channel T^2 admits arithmetic and Hilbert-space estimates strong enough to provide information beyond the classical circle-method formulation.