A STRENGTHENING OF THE NYMAN-BEURLING CRITERION FOR THE RIEMANN HYPOTHESIS

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Abstract. Let $\rho(x) = x - [x]$, $\chi = \chi_{(0,1)}$. In $L_2(0,\infty)$ consider the subspace $\mathcal{B}$ generated by $\{\rho_a | a \geq 1\}$ where $\rho_a(x) := \rho\left(\frac{1}{ax}\right)$. By the Nyman-Beurling criterion the Riemann hypothesis is equivalent to the statement $\chi \in \overline{\mathcal{B}}$. For some time it has been conjectured, and proved in this paper, that the Riemann hypothesis is equivalent to the stronger statement that $\chi \in \overline{\mathcal{B}^{nat}}$ where $\mathcal{B}^{nat}$ is the much smaller subspace generated by $\{\rho_a | a \in \mathbb{N}\}$.

1. Introduction

We denote the fractional part of x by $\rho(x) = x - [x]$, and let χ stand for the characteristic function of the interval $(0, 1]$. μ denotes the Möbius function. We shall be working in the Hilbert space

$$\mathcal{H} := L_2(0,\infty),$$

where the main object of interest is the subspace of *Beurling functions*, defined as the linear hull of the family $\{\rho_a | 1 \leq a \in \mathbb{R}\}$ with

$$\rho_a(x) := \rho\left(\frac{1}{ax}\right).$$

The much smaller subspace $\mathcal{B}^{nat}$ of *natural Beurling functions* is generated by $\{\rho_a | a \in \mathbb{N}\}$. The Nyman-Beurling criterion ([13], [6]) states, in a slightly modified form [4] (the original formulation is related to $L_2(0,1)$), that the Riemann hypothesis is equivalent to the statement that

$$\chi \in \overline{\mathcal{B}},$$

but it has recently been conjectured by several authors[1] that this condition could be substituted by $\chi \in \overline{\mathcal{B}^{nat}}$. We state this as a theorem to be proved below.

Theorem 1.1. *The Riemann hypothesis is equivalent to the statement that*

$$\chi \in \overline{\mathcal{B}^{nat}}.$$

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[1]see [1], [2], [3], [4], [5], [8], [9], [10], [11], [12], [16], [17]

To properly gauge the strength of this theorem note this: not only is $\mathcal{B}^{nat}$ a rather thin subspace of $\mathcal{B}$ but, as is easily seen, it is also true that $\overline{\mathcal{B}}$ is much larger than $\overline{\mathcal{B}^{nat}}$.

By necessity all authors have been led in one way or another to the *natural approximation*

$$F_n := \sum_{a=1}^{n} \mu(a)\rho_a, \tag{1.1}$$

which tends to $-\chi$ both a.e. and in L_1 norm when restricted to $(0,1)$ (see [1]), but which has been shown ([2], [3]) to diverge in $\mathcal{H}$. In unpublished work one has tried to prove convergence under the Riemann hypothesis of subsequences of $\{F_n\}$ such as when n is restricted to run along the solutions of $\sum_{a=1}^{n} \mu(n) = 0$. Another attempt by J. B. Conrey and G. Myerson [8] relates to a mollification of F_n, the *Selberg approximation*, defined in [4] by

$$S_n := \sum_{a=1}^{n} \mu(a) \left(1 - \frac{\log a}{\log n}\right) \rho_a.$$

A common problem to these sequences is that if they converge at all to $-\chi$ in $\mathcal{H}$ they must do so very slowly: it is known [4] that for any $F = \sum_{k=1}^{n} c_k \rho_{a_k}$, $a_k \geq 1$, if $N = \max a_k$, then

$$\|F - \chi\|_{\mathcal{H}} \geq \frac{C}{\sqrt{\log N}}, \tag{1.2}$$

for an absolute constant C that has recently been sharpened by J. F. Burnol [7]. This, as well as considerations of *summability* of series, led the author in [3] as well as here to try to employ symultaneously, as it were, the whole range of $a \in [1, \infty)$. Thus we define for complex s and $x > 0$ the functions

$$f_s(x) := \sum_{a=1}^{\infty} \frac{\mu(a)}{a^s} \rho_a(x). \tag{1.3}$$

For fixed $x > 0$ this is a meromorphic functions of s in the complex plane since

$$f_s(x) = \frac{1}{x\zeta(s+1)} - \sum_{a \leq 1/x} \frac{\mu(a)}{a^s} \left[\frac{1}{ax}\right],$$

where the finite sum on the right is an entire function; thus f_s is seen to be a sort of correction of $1/\zeta(s)$. Assuming the Riemann hypothesis we shall prove for small positive ϵ that

$$f_\epsilon \in \overline{\mathcal{B}^{nat}},$$

and then, *unconditionally*, that

$$f_\epsilon \overset{\mathcal{H}}{\to} -\chi, \quad (\epsilon \downarrow 0),$$

so that $\chi \in \overline{\mathcal{B}^{nat}}$.

2. The Proof

2.1. **Two technical lemmae.** Here $s = \sigma + i\tau$ with σ and τ real. The well-known theorem of Littlewood (see [15] Theorem 14.25 (A)) to the effect that under the Riemann hypothesis $\sum_{a=1}^{\infty} \mu(a)a^{-s}$ converges to $1/\zeta(s)$ for $\Re(s) > 1/2$ has been provided in the more general setting of $\Re(s) > \alpha$ with a precise error term by M. Balazard and E. Saias ([5], Lemme 2). We quote their lemma here for the sake of convenience.

Lemma 2.1. *Let $1/2 \leq \alpha < 1$, $\delta > 0$, and $\epsilon > 0$. If $\zeta(s)$ does not vanish in the half-plane $\Re(s) > \alpha$, then for $n \geq 2$ and $\alpha + \delta \leq \Re(s) \leq 1$ we have*

$$\sum_{a=1}^{n} \frac{\mu(a)}{a^s} = \frac{1}{\zeta(s)} + O_{\alpha,\delta,\epsilon}\left(n^{-\delta/3}(1+|\tau|)^\epsilon\right) \tag{2.1}$$

It is important to note that the next lemma is independent of the Riemann or even the Lindelöf hypothesis.

Lemma 2.2. *For $0 \leq \epsilon \leq \epsilon_0 < 1/4$ there is a positive constant $C = C(\epsilon_0)$ such that for all τ*

$$\left|\frac{\zeta(\frac{1}{2} - \epsilon + i\tau)}{\zeta(\frac{1}{2} + \epsilon + i\tau)}\right| \leq C\,(1+|\tau|)^\epsilon. \tag{2.2}$$

Proof. We bring in the functional equation of $\zeta(s)$ to bear as follows

$$\begin{aligned}\left|\frac{\zeta(\frac{1}{2} - \epsilon + i\tau)}{\zeta(\frac{1}{2} + \epsilon + i\tau)}\right| &= \left|\frac{\zeta(\frac{1}{2} - \epsilon - i\tau)}{\zeta(\frac{1}{2} + \epsilon + i\tau)}\right| \\ &= \pi^{-\epsilon}\left|\frac{\Gamma(\frac{1}{4} + \frac{1}{2}\epsilon + \frac{1}{2}i\tau)}{\Gamma(\frac{1}{4} + \frac{1}{2}\epsilon + \frac{1}{2}i\tau)}\right|,\end{aligned}$$

then the conclusion easily follows from well-known asymptotic formulae for the gamma function in a vertical strip ([14] (21.51), (21.52)). ∎

2.2. **The proof proper of Theorem 1.1.** It is clear that we need not prove the *if* part of Theorem 1.1. So let us assume that the Riemann hypothesis is true. We define

$$f_{\epsilon,n} := \sum_{a=1}^{n} \frac{\mu(a)}{a^\epsilon}\rho_a, \quad (\epsilon > 0).$$

It is easy to see that

$$f_{\epsilon,n}(x) = \frac{1}{x}\sum_{a=1}^{n}\frac{\mu(a)}{a^{1+\epsilon}} - \sum_{a=1}^{n}\frac{\mu(a)}{a^{\epsilon}}\left[\frac{1}{ax}\right], \tag{2.3}$$

then, noting that the terms of the right-hand sum drop out when $a > 1/x$, we obtain the pointwise limit

$$f_{\epsilon}(x) = \lim_{n\to\infty} f_{\epsilon,n}(x) = \frac{1}{x\zeta(1+\epsilon)} - \sum_{a\le 1/x}\frac{\mu(a)}{a^{\epsilon}}\left[\frac{1}{ax}\right]. \tag{2.4}$$

Then again for fixed $x > 0$ we have

$$\lim_{\epsilon\downarrow 0} f_{\epsilon}(x) = -\sum_{a\le 1/x}\mu(a)\left[\frac{1}{ax}\right] = -\chi(x), \tag{2.5}$$

by the fundamental property on Möbius numbers. The task at hand now is to prove these pointwise limits are also valid in the $\mathcal{H}$-norm. To this effect we introduce a new Hilbert space

$$\mathcal{K} := L_2((\infty,\infty),(2\pi)^{-1/2}dt),$$

and note that by virtue of Plancherel's theorem the Fourier-Mellin map $\mathbf{M}$ defined by

$$\mathbf{M}(f)(\tau) := \int_0^{\infty} x^{-\frac{1}{2}+i\tau}f(x)dx, \tag{2.6}$$

is an invertible isometry from $\mathcal{H}$ to $\mathcal{K}$. A well-known identity, which is at the root of the Nyman-Beurling formulation, probably due to Titchmarsh ([15], (2.1.5)), namely

$$-\frac{\zeta(s)}{s} = \int_0^{\infty} x^{s-1}\rho_1(x)dx, \quad (0 < \Re(s) < 1),$$

immediately yields, denoting $X_{\epsilon}(x) = x^{-\epsilon}$,

$$\mathbf{M}(X_{\epsilon}f_{2\epsilon,n})(\tau) = -\frac{\zeta(\frac{1}{2}-\epsilon+i\tau)}{\frac{1}{2}-\epsilon+i\tau}\sum_{a=1}^{n}\frac{\mu(a)}{a^{\frac{1}{2}+\epsilon+i\tau}}, \quad (0<\epsilon<1/2). \tag{2.7}$$

By a theorem of Littlewood ([15], Theorem 14.25 (A)) if we let $n\to\infty$ in the right-hand side of (2.7) we get the pointwise limit

$$-\frac{\zeta(\frac{1}{2}-\epsilon+i\tau)}{\frac{1}{2}-\epsilon+i\tau}\sum_{a=1}^{n}\frac{\mu(a)}{a^{\frac{1}{2}+\epsilon+i\tau}} \to -\frac{\zeta(\frac{1}{2}-\epsilon+i\tau)}{\zeta(\frac{1}{2}+\epsilon+i\tau)}\frac{1}{\frac{1}{2}-\epsilon+i\tau}. \tag{2.8}$$

To see that this limit also takes place in $\mathcal{H}$ we choose the parameters in Lemma 2.1 as $\alpha = 1/2$, $\delta = \epsilon > 0$, $\epsilon \leq 1/2$, and $n \geq 2$ to obtain

$$\sum_{a=1}^{n} \frac{\mu(a)}{a^{\frac{1}{2}+\epsilon+i\tau}} = \frac{1}{\zeta(\frac{1}{2}+\epsilon+i\tau)} + O_\epsilon\left((1+|\tau|)^\epsilon\right).$$

If we now use Lemma 2.2 and the Lindelöf hypothesis applied to the abcissa $1/2-\epsilon$, which follows from the Riemann hypothesis, we obtain a positive constant K_ϵ such that for all real τ

$$\left| -\frac{\zeta(\frac{1}{2}-\epsilon+i\tau)}{\frac{1}{2}-\epsilon+i\tau} \sum_{a=1}^{n} \frac{\mu(a)}{a^{\frac{1}{2}+\epsilon+i\tau}} \right| \leq K_\epsilon (1+|\tau|)^{-1+2\epsilon}.$$

It is then clear that for $0 < \epsilon < 1/4$ the left-hand side of (2.8) is uniformly majorized by a function in $\mathcal{K}$. Thus the convergence does take place in $\mathcal{K}$ which implies that

$$X_\epsilon f_{2\epsilon,n} \xrightarrow{\mathcal{H}} X_\epsilon f_{2\epsilon}.$$

But $x^{-\epsilon} > 1$ for $0 < x < 1$, and for $x > 1$

$$f_{2\epsilon,n}(x) = \frac{1}{x}\sum_{a=1}^{n} \frac{\mu(a)}{a^{1+\epsilon}} \ll \frac{1}{x}, \quad (x > 1), \tag{2.9}$$

which easily implies that one also has $\mathcal{H}$-convergence for $f_{2\epsilon,n}$ as $n \to \infty$. The factor 2 in the subindex is unessential, so that we now have for sufficiently small $\epsilon > 0$ that

$$f_{\epsilon,n} \xrightarrow{\mathcal{H}} f_\epsilon \in \overline{\mathcal{B}^{nat}},$$

as was announced above. Moreover, since we have identified the pointwise limit in (2.8) we now have

$$\mathbf{M}(X_\epsilon f_{2\epsilon})(t) = -\frac{\zeta(\frac{1}{2}-\epsilon+i\tau)}{\zeta(\frac{1}{2}+\epsilon+i\tau)} \frac{1}{\frac{1}{2}-\epsilon+i\tau}.$$

Now we apply Lemma 2.2 and obtain, *without the assumption of the Riemann hypothesis*, that $\mathbf{M}(X_\epsilon f_{2\epsilon})$ converges in $\mathcal{K}$, thus $X_\epsilon f_{2\epsilon}$ converges in $\mathcal{H}$, and this means that f_ϵ also converges in $\mathcal{H}$ as $\epsilon \downarrow 0$ by an argument entirely similar to that used for $f_{\epsilon,n}$. The identification of the pointwise limit in (2.5) finally gives

$$f_\epsilon \xrightarrow{\mathcal{H}} -\chi,$$

which concludes the proof.

3. Some comments and a corollary

The proof of Theorem 1.1 provides in turn a new proof, albeit of a stronger theorem, of the Nyman-Beurling criterion which bypasses the deep and complicated Hardy space techniques. One should extend it to the L_p case, that is, to the condition that $\zeta(s)$ does not vanish in a half-plane $\Re(s) > 1/p$. It should be clear also that we have shown this special equivalence criterion to be true:

Corollary 3.1. *The Riemann hypothesis is equivalent to the $\mathcal{H}$-convergence of $f_{\epsilon,n}$ as $n \to \infty$ for all sufficiently small $\epsilon > 0$.*

In essence what has been done is to apply a summability method to the old natural approximation. The convergence on special subsequences both of n and of ϵ is also necessary and sufficient, and it is proposed here to study this alongside with other summability methods for the natural approximation.
A final remark is in order. Note that we did not employ the dependence on n in the Balazard-Saias Lemma 2.1. This dependence would seem to be closely connected to the slowness of approximation to $-\chi$ indicated in (1.2).

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