A PROOF OF THE RIEMANN HYPOTHESIS

enabled by

a new integral and series representation of the meromorphic Zeta function occuring in the symmetrical form of the Riemann functional equation

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Dedicated to my wife Vibhuta
on the occasion of her 62th birthday, August 25, 2023 (*)

Abstract

For $s \neq \nu, \nu \in Z$, Riemann's meromorphic Zeta function

$$\xi^{*}(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^{*}(1-s)$$

is represented in the form

$$\xi^{*}(s) = \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s-\frac{1}{2})^{2n}$$

with

$$b_{2n} := \int_1^\infty \Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}} \quad \text{and} \quad \Phi(x) := \sum_{n=1}^{\infty}(e^{-2\pi n x} - e^{-\pi n^2 x^2}) .$$

The non-trivial zeros $\left\{s_n = \frac{1}{2} + it_n\right\}$ of the Zeta function are characterized by the identity of two convergent series representations in the form

$$\sum_{n=0}^{\infty}(-1)^n b_{2n} t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{4n-1}{(2n-\frac{1}{2})^2 + t_n^2}\right],$$

which do not allow negative values $t_n^{2n} < 0$. The correspondingly defined entire Zeta function $\xi^{**}(s) := \sin(\pi s)\,\xi^{*}(s)$ can be represented in the form

$$\xi^{**}(s) = \xi^{*}(s)\sin(\pi s) = (1-s)\frac{\sin(\pi s)}{\pi s}\left[\pi^{-\frac{s}{2}}\frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s}\right]\zeta(s) \text{ (**)}.$$

In the critical stripe the term in bracket is the Mellin transform of a Kummer function

$$M\left[{}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi x^2\right)\right](s) = \pi^{-\frac{s}{2}}\frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s} \quad , \quad 0 < Re(s) < 1 \text{ (***)} .$$

(*) error correction in December 2024: the term (**)

(**) the representation is in line with the concept of *„a self-adjoint operator with transform $\bar{\xi}(s)$"*, (EdH) 10.3. The concept is also in line with the proposed Kummer function based Zeta function theory and a related alternatively proposed two-semicircle method to the Hardy-Littewood (major/minor arcs based) circle method in (BrK).

(***) the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2},z\right)$ is accompanied by the product representation ${}_1F_1\left(\frac{1}{2},\frac{3}{2},z\right) = \frac{\sqrt{\pi}}{2}e^{\frac{z}{3}}\prod(1-\frac{z}{z_n})e^{z/z_n}$ with only complex valued zeros with $Re(z_n) > 1/2$ and imaginary parts lying in the horizontal stripes $(2n-1)\pi < |Im(z_n)| < 2\pi n, n \in N$.

1. Notations and Main Theorem

For the notations we refer to (EdH). The baseline function for the Zeta function theory is given by $\psi(x) := \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, (EdH) 1.7. It is related to Jacobi's functional equation of the theta function ϑ enabling the symmetrical form of Riemann's functional equation in the form, (EdH) 1.7,

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s) .$$

Riemann's related entire Zeta function is given by $\xi(s) = \frac{s}{2}(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$, (EdH) 1.8. Alternatively to $\psi(x^2)$ we shall apply the function

$$\Phi(x) := \varphi(x) - \psi(x^2) := \sum_{n=1}^{\infty} \Phi_n(x) := \sum_{n=1}^{\infty}(e^{-2\pi n x} - e^{-\pi n^2 x^2}) , x \geq 1 \, ^{(*)}.$$

The main result of our paper is an alternative $\xi^*(s)$ –function representation with three $s \leftrightarrow (1-s)$ symmetric summands in the form

$$\xi^*(s) = \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s-\tfrac{1}{2})^{2n}$$

with

$$b_{2n} := \int_1^{\infty} \Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}} .$$

A correspondingly defined entire Zeta function $\xi^{**}(s)$ is built by multiplication of $\xi^*(s)$ with $\frac{\sin(\pi s)}{\pi}$ (**), i.e.

$$\xi^{**}(s) = \xi^*(s)\sin(\pi s) = (1-s)\frac{\sin(\pi s)}{\pi s}\left[\pi^{-\frac{s}{2}}\frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s}\right]\zeta(s).$$

The term $(1-s)$ corresponds to the principle term of the Riemann density function $J(x)$ (***); the term $\frac{\sin(\pi s)}{\pi s}$ is an entire function of order one and order type $\sigma = \pi/2$ with $\frac{\sin(\pi s)}{\pi s} = \prod_{n=1}^{\infty}\left(1 - \frac{s^2}{n^2}\right) = \prod'^{\,n=\infty}_{n=-\infty}\left(1 - \frac{s}{n}\right)e^{s/n}$, (LeB) p. 32. For $0 < \mathrm{Re}(s) < 1$ the term in bracket is the Mellin transform of the Kummer function ${}_1F_1\left(\frac{1}{2};\frac{3}{2}, -\pi x^2\right)$ (****) i.e., *formally* with $\omega(x) := \sum_{n=1}^{\infty} {}_1F_1\left(\frac{1}{2};\frac{3}{2}, -n^2 x^2\right)$ the zeta function $\zeta(s)$ can be represented in the form $\zeta(s) := \frac{1-s}{\Gamma\left(1+\frac{s}{2}\right)}\int_0^{\infty} x^s\omega(x)\frac{dx}{x}$ (*****) . For further details including the non-negative zeros of the Digamma function we refer to the concept of „telescoping product representations" in (BrK).

(*) Note: $\Phi_n(x) \geq 0$ for $n \geq 2, x \geq 1$; $\Phi_1(x) < 0$ for $1 \leq x < 2$; $\Phi_1(2) = 0$; $\Phi_1(x) > 0$ for $x > 2$. Putting $\Phi_{1,2}^*(x) := -\Phi_1(x)$, $\Phi_{0,1}^*(x) := -\Phi_1\left(\frac{1}{x}\right)$, for $1 \leq x < 2$, $\Phi_{1,2}^*(x) = \Phi_{0,1}^*(x) = 0$ for $x \geq 2$, $\Phi_{2,\infty}^*(x) := \Phi(x)$ for $x \geq 2$, $\Phi_{2,\infty}^*(x) = 0$ for $x < 2$, the three terms of the sum $\Phi_{0,1}^*(x) + \Phi_{1,2}^*(x) + \Phi_{2,\infty}^*(x) > 0$ have the disjunct domains $0 < x < 1$, $1 \leq x < 2$, $2 \leq x < \infty$; we note Riemann's density function connection between $\zeta(s)$ and the primes given by $J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^s\frac{ds}{s}$, $(a > 1)$, which is zero for $0 \leq x < 2$, (EdH) 1.11;
we also note the formula $\int_1^{\infty} x^{2m}\varphi(x)\frac{dx}{x} = \frac{|B_{2m}|}{2m}$, (GrI) 3.552

(**) Riemann built his famous power series representation of his entire Zeta function $\xi(s) := \pi^{-\frac{s}{2}}\frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\zeta(s)$ by multiplication of $\xi^*(s) = \int_1^{\infty}\psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}$ with $s(s-1)$ to govern the two poles of the term $-\frac{1}{2}\frac{1}{s(1-s)} = -\frac{1}{2}\frac{1}{s(1-s)} = -\frac{1}{2}\left[\frac{1}{s} + \frac{1}{1-s}\right]$; multiplication of this term with $\sin(\pi s)$ gives $\frac{1}{2}\frac{\sin(\pi s)}{s(1-s)} = \sin\left(\frac{\pi}{2}s\right)\cos\left(\frac{\pi}{2}s\right)\left[\frac{1}{s} + \frac{1}{1-s}\right] = \cos\left(\frac{\pi}{2}s\right)\left[\frac{\sin\left(\frac{\pi}{2}s\right)}{s} + \frac{\sin\left(\frac{\pi}{2}(1-s)\right)}{1-s}\right]$; see also (EdH) 10.3 *„A self-adjoint operator with transform $\xi(s)$"*, and (EdH) 10.5 *„$\frac{2\xi(s)}{s(s-1)}$ as a transform"*.

(***) For Riemann's density function $J(x)$ the term $\log(s-1)$ gives $li_1(x) = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log(s-1)}{s}\right]x^s ds$ $(a > 1)$, (EdH) 1.14.

(****) the Mellin transform of the Kummer function in the critical stripe is given by $M\left[{}_1F_1\left(\frac{1}{2};\frac{3}{2}, -\pi x^2\right)\right](s) = \pi^{-\frac{s}{2}}\frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)}$, (GrI) 7.612; ${}_1F_1\left(\frac{1}{2},\frac{3}{2}, x\right) \sim \frac{1}{\sqrt{\pi}}\frac{e^x}{x}$ as $x \to \infty$, (OlF) p. 257); the zeros z_ν, $\nu \in Z - \{0\}$, of ${}_1F_1\left(\frac{1}{2},\frac{3}{2}, z\right)$ are all simple, complex valued with $\mathrm{Re}(z_\nu) > 1/2$, and $(2n-1)\pi < |\mathrm{Im}(z_\nu)| < 2\pi n$, $n \in N$, (SeA); The related product representation is given by ${}_1F_1\left(\frac{1}{2},\frac{3}{2}, z\right) = \frac{\sqrt{\pi}}{2}e^{\frac{z}{3}}\prod(1 - \frac{z}{z_n})e^{z/z_n}$, (BuH) p.184; we note that the „density" properties of the zeros are very much related to the negative zeros w_n of the Digamma function accompanied by the additional asymptotics $y_n := n + w_n \sim \ln(n)^{-1}$; the latter property implies that the negative zeros of the Digamma function fullfill the Kadec-$1/4$ condition, enabling the definition of non-harmonic Fourier series based Riesz bases and related Paley-Wiener functions, (BrK), (LeN), (YoR); regarding Riemann's method deriving the formula for his prime number density function $J(x)$, (EdH) 1.13.

(*****) $M[-xh'](s) = sM[h](s)$; $M[(xh)'](s) = (1-s)M[h](s)$; (TiE) 2.12.6: $(s-1)\zeta(s) = e^{bs}\frac{1}{\Gamma\left(1+\frac{s}{2}\right)}\prod_\rho\left(1 - \frac{s}{\rho}\right)e^{\frac{s}{\rho}}$, with $b := \log(2\pi) - 1 - \gamma/2$.

Main Theorem: For $s \neq \nu$, $\nu \in \mathbb{Z}$, it holds

$$\xi^*(s) = \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right)+\zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s-\tfrac{1}{2})^{2n}$$

and

$$b_{2n} = \int_1^{\infty}\Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}}.$$

In proving the Main Theorem the essential step (which is proven in the next section) is

Lemma MT: For $s \neq \nu$, $\nu \in \mathbb{Z}$, it holds

$$-\frac{1}{2}\frac{1}{s(1-s)} = \frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}+\frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right] - \int_1^{\infty}[x^s+x^{1-s}]\frac{1}{2}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}.$$

Proof of the Main Theorem: With

$$\xi^*(s) = \int_1^{\infty}\psi(x^2)[x^s+x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}$$

and

$$\Phi(x) := \varphi(x) - \psi(x^2) = \sum_{n=1}^{\infty}(e^{-2\pi n x} - e^{-\pi n^2 x^2}),\ x > 1$$

the Lemma MT gives

$$\xi^*(s) = -\int_1^{\infty}\Phi(x)[x^s+x^{1-s}]\frac{dx}{x} + \frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}+\frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right].$$

Analogue to Riemann's approach deriving his famous power series representation for $\xi(s)$, (EdH) 1.8, (*),
the first term can be represented in the form

$$-\int_1^{\infty}\Phi(x)[x^s+x^{1-s}]\frac{dx}{x} = -2\sum_{n=0}^{\infty} b_{2n}(s-\tfrac{1}{2})^{2n}.$$

Corollary: The set of non-trivial zeros $\left\{s_n = \frac{1}{2}+it_n\right\}$ of the zeta function are characterized by the identity of two convergent series representations

$$\sum_{n=0}^{\infty} b_{2n}(s_n-\tfrac{1}{2})^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-z_n}+\frac{\zeta(2n)}{(2n-1)+s_n}\right]$$

resp.

$$\sum_{n=0}^{\infty}(-1)^n b_{2n}t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{4n-1}{(2n-\frac{1}{2})^2+t_n^2}\right].$$

Note: In case there would exist a negative value $t_n^{2n} < 0$, the affected term on the left side changes its sign, while the corresponding term on the right side would not.

(*) $[x^s+x^{1-s}] = 2\sqrt{x}\left[\cosh\left(s-\frac{1}{2}\right)\log x\right]$ and $\cosh(y) = \sum_{n=0}^{\infty}\frac{y^{2n}}{(2n)!}$ with $y := \left(s-\frac{1}{2}\right)\log x$.

2. Proof of the Lemma MT

With

$$\varphi(x)=\frac{1}{2}\frac{e^{-\pi x}}{\sinh(\pi x)}=\frac{1}{e^{2\pi x}-1}=\sum_{n=1}^{\infty}e^{-2\pi nx}\ ,\ x>1$$

the Lemma MT takes the form

Lemma MT: For $s\neq\nu$, $\nu\in Z$, it holds

$$-\frac{1}{2}\frac{1}{s(1-s)}=-\int_1^{\infty}[x^s+x^{1-s}]\varphi(x)+\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}+\frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right]+\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right].$$

Proof:

As $\frac{1}{s-1}+\frac{1}{-s}=\frac{1}{s(s-1)}$ the Lemma MT is a consequence of the integral and series representations as provided in (MiM) in section 4 (*) in the form

$$\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}=\frac{1}{s-1}-\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s}+\int_1^{\infty}x^{1-s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}$$

resp.

$$\frac{\zeta(1-s)}{\sin\left(\frac{\pi}{2}(1-s)\right)}=\frac{1}{-s}-\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{(2n-1)+s}+\int_1^{\infty}x^{s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}.$$

(*) (MiM) p. 7: *Special cases, 4.1 The case $c=0$*

For the special case $c=0$ the integral

$$\zeta(s)=-\pi^{s-1}\frac{\sin\left(\frac{\pi}{2}s\right)}{s-1}\int_0^{\infty}\frac{x^{1-s}}{\sinh^2(x)}dx\ ,\qquad Re(s)<0 \qquad \textit{(MiM) (4.1)}$$

can be broken into two parts $\zeta(s)=\zeta_0(s)+\zeta_1(s)$ where

$$\zeta_1(s)=\frac{\sin\left(\frac{\pi}{2}s\right)}{s-1}+\sin\left(\frac{\pi}{2}s\right)\int_1^{\infty}x^{1-s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x} \qquad \textit{(MiM) (4.6)}$$

$$\zeta_0(s)=-\frac{2}{\pi}\sin\left(\frac{\pi}{2}s\right)\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s} \qquad \textit{(MiM) (4.8)}$$

which are both valid for all s.

3. References

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