

**A Bessel function based proof  
that the Euler-Mascheroni constant  $\gamma$  is irrational**

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Dedicated to my son Mario  
on the occasion of his 30th birthday

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**Abstract**

A strictly monotonous sequence  $\{a_n\}_{n \in \mathbb{N}}$  with  $a_1 = 0$ ,  $a_\infty = \gamma$ , and  $a_n$  transcendental für  $n \geq 2$  is provided which proves the irrationality of the Euler-Mascheroni constant.

**Theorem**

For the Bessel functions  $J_\nu(x)$  we consider the generalized Bessel functions in the form

$$B_\nu(x) := \frac{J_\nu(2x)}{x^\nu} - \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}}$$

with

$$\frac{J_\nu(2x)}{x^\nu} := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k}}{\Gamma(\nu+k+1)} .$$

**Theorem:** The sequence

$$a_n := \int_0^\infty x^{1/n} B_{1/n}(x) \frac{dx}{x} = \int_0^1 \left[ x^{1/n} B_{1/n}(x) + x^{-1/n} B_{1/n}\left(\frac{1}{x}\right) \right] \frac{dx}{x}$$

is strictly monotonous with  $a_1 = 0$ ,  $a_\infty = \gamma$ , and all  $a_n$  are transcendental für  $n \geq 2$ .

### Proof of the theorem

The theorem is the consequence of the following lemmata.

**Lemma 1:** For  $\nu \in \mathbb{C}$ , and  $\alpha$  algebraic with  $\alpha \neq \pm \frac{1}{2}, -1, \pm \frac{3}{2}, -2 \dots$ ,  $J_\nu(\alpha)$  are transcendental.

**Main lemma:**

- i)  $\gamma = \int_0^\infty [J_0(2x) - J_0(2\sqrt{x})] \frac{dx}{x} = \int_0^\infty B_0(x) \frac{dx}{x}$ , with  $J_0(x) = O(x^{-\frac{1}{2}})$  for large  $x$
- ii) for  $0 < \operatorname{Re}(\mu) < \operatorname{Re}(\nu) + \frac{3}{2}$  it holds  $\int_0^\infty x^\mu B_\nu(x) \frac{dx}{x} = \frac{1}{\mu} \left[ \frac{\Gamma(1+\frac{\mu}{2})}{\Gamma(1+\nu-\frac{\mu}{2})} - \frac{\Gamma(1+\mu)}{\Gamma(1+\nu-\mu)} \right]$
- iii) for  $0 \leq \alpha < \beta$ ,  $0 \leq x < 1$  it holds  $B_\beta(x) - B_\alpha(x) > 0$ .

Proof: i) - iii) see lemmata 2-4 below.

Putting  $\mu = \nu = \frac{1}{n}$  and

$$a_n := \int_0^\infty x^{1/n} B_{1/n}(x) \frac{dx}{x} = \int_0^1 \left[ x^{1/n} B_{1/n}(x) + x^{-1/n} B_{1/n}\left(\frac{1}{x}\right) \right] \frac{dx}{x}$$

it follows

- i)  $a_1 = 0$ ,  $a_\infty = \int_0^\infty B_0(x) \frac{dx}{x} = \gamma$
- ii)  $a_n = n \left[ 1 - \Gamma\left(1 + \frac{1}{n}\right) \right] < a_{n+1}$  (\*)
- iii) for  $n > 1$  all  $a_n$  are transcendental.

(\*) (AbM) p. 255: for  $1 \leq x \leq 2$  it holds

- i)  $\max\{\Gamma(x)\} = \Gamma(1) = \Gamma(2) = 1$ ,
- ii)  $\Gamma(x)$  is convex with minimum at  $\Gamma(3/2) \sim 0,88622\dots = 1 - 0,11378\dots$
- iii)  $1/\Gamma(x)$  is concave with maximum at  $\frac{1}{\Gamma(3/2)} \sim 1 + 0,11378\dots$

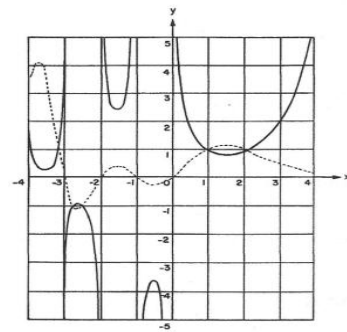


FIGURE 6.1. Gamma function. \*

—,  $y = \Gamma(x)$ , - - -,  $y = 1/\Gamma(x)$

**Lemma 2** (BrR), (WaG) 13-24, (WeH): For  $0 < \operatorname{Re}(\mu) < \operatorname{Re}(\nu) + \frac{3}{2}$  it holds

$$\int_0^\infty x^\mu \frac{J_\nu(x)}{x^\nu} \frac{dx}{x} = \frac{\Gamma\left(\frac{\mu}{2}\right)}{2^{\nu-\mu+1} \Gamma\left(1+\nu-\frac{\mu}{2}\right)}.$$

From

$$\frac{1}{2} \int_0^\infty x^{\mu/2} \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}} \frac{dx}{x} = \frac{\frac{1}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(1+\nu-\frac{\mu}{2}\right)} \text{ resp. } \int_0^\infty x^\mu \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}} \frac{dx}{x} = \frac{\Gamma(\mu)}{\Gamma(1+\nu-\mu)} = \frac{1}{\mu} \frac{\Gamma(1+\mu)}{\Gamma(1+\nu-\mu)}$$

it therefore follows

**Lemma 3:** for  $0 < \operatorname{Re}(\mu) < \operatorname{Re}(\nu) + \frac{3}{4}$  it holds

$$\int_0^\infty x^\mu B_\nu(x) \frac{dx}{x} = \frac{1}{\mu} \left[ \frac{\Gamma\left(1+\frac{\mu}{2}\right)}{\Gamma\left(1+\nu-\frac{\mu}{2}\right)} - \frac{\Gamma(1+\mu)}{\Gamma(1+\nu-\mu)} \right].$$

**Corollary:** It holds

$$\text{i) } \int_0^\infty t^{\mu-1} J_0(2t) dt = \frac{\frac{\mu}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(1-\frac{\mu}{2}\right)} \text{ for } 0 < \operatorname{Re}(\mu) < \frac{3}{2}$$

$$\text{ii) } \gamma = \int_0^\infty [J_0(2x) - J_0(2\sqrt{x})] \frac{dx}{x}.$$

Proof: i) follows from the lemma 2 above; in (BrR) it is proven that  $\int_0^\infty [e^{-t} - J_0(2t)] \frac{dt}{t} = 0$  based on the asymptotics formula

$$\Gamma(\mu) - \frac{\frac{1}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(1-\frac{\mu}{2}\right)} = \frac{1}{\mu} \left[ \Gamma(1+\mu) - \frac{\Gamma\left(1+\frac{\mu}{2}\right)}{\Gamma\left(1-\frac{\mu}{2}\right)} \right] = \frac{1}{\mu} \left[ (1-\gamma\mu) - \frac{(1-\gamma\mu)}{(1+\gamma\mu)} + O(\mu^2) \right] = O(\mu) \rightarrow_{\mu \rightarrow 0^+} 0.$$

The corresponding asymptotics

$$\Gamma(\mu) - \frac{\Gamma(\mu)}{\Gamma(1-\mu)} = \frac{1}{\mu} \left[ \Gamma(1+\mu) - \frac{\Gamma(1+\mu)}{\Gamma(1-\mu)} \right] = \frac{1}{\mu} \left[ (1-\gamma\mu) - \frac{(1-\gamma\mu)}{(1+\gamma\mu)} + O(\mu^2) \right] = \gamma + O(\mu) \rightarrow_{\mu \rightarrow 0^+}$$

result into  $\int_0^\infty [e^{-t} - J_0(2\sqrt{t})] \frac{dt}{t} = \gamma$ , with same asymptotics as  $\frac{\frac{1}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(1-\frac{\mu}{2}\right)} - \frac{\Gamma(\mu)}{\Gamma(1-\mu)}$ .

From the representation

$$B_\nu(x) := \frac{J_\nu(2x)}{x^\nu} - \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}} = \sum_{k=0}^\infty (-1)^{k+1} \frac{x^k}{k!} \frac{(1-x^k)}{\Gamma(\nu+k+1)}$$

one gets

**Lemma 4:** for  $0 \leq \alpha < \beta$ ,  $0 < x < 1$  it holds  $B_\beta(x) - B_\alpha(x) > 0$ .

Proof: The lemma follows from the equation

$$\begin{aligned} B_\beta(x) - B_\alpha(x) &= \sum_{k=0}^\infty (-1)^{k+1} \frac{x^k}{k!} \frac{(1-x^k)}{\Gamma(\beta+k+1)} - \sum_{k=0}^\infty (-1)^{k+1} \frac{x^k}{k!} \frac{(1-x^k)}{\Gamma(\alpha+k+1)} \\ &= \sum_{k=0}^\infty (-1)^k \frac{x^k (1-x^k)}{k!} \left[ \frac{1}{\Gamma(\alpha+k+1)} - \frac{1}{\Gamma(\beta+k+1)} \right] > 0. \end{aligned}$$

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