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A Representation of the Fundamental Solution for the Fokker–Planck Equation and Its Application

Chisato Iwasaki

Abstract. The aim of this paper is to construct the fundamental solution for the Fokker–Planck equation by calculus of pseudo-differential operators. Eigenfunction expansion of the Fokker–Planck operator is obtained by this exact construction.

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1. Introduction

A. Melin [8] characterized the symbol of differential operators which satisfy a sub-elliptic estimate. C. Iwasaki and N. Iwasaki [6], [7] constructed the fundamental solution for the heat equations corresponding to degenerate operators which A. Melin characterized. They showed that the fundamental solution is obtained as a pseudo-differential operator of the Weyl symbol and they gave the exact expression of its main part represented in terms of derivatives of the second order of the principal symbol and the sub-principal symbol of degenerate operators. If the symbol of operators are quadratic polynomials with respect of x and ξ, the main part of the fundamental solution coincides with the fundamental solution itself (see also O. Calin, D.-C. Chang, K. Furutani, C. Iwasaki [2]). This fact is useful to show that the spectral zeta function for sub-Laplacians on nilmanifolds has only one singularity (see W. Bauer, K. Furutani, C. Iwasaki [1]).

In this paper, we apply this expression of the fundamental solution to the Fokker–Planck equation. We obtain the exact symbol of the fundamental solution

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and the eigenfunction expansion to the Fokker–Planck operator as its application. The conclusions are different corresponding to the potentials of operators. It is shown that the generalized eigenfunctions appear under some potentials.

The plan of this paper is as follows: In Section 2 we introduce the Fokker–Planck operator which will be studied in this paper and give some remarks. Section 3 is devoted to the exact symbol of the fundamental solution for the heat equation in case the symbol is a polynomial. In the next section we apply this expression to the Fokker–Planck equation and obtain one of the main theorems of this paper. The following section is devoted to its proof. In Section 6 a key proposition is stated which is useful if one gets the eigenfunction expansion by the symbol of the fundamental solution as a pseudo-differential operator of Weyl symbol. The last section is devoted to obtain the eigenfunction expansion of the Fokker–Planck operator, applying the key proposition.

2. Fokker–Planck equation

We call the following operator K the Fokker–Planck (Kramers) operator on $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$ (see H. Riskin [9], Helffer and Nier [3]):

$$\begin{aligned} K &= v \cdot \partial_x - \big(\partial_x V(x)\big) \cdot \partial_v - \triangle_v + \frac{|v|^2}{4} - \frac{n}{2} \\ &= X_0 - \triangle_v + \frac{|v|^2}{4} - \frac{n}{2}. \end{aligned}$$

We note that the operator K is hypo-elliptic but not self-adjoint. $\Phi(x,v)$ is a classical Hamiltonian

$$\Phi(x,v) = \frac{|v|^2}{2} + V(x)$$

and X_0 is the corresponding Hamiltonian vector field

$$X_0 = v \cdot \partial_x - \big(\partial_x V(x)\big) \cdot \partial_v.$$

There are many studies about relations between the Fokker–Planck operators and the Witten Laplacian

$$\triangle_{\Phi/2} = -\triangle + \frac{1}{4}|\nabla \Phi|^2 - \frac{1}{2}\triangle \Phi$$

(see [3]).

Helffer and Nier [3] showed that the following inequality holds under some assumption for $V(x)$

$$\|\Lambda_x^{2/3} u\|^2 + \|\Lambda_v^2 u\|^2 \le C(\|Ku\|^2 + \|u\|^2), \tag{2.1}$$

where $\Lambda_x = (-\triangle_x + x^2/4)^{1/2}, \Lambda_v = (-\triangle_v + v^2/4)^{1/2}$.

In F. Hérau and F. Nier [5] the following estimate was studied

$$||u(t) - u^*||_{L^2} \le Ce^{-\tau t} \tag{2.2}$$

for the following solution $u(t)$ under some assumptions on $V(x)$:

$$\frac{d}{dt}u(t) + Ku(t) = 0, \quad t > 0, \quad u(0) = u_0.$$

We call an operator $E(t)$ the fundamental solution of the Fokker–Planck equation if $E(t)$ satisfies

$$\frac{d}{dt}E(t) + KE(t) = 0, \quad t > 0, \quad E(0) = I.$$

Our aim is to give the precise expression of the fundamental solution $E(t)$ as a pseudo-differential operator of Weyl symbol and to get the eigenfunction expansion of K as its application. The estimates (2.1) is obtained by the construction of the fundamental solution as we will mention in Section 4. Because we can choose $\int_0^T E(t)dt \quad (T > 0)$ as a parametrix for K. We have also (2.2) by the eigenfunction expansion. We note that K may have the generalized eigenvalues if $V(x)$ is chosen suitably, which was avoided in [9].

3. Expression of the fundamental solution

A pseudo-differential operator P on $\mathbb{R}^d$ of a Weyl symbol $p(x,\xi)$ is defined by:

$$Pu(x) = p(x,D)u(x) = \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{i(x-y)\cdot\xi} p(\frac{x+y}{2},\xi)u(y)d\xi dy$$

We will construct the following fundamental solution $E(t)$ for a heat equation for an operator P whose the principal symbol is nonnegative:

$$\left(\frac{d}{dt} + P\right)E(t) = 0, \quad E(0) = I.$$

The following Theorem 3.1 has been obtained (see [7], [2]).

Theorem 3.1. *Suppose $p(x,\xi)$ is a quadratic polynomial with respect $X^t = (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$*

$$p = \frac{1}{2}\langle X, HX\rangle + i\langle X, p_0\rangle + b$$

with a nonnegative real symmetric $2d \times 2d$ matrix H, $p_0 \in \mathbb{C}^{2d}$ and $b \in \mathbb{C}$. Then $E(t)$ is constructed as pseudo-differential operator whose symbol $e(t;x,\xi)$ is given by

$$e(t;x,\xi) = \frac{e^{-bt}}{\sqrt{\det\cosh(At/2)}}\exp\Big[-i\Big\{\langle J\tanh(At/2)X, X\rangle + t\langle J\tanh(At/2)(At/2)^{-1}X, Jp_0\rangle + \frac{t^2}{4}\langle JG(At/2)Jp_0, Jp_0\rangle\Big\}\Big],$$

where

$$G(x) = (1 - x^{-1}\tanh x)/x$$

and $A = iJH$ *is a* $2d \times 2d$ matrix*:*

$$J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}, \quad I_d \text{ is the } d \times d \text{ identity matrix.}$$

In our case $d = 2n$, $X =^t (x, v, \xi, \rho)$

$$\sigma(K) = p(x, v, \xi, \rho) = i\Big(\langle v, \xi\rangle - \langle \partial_x V(x), \rho\rangle\Big) + |\rho|^2 + \frac{|v|^2}{4} - \frac{n}{2}.$$

4. The exact form of the fundamental solution to the Fokker–Planck equation

Suppose $V(x) = a \cdot x + \frac{1}{2}\langle \varepsilon x, x\rangle$ with a symmetric matrix ε. We have the following theorem by Theorem 3.1, setting

$$H = \begin{pmatrix} 0 & 0 & 0 & -i\varepsilon \\ 0 & \frac{1}{2}I_n & iI_n & 0 \\ 0 & iI_n & 0 & 0 \\ -i\varepsilon & 0 & 0 & 2I_n \end{pmatrix}, \quad p_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -a \end{pmatrix}, \quad b = -\frac{n}{2}$$

and A is a $4n \times 4n$ matrix of the form

$$A = \begin{pmatrix} 0 & -I_n & 0 & 0 \\ \varepsilon & 0 & 0 & 2iI_n \\ 0 & 0 & 0 & -\varepsilon \\ 0 & -\frac{i}{2}I_n & I_n & 0 \end{pmatrix}.$$

We have the following result for the symbol $e(t; x, v, \xi, \rho)$ of the fundamental solution $E(t)$ for the Fokker–Planck equation.

Theorem 4.1. (I) *If* $V(x) = a \cdot x$, *we have*

$$e(t; x, v, \xi, \rho) = \Big(\frac{1 + e^{-t}}{2}\Big)^{-n} \exp\Big[- 2\tanh(t/2)\left\{|\rho|^2 + |v|^2/4 + i(\langle v, \xi\rangle - \langle a, \rho\rangle)\right\} - 2\,(t/2 - \tanh(t/2))\,(|\xi|^2 + |a|^2/4)\Big].$$

(II) *Assume* $n = 1$ *and* $\varepsilon = \mu \neq 0$. *Let* δ *be the following constant depending on* μ*:*

$$\delta = \begin{cases} \sqrt{1 - 4\mu}, & \text{for } -\infty < \mu \leq \frac{1}{4}, \\ i\sqrt{4\mu - 1}, & \text{for } \mu > \frac{1}{4}. \end{cases}$$

(II-1) *If* $\mu \neq \frac{1}{4}$, *then*

$$e(t; x, v, \xi, \rho) = C_\delta(t) \exp\big\{-2\phi(t)\big\},$$

where $\phi(t)$ and $C_\delta(t)$ are given by the following formula:

$$
\begin{aligned}
C_\delta(t) &= 4\Big\{1 + e^{-t} + 2e^{-t/2}\cosh(\delta t/2)\Big\}^{-1}, \\
\phi(t) &= F_1(t)\left(|\rho|^2 + |v|^2/4\right) + F_2(t)\left(|\xi|^2 + |\mu x + a|^2/4\right) \\
&\quad + iF_3(t)\left(\langle v, \xi\rangle - \langle \mu x + a, \rho\rangle\right),
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(t) &= \frac{1}{\delta}\Big(\lambda_1 g(\lambda_1 t/2) - \lambda_2 g(\lambda_2 t/2)\Big), \\
F_2(t) &= \frac{1}{\delta}\Big(\frac{1}{\lambda_2} g(\lambda_2 t/2) - \frac{1}{\lambda_1} g(\lambda_1 t/2)\Big), \\
F_3(t) &= \frac{1}{\delta}\Big(g(\lambda_1 t/2) - g(\lambda_2 t/2)\Big)
\end{aligned}
$$

with $g(x) = \tanh x$ and

$$
\lambda_1 = \frac{1}{2}(1+\delta), \quad \lambda_2 = \frac{1}{2}(1-\delta).
$$

(II-2) *If $\mu = \frac{1}{4}$, then*

$$
\begin{aligned}
&e(t; x, v, \xi, \rho) \\
&\quad = 4(1+e^{-t/2})^{-2}\exp\Bigg[-2\Big(\tanh(t/4) + \frac{t/4}{\cosh^2(t/4)}\Big)\left(|\rho|^2 + |v/2|^2\right) \\
&\qquad\qquad - i\frac{t}{\cosh^2(t/4)}\left(\langle v, \xi\rangle - \langle x/4 + a, \rho\rangle\right) \\
&\qquad\qquad - 8\Big(\tanh(t/4) - \frac{t/4}{\cosh^2(t/4)}\Big)(|\xi|^2 + |x/4 + a|^2/4)\Bigg].
\end{aligned}
$$

Remark 4.2. We note that

$$
\begin{aligned}
t/2 - \tanh(t/2) &= \frac{1}{24}t^3 + O(t^5), \\
F_2(t) &= \frac{1}{24}t^3 + O(t^5), \\
4\tanh(t/4) - \frac{t}{\cosh^2(t/4)} &= \frac{1}{24}t^3 + O(t^5).
\end{aligned}
$$

By the above behavior of the symbol of the fundamental solution it is easy to see that the estimate (2.1) for $\mu \neq 0$ holds and that if $\varepsilon = 0$ we have the following estimate

$$
\|(-\Delta_x)^{2/3}u\|^2 + \|\Lambda_v^2 u\|^2 \le C(\|Ku\|^2 + \|u\|^2).
$$

Remark 4.3. Since $xg(x)$ is an increasing function and $\dfrac{g(x)}{x}$ is a decreasing functions, $F_1(t)$ and $F_2(t)$ are non-negative functions. $F_3(t)$ is a real function.

5. Proof of Theorem 4.1

For the proof of (I) we apply Theorem 3.1 to the following the $2n \times 2n$ matrix

$$H = \begin{pmatrix} \frac{1}{2}I_n & 0 \\ 0 & 2I_n \end{pmatrix}, \quad p_0 = \begin{pmatrix} \xi \\ -a \end{pmatrix}, \quad b = -\frac{n}{2}.$$

Then $2n \times 2n$ matrix A is of the form

$$A = \begin{pmatrix} 0 & 2iI_n \\ -\frac{i}{2}I_n & 0 \end{pmatrix}.$$

Let $e_j = {}^t(0, 0, \dots, 1, 0, \dots, 0)$ be n-vector and $2n$-vectors E_j^+, E_j^- be as follow:

$$E_j^+ = \begin{pmatrix} ie_j \\ \frac{1}{2}e_j \end{pmatrix}, \quad E_j^- = \begin{pmatrix} -ie_j \\ \frac{1}{2}e_j \end{pmatrix}.$$

Then the following equalities hold:

$$AE_j^+ = E_j^+, \quad AE_j^- = -E_j^-, \quad i\langle JE_j^+, E_k^- \rangle = \delta_{jk}.$$

For any $2n$ vector $\begin{pmatrix} v \\ \rho \end{pmatrix}$ we have

$$\begin{pmatrix} v \\ \rho \end{pmatrix} = \sum_{j=1}^{n} \left(\rho_j - \frac{i}{2}v_j\right) E_j^+ + \sum_{j=1}^{n} \left(\rho_j + \frac{i}{2}v_j\right) E_j^-.$$

So if $h(x)$ is an odd function, then

$$i \left\langle Jh(At/2) \begin{pmatrix} v \\ \rho \end{pmatrix}, \begin{pmatrix} v \\ \rho \end{pmatrix} \right\rangle = 2h(t/2)(|\rho|^2 + |v/2|^2).$$

On the other hand if $h(x)$ is an even function, then

$$i \left\langle Jh(At/2) \begin{pmatrix} v \\ \rho \end{pmatrix}, \begin{pmatrix} a \\ \xi \end{pmatrix} \right\rangle = ih(t/2)(\langle \rho, a \rangle - \langle \xi, v \rangle).$$

So we obtain the assertion (I), noting

$$\frac{t^2}{4}G(t/2) = \frac{t}{2} - \tanh(t/2),$$

$$(\det[\cosh(At/2)])^{-1} = (\cosh(t/2))^{-n} = e^{-nt/2}\left(\frac{1 + e^{-t}}{2}\right)^{-n}.$$

(II) We note that

$$F_1(t) = \frac{1}{2}\{g(\lambda_1 t/2) + g(\lambda_2 t/2)\} + \frac{1}{2}F_3(t) \tag{5.1}$$

$$\mu F_2(t) = \frac{1}{2}\{g(\lambda_1 t/2) + g(\lambda_2 t/2)\} - \frac{1}{2}F_3(t) \tag{5.2}$$

by the definition. It is sufficient to show (II-1) because if $\mu = 1/4$, we have $\lambda_1 = \lambda_2 = 1/2$, $\delta = 0$ and

$$\lim_{\delta\to 0} F_3(t) = \lim_{\delta\to 0} \frac{\bigl(\tanh((1+\delta)t/4) - \tanh((1-\delta)t/4)\bigr)}{\delta} = \frac{t/2}{\cosh^2(t/4)}.$$

By the above equation and (5.1), (5.2) we get the assertion for $\mu = 1/4$.

In case $\mu \neq 1/4$ the proof for (II-2) is complete by Theorem 3.1 and the following Lemma 5.1.

Lemma 5.1.

(1) *If* $U =^t (x, v, \xi, \rho)$, *then*

$$\begin{aligned} i\langle J\tanh(At/2)U, U\rangle &= 2F_1(t)(|\rho|^2 + |v|^2/4) + 2F_2(t)(|\xi|^2 + |\mu x|^2/4) \\ &\quad + 2iF_3(t)\bigl(\langle v, \xi\rangle - \langle \mu x, \rho\rangle\bigr). \end{aligned}$$

(2) *If* $W =^t (0, -a, 0, 0)$, *then*

$$i\frac{t^2}{4}\langle JG(At/2)W, W\rangle = \frac{a^2}{2}F_2(t).$$

(3) *If* $U =^t (x, v, \xi, \rho)$, $W =^t (0, -a, 0, 0)$, *then*

$$it\langle J\tanh(At/2)(At/2)^{-1}U, W\rangle = 2a\left(\frac{\mu x}{2}F_2(t) - i\rho F_3(t)\right).$$

(4)
$$\frac{1}{\sqrt{\det\cosh(At/2)}} = e^{-t/2}C_\delta(t).$$

Proof. (I) If $\mu \neq 1/4$, then eigenvalues $\pm\lambda_1, \pm\lambda_2$ of A are distinct. The corresponding eigenvectors are as follow:

$$\begin{aligned} w_j &= {}^t(-1/\lambda_j, 1, i\mu/(2\lambda_j), -i/2)/\left(\sqrt{1-\frac{\mu}{\lambda_j^2}}\right), \quad j = 1, 2, \\ \tilde{w}_j &= {}^t(1/\lambda_j, 1, i\mu/(2\lambda_j), i/2)/\left(\sqrt{1-\frac{\mu}{\lambda_j^2}}\right), \quad j = 1, 2, \\ Aw_j &= \lambda_j w_j, \quad A\tilde{w}_j = -\lambda_j \tilde{w}_j. \end{aligned}$$

Note that λ_j are the solutions of $\lambda^2 - \lambda + \mu = 0$ and

$$i\langle Jw_j, w_k\rangle = 0 \quad i\langle Jw_j, \tilde{w}_k\rangle = \delta_{jk}, \quad j, k = 1, 2.$$

Then for any $U \in \mathbb{R}^4$ we can write

$$U = a_1 w_1 + a_2 w_2 + \tilde{a}_1\tilde{w}_1 + \tilde{a}_2\tilde{w}_2,$$

where

$$a_j = -i\langle J\tilde{w}_j, U\rangle, \tilde{a}_j = i\langle Jw_j, U\rangle, \quad j = 1, 2.$$

For any analytic function $h(x)$ we have

$$i\langle Jh(A)U,W\rangle = h(\lambda_1)\langle J\tilde{w}_1,U\rangle\langle Jw_1,W\rangle + h(\lambda_2)\langle J\tilde{w}_2,U\rangle\langle Jw_2,W\rangle \\ - h(-\lambda_1)\langle Jw_1,U\rangle\langle J\tilde{w}_1,W\rangle - h(-\lambda_2)\langle Jw_2,U\rangle\langle J\tilde{w}_2,W\rangle.$$

Especially we have the following results (i) and (ii).

(i) If $h(x)$ is an odd function, we have

$$i\langle Jh(A)U,U\rangle = 2\sum_{j=1}^{2} h(\lambda_j)\langle J\tilde{w}_j,U\rangle\langle Jw_j,U\rangle. \tag{5.3}$$

(ii) If $h(x)$ is an even function, we have

$$i\langle Jh(A)U,W\rangle = \sum_{j=1}^{2} h(\lambda_j)\Big\{\langle J\tilde{w}_j,U\rangle\langle Jw_j,W\rangle - \langle Jw_j,U\rangle\langle J\tilde{w}_j,W\rangle\Big\}. \tag{5.4}$$

By the following equations

$$\langle J\tilde{w}_j,U\rangle = \left(\frac{i}{2}\frac{\mu}{\lambda_j}x + \frac{i}{2}v - \frac{\xi}{\lambda_j} - \rho\right)\Bigg/\left(\sqrt{1-\frac{\mu}{\lambda_j^2}}\right),$$

$$\langle Jw_j,U\rangle = \left(\frac{i}{2}\frac{\mu}{\lambda_j}x - \frac{i}{2}v + \frac{\xi}{\lambda_j} - \rho\right)\Bigg/\left(\sqrt{1-\frac{\mu}{\lambda_j^2}}\right)$$

and noting that

$$1\Bigg/\left(1-\frac{\mu}{\lambda_1^2}\right) = \frac{\lambda_1}{\delta}, \quad 1\Bigg/\left(1-\frac{\mu}{\lambda_2^2}\right) = -\frac{\lambda_2}{\delta},$$

we get

$$\langle J\tilde{w}_1,U\rangle\langle Jw_1,U\rangle = \frac{\lambda_1}{\delta}\left\{\left(\frac{i}{2}\frac{\mu}{\lambda_1}x-\rho\right)^2 - \left(\frac{i}{2}v-\frac{\xi}{\lambda_1}\right)^2\right\} \\ = \frac{\lambda_1}{\delta}\left(|\rho|^2+|v|^2/4\right) - \frac{1}{\lambda_1\delta}\left(|\xi|^2+|\mu x|^2/4\right) + \frac{i}{\delta}(\langle v,\xi\rangle - \langle\mu x,\rho\rangle)$$

and

$$\langle J\tilde{w}_2,U\rangle\langle Jw_2,U\rangle = -\frac{\lambda_2}{\delta}\left\{\left(\frac{i}{2}\frac{\mu}{\lambda_2}x-\rho\right)^2 - \left(\frac{i}{2}v-\frac{\xi}{\lambda_2}\right)^2\right\} \\ = -\frac{\lambda_2}{\delta}\left(|\rho|^2+|v|^2/4\right) + \frac{1}{\lambda_2\delta}\left(|\xi|^2+|\mu x|^2/4\right) - \frac{i}{\delta}(\langle v,\xi\rangle - \langle\mu x,\rho\rangle).$$

We obtain the following equation by (5.3)

$$i\langle J\tanh(At/2)U,U\rangle = 2\sum_{j=1}^{2}\tanh(\lambda_j t/2)\langle J\tilde{w}_j,U\rangle\langle Jw_j,U\rangle.$$

The proof of the assertion (1) is complete.

(2) Noting

$$\langle Jw_j, W\rangle = ia/\left(2\sqrt{1-\frac{\mu}{\lambda_j^2}}\right), \quad \langle J\tilde{w}_j, W\rangle = -ia/\left(2\sqrt{1-\frac{\mu}{\lambda_j^2}}\right),$$

we have by (5.3)

$$i\frac{t^2}{4}\langle JG(At/2)W, W\rangle = \frac{a^2t^2}{8\delta}\Big\{\lambda_1 G(\lambda_1 t/2) - \lambda_2 G(\lambda_2 t/2)\Big\}.$$

Set $G(x) = (1 - G_1(x))/x$ with $G_1(x) = g(x)/x$. Then the following equation holds:

$$\begin{aligned} i\frac{t^2}{4}\langle JG(At/2)W, W\rangle &= \frac{a^2t}{4\delta}\Big\{-G_1(\lambda_1 t/2) + G_1(\lambda_2 t/2)\Big\}. \\ &= \frac{a^2t}{4\delta}\left\{-\frac{2g(\lambda_1 t/2)}{\lambda_1 t} + \frac{2g(\lambda_2 t/2)}{\lambda_2 t}\right\} \\ &= \frac{a^2}{2}F_2(t). \end{aligned}$$

(3) We have by (5.4)

$$\begin{aligned} it\langle J\tanh(At/2)(At/2)^{-1}U, W\rangle &= \frac{iat}{2\delta}\bigg[-\frac{4\rho}{t}\big\{\tanh(\lambda_1 t/2) - \tanh(\lambda_2 t/2)\big\} \\ &\qquad + i\mu x\Big\{\frac{\tanh(\lambda_1 t/2)}{\lambda_1 t/2} - \frac{\tanh(\lambda_2 t/2)}{\lambda_2 t/2}\Big\}\bigg] \\ &= 2a\left(\frac{\mu x}{2}F_2(t) - i\rho F_3(t)\right). \end{aligned}$$

(4) Eigenvalues of A are $\pm\lambda_1$ and $\pm\lambda_2$. So we have

$$\det\cosh(At/2) = \cosh^2(\lambda_1 t/2)\cosh^2(\lambda_2 t/2),$$

$$\begin{aligned} \sqrt{\det\cosh(At/2)} &= \cosh(\lambda_1 t/2)\cosh(\lambda_2 t/2) \\ &= e^{\lambda_1 t/2}e^{\lambda_2 t/2}\Big(\frac{1+e^{-\lambda_1 t}}{2}\Big)\Big(\frac{1+e^{-\lambda_2 t}}{2}\Big) \\ &= \frac{1}{4}e^{t/2}\Big(1 + e^{-\lambda_1 t} + e^{-\lambda_2 t} + e^{-t}\Big) \\ &= \frac{1}{4}e^{t/2}\Big(1 + e^{-t} + 2e^{-t/2}\cosh(\delta t/2)\Big). \end{aligned}$$

Therefore we obtain

$$\frac{1}{\sqrt{\det\cosh(At/2)}} = C_\delta(t)e^{-t/2}. \qquad \square$$

6. The key proposition for the eigenfunction expansion

In the rest of this paper we assume that $n = 1$ and $\mu > 0$. It is easy to see for $\mu > 0$

$$\begin{aligned} e_\infty(x, v, \xi, \rho) &= \lim_{t\to\infty} e(t; x, v, \xi, \rho) \\ &= 4\exp\left[-2(|\rho|^2 + |v|^2/4) - \frac{2}{\mu}(|\xi|^2 + |\mu x + a|^2/4)\right] \end{aligned}$$

and we have

$$\begin{aligned} &(2\pi)^{-2}\int_{\mathbb{R}^2} e^{i(x-x')\cdot\xi + i(v-v')\cdot\rho} e_\infty\left(\frac{x+x'}{2}, \frac{v+v'}{2}, \xi, \rho\right) d\xi d\rho \\ &= \frac{\sqrt{\mu}}{2\pi}\exp\left[-\frac{1}{4}\left\{v^2 + v'^2 + \mu(x + a/\mu)^2 + \mu(x' + a/\mu)^2\right\}\right]. \end{aligned}$$

We show a method of obtaining eigenfunctions due to the fundamental solution represented by pseudo-differetial operator of Weyl symbols. The following proposition is the key of a proof of an expansipn of the kernel of the fundamental solution obtained as a pseudo-differential operator.

Proposition 6.1. *If the symbol of a pseudo-differential operator P is of the form*

$$p(x, \xi) = k(x, \xi)g(x, \xi),$$

then the kernel of the operator P is

$$(2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x-x')\cdot\xi} p\left(\frac{x+x'}{2}, \xi\right) d\xi = k\left(\frac{r}{2}, -i\partial_q\right)\tilde{g}\left(\frac{r}{2}, q\right)\Big|_{r=x+x', q=x-x'},$$

where

$$\tilde{g}\left(\frac{r}{2}, q\right) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{iq\cdot\xi} g\left(\frac{r}{2}, \xi\right) d\xi.$$

Proof. Formally if the symbol $k(x, \xi) = \sum_{j=0}^\infty a_j(x)\xi^j$ has the expansion, we have with $r = x + x', q = x - x'$

$$\begin{aligned} &(2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x-x')\cdot\xi} p\left(\frac{x+x'}{2}, \xi\right) d\xi \\ &= (2\pi)^{-d}\int_{\mathbb{R}^d}\sum_{j=0}^\infty a_j\left(\frac{r}{2}\right)\xi^j e^{iq\cdot\xi} g\left(\frac{r}{2}, \xi\right) d\xi \\ &= \sum_{j=0}^\infty a_j\left(\frac{r}{2}\right)(-i\partial_q)^j (2\pi)^{-d}\int_{\mathbb{R}^d} e^{iq\cdot\xi} g\left(\frac{r}{2}, \xi\right) d\xi \\ &= k\left(\frac{r}{2}, -i\partial_q\right)\tilde{g}\left(\frac{r}{2}, q\right)\Big|_{r=x+x', q=x-x'}. \end{aligned}$$

□

7. The eigenfunction expansion for K

Apply Proposition 6.1 for $d = 2$, $g = e_\infty(x, v, \xi, \rho)$, $k = e(t; x, v, \xi, \rho)/e_\infty(x, v, \xi, \rho)$. In our case we have

$$\tilde{e}_\infty\left(\frac{x+x'}{2}, \frac{v+v'}{2}, x-x', v-v'\right) = \psi(x,v)\psi(x',v'),$$

where

$$\psi(x,v) = \left(\frac{\sqrt{\mu}}{2\pi}\right)^{1/2} \exp\left[-\frac{1}{4}\left\{v^2 + \mu\left(x + \frac{a}{\mu}\right)^2\right\}\right] \tag{7.1}$$

by the argument of the previous section.

Set $z_j = e^{-\lambda_j t}$ $\quad j = 1, 2$. Then

$$\frac{C_\delta(t)}{4} = \frac{1}{(1+z_1)(1+z_2)}.$$

The methods to obtain the eigenfunction expansion are similar for any positive μ. But the symbol $k(t; x, v, \xi, \rho)$ for $\mu = 1/4$ is different from that for $\mu \neq 1/4$. In fact there are generalized eigenfunctions in case $\mu = 1/4$ as discussed later.

7.1. K in case of positive $\mu \neq \frac{1}{4}$

Set

$$h(z) = \frac{z}{1+z}.$$

Then we have

$$\frac{1}{2}\left\{\tanh(\lambda_j t/2) - 1\right\} = -h(z_j), \quad j = 1, 2.$$

In our case the symbol $k(t; x, v, \xi, \rho)$ is obtained as

$$k(t;x,v,\xi,\rho) = \frac{1}{(1+z_1)(1+z_2)} \exp\left\{\frac{1}{\delta}h(z_1)\tilde{k}_{(\lambda_1)}(x,v,\xi,\rho) - \frac{1}{\delta}h(z_2)\tilde{k}_{(\lambda_2)}(x,v,\xi,\rho)\right\},$$

where

$$\begin{aligned}
&\tilde{k}_{(\lambda_j)}(x,v,\xi,\rho) \\
&\quad = 4\left[\lambda_j\left(\rho^2 + \frac{v^2}{4}\right) + i\left(\langle v,\xi\rangle - \langle \mu x + a, \rho\rangle\right) - \frac{1}{\lambda_j}\left\{\xi^2 + \frac{1}{4}(\mu x + a)^2\right\}\right] \\
&\quad = 4\left\{\sqrt{\lambda_j}\left(i\rho + \frac{v}{2}\right) + \frac{1}{\sqrt{\lambda_j}}\left(i\xi + \frac{1}{2}(\mu x + a)\right)\right\} \\
&\qquad \times \left\{\sqrt{\lambda_j}\left(-i\rho + \frac{v}{2}\right) - \frac{1}{\sqrt{\lambda_j}}\left(-i\xi + \frac{1}{2}(\mu x + a)\right)\right\}.
\end{aligned}$$

So we have with $r_1 = x + x', r_2 = v + v', q_1 = x - x', q_2 = v - v'$

$$\begin{aligned}
&\tilde{k}_{(\lambda_j)}\left(\frac{r_1}{2}, \frac{r_2}{2}, -i\partial_{q_1}, -i\partial_{q_2}\right) \\
&\quad = 4\left\{\sqrt{\lambda_j}\ \left(\partial_{q_2} + \frac{r_2}{4}\right) + \frac{1}{\sqrt{\lambda_j}}\left(\partial_{q_1} + \frac{1}{4}\mu r_1 + \frac{a}{2}\right)\right\} \\
&\qquad \times \left\{\sqrt{\lambda_j}\ \left(-\partial_{q_2} + \frac{r_2}{4}\right) - \frac{1}{\sqrt{\lambda_j}}\left(-\partial_{q_1} + \frac{1}{4}\mu r_1 + \frac{a}{2}\right)\right\} \\
&\quad = \left\{\sqrt{\lambda_j}(b_v + b^*_{v'}) + \frac{1}{\sqrt{\lambda_j}}(a_x + a^*_{x'})\right\}\left\{\sqrt{\lambda_1}(b^*_v + b_{v'}) - \frac{1}{\sqrt{\lambda_j}}(a^*_x + a_{x'})\right\},
\end{aligned}$$

where $a_x, b_v, a_{x'}, b_{v'}$ are differential operators of the first order:

$$b_v = \partial_v + \frac{v}{2}, \qquad a_x = \partial_x + \frac{\mu x}{2} + \frac{a}{2},$$
$$b_{v'} = \partial_{v'} + \frac{v'}{2}, \quad a_{x'} = \partial_{x'} + \frac{\mu x'}{2} + \frac{a}{2}.$$

Definition 7.1. For $\lambda = \lambda_1, \lambda_2$ we define differential operators:

$$M_\lambda = \frac{1}{\sqrt{\delta}}\left(\sqrt{\lambda}b_v - \frac{1}{\sqrt{\lambda}}a_x\right), \qquad N_\lambda = \frac{1}{\sqrt{\delta}}\left(\sqrt{\lambda}b_v + \frac{1}{\sqrt{\lambda}}a_x\right),$$
$$M'_\lambda = \frac{1}{\sqrt{\delta}}\left(\sqrt{\lambda}b_{v'} - \frac{1}{\sqrt{\lambda}}a_{x'}\right), \qquad N'_\lambda = \frac{1}{\sqrt{\delta}}\left(\sqrt{\lambda}b_{v'} + \frac{1}{\sqrt{\lambda}}a_{x'}\right),$$
$$S = \left(N_{\lambda_1} + (N'_{\lambda_1})^*\right)(M^*_{\lambda_1} + M'_{\lambda_1}), \qquad \tilde{S} = -\left(N_{\lambda_2} + (N'_{\lambda_2})^*\right)(M^*_{\lambda_2} + M'_{\lambda_2}).$$

Using the Definition 7.1, we can write

$$\frac{1}{\delta}\tilde{k}_{(\lambda_1)}\left(\frac{r_1}{2}, \frac{r_2}{2}, -i\partial_{q_1}, -i\partial_{q_2}\right) = S$$

and

$$\frac{1}{\delta}\tilde{k}_{(\lambda_2)}\left(\frac{r_1}{2}, \frac{r_2}{2}, -i\partial_{q_1}, -i\partial_{q_2}\right) = -\tilde{S}.$$

So we have by Proposition 6.1

$$e(t; x, v, x', v') = \frac{1}{(1 + z_1)(1 + z_2)}\exp\left\{h(z_1)S + h(z_2)\tilde{S}\right\}\left(\psi(x, v)\psi(x', v')\right).$$

It is clear that

$$[a_x, a^*_x] = \mu, \quad [b_v, b^*_v] = 1.$$

So we have

$$\begin{aligned}
&[N_{\lambda_1}, M^*_{\lambda_1}] = [M_{\lambda_1}, N^*_{\lambda_1}] = 1, \\
&[N_{\lambda_2}, M^*_{\lambda_2}] = [M_{\lambda_2}, N^*_{\lambda_2}] = -1, \\
&[N_{\lambda_1}, M^*_{\lambda_2}] = [N_{\lambda_2}, M^*_{\lambda_1}] = [M_{\lambda_1}, N^*_{\lambda_2}] = [M_{\lambda_2}, N^*_{\lambda_1}] = 0.
\end{aligned}$$

Then $M^*_{\lambda_1} + M'_{\lambda_1}$, $N_{\lambda_1} + (N'_{\lambda_1})^*$, $M^*_{\lambda_2} + M'_{\lambda_2}$, $N_{\lambda_2} + (N'_{\lambda_2})^*$ are commuting. Also S and $\tilde{S}$ are commuting. So we have

$$e(t; x, v, x', v') = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{j!k!} C_j(S) C_k(\tilde{S}) \Big(\psi(x,v)\psi(x',v')\Big) z_1^j z_2^k,$$

where the operators $C_j(P)$ are defined as

$$\frac{1}{(1+z)} \exp\left(\frac{z}{1+z}P\right) = \sum_{j=0}^{\infty} \frac{C_j(P)}{j!} z^j. \tag{7.2}$$

Definition 7.2. For a pair of nonnegative integers m and k we set

$$\psi_{m,k}(x,v) = \frac{(M^*_{\lambda_1})^m}{\sqrt{m!}} \frac{(-M^*_{\lambda_2})^k}{\sqrt{k!}} \psi(x,v), \quad \psi^{m,k}(x,v) = \frac{(N^*_{\lambda_1})^m}{\sqrt{m!}} \frac{(N^*_{\lambda_2})^k}{\sqrt{k!}} \psi(x,v).$$

The following expansion of the kernel of the fundamental solution is obtained. The assertion of (2) and (3) Theorem 7.3 means that $\psi_{m,k}(x,v)$ are the orthonormal eigenfunctions whose eigenvalues are $\lambda_1 m + \lambda_2 k$.

Theorem 7.3.

(1) *We have the eigenfunction expansion for the kernel of the heat operator:*

$$e(t; x, v, x', v') = \sum_{n_1, n_2 = 0}^{\infty} e^{-(\lambda_1 n_1 + \lambda_2 n_2)t} \psi_{n_1,n_2}(x,v) \psi^{n_1,n_2}(x',v').$$

(2) *For any* $(m,k) \in \mathbb{N}_0 \times \mathbb{N}_0$ *it holds that*

$$K\psi_{m,k}(x,v) = (\lambda_1 m + \lambda_2 k)\psi_{m,k}(x,v).$$

(3) *For any* $(m,k), (n,\ell) \in \mathbb{N}_0 \times \mathbb{N}_0$ *we have*

$$\langle \psi_{m,k}, \psi^{n,\ell} \rangle = \delta_{m,n}\delta_{k,\ell}.$$

For the proof of Theorem 7.3 we prepare two propositions.

Proposition 7.4. *The operator* K *is written by* M_λ *and* N_λ *as follows:*

$$K = \lambda_1 M^*_{\lambda_1} N_{\lambda_1} - \lambda_2 M^*_{\lambda_2} N_{\lambda_2}.$$

Proof. It is easy to see

$$K = b_v^* b_v + \big(b_v^* a_x - a_x^* b_v\big).$$

We can show the following equation by the direct calculation.

$$b_v^* b_v + \big(b_v^* a_x - a_x^* b_v\big) = \lambda_1 M^*_{\lambda_1} N_{\lambda_1} - \lambda_2 M^*_{\lambda_2} N_{\lambda_2}. \qquad \square$$

Proposition 7.5.

(1) $\psi_{m,k}(x,v)$ *and* $\psi^{m,k}(x,v)$ *have the following properties:*

$$a_x\psi(x,v) = b_v\psi(x,v) = 0,$$
$$M_{\lambda_1}\psi(x,v) = M_{\lambda_2}\psi(x,v) = N_{\lambda_1}\psi(x,v) = N_{\lambda_2}\psi(x,v) = 0.$$

Moreover

$$N_{\lambda_1}\psi_{m,\ell}(x,v)=\sqrt{m}\psi_{m-1,\ell}(x,v),\qquad N_{\lambda_2}\psi_{m,\ell}(x,v)=\sqrt{\ell}\psi_{m,\ell-1}(x,v),$$
$$M_{\lambda_1}\psi^{m,\ell}(x,v)=\sqrt{m}\psi^{m-1,\ell}(x,v),\quad -M_{\lambda_2}\psi^{m,\ell}(x,v)=\sqrt{\ell}\psi^{m,\ell-1}(x,v).$$

(2) *For any nonnegative integers* k *we have*

$$N^k_{\lambda_1}(M^*_{\lambda_1})^k\psi(x,v)=N^k_{\lambda_2}(-M^*_{\lambda_2})^k\psi(x,v)=k!\psi(x,v)\qquad(k\in\mathbb{N}_0),$$
$$M^k_{\lambda_1}(N^*_{\lambda_1})^k\psi(x,v)=(-M_{\lambda_2})^k(N^*_{\lambda_2})^k\psi(x,v)=k!\psi(x,v)\quad(k\in\mathbb{N}_0).$$

(3) $C_\ell(S)$ *defined by* (7.2) *satisfies*

$$C_\ell(S)\Big(\psi_{0,m}(x,v)\psi^{0,r}(x',v')\Big)=\ell!\psi_{\ell,m}(x,v)\psi^{\ell,r}(x',v')\quad(\ell,m,r\in\mathbb{N}_0).$$

(4) $C_\ell(\tilde{S})$ *defined by* (7.2) *satisfies*

$$C_\ell(\tilde{S})\Big(\psi_{m,0}(x,v)\psi^{r,0}(x',v')\Big)=\ell!\psi_{m,\ell}(x,v)\psi^{r,\ell}(x',v')\quad(\ell,m,r\in\mathbb{N}_0).$$

Proof. (1) By the following equations the assertions are obtained.

$$\begin{aligned}N_{\lambda_1}\psi_{m,\ell}(x,v)&=\left\{[N_{\lambda_1},(M^*_{\lambda_1})^m]\psi_{0,\ell}(x,v)+(M^*_{\lambda_1})^mN_{\lambda_1}\psi_{0,\ell}(x,v)\right\}/\sqrt{m!}\\&=m(M^*_{\lambda_1})^{m-1}\psi_{0,\ell}(x,v)/\sqrt{m!}\\&=\sqrt{m}\psi_{m-1,\ell}(x,v).\end{aligned}$$

$$\begin{aligned}N_{\lambda_2}\psi_{m,\ell}(x,v)&=\left\{[N_{\lambda_2},(-M^*_{\lambda_2})^\ell]\psi_{m,0}(x,v)+(-M^*_{\lambda_2})^\ell N_{\lambda_2}\psi_{m,0}(x,v)\right\}/\sqrt{\ell!}\\&=\ell(-M^*_{\lambda_2})^{\ell-1}\psi_{m,0}(x,v)/\sqrt{\ell!}\\&=\sqrt{\ell}\psi_{m,\ell-1}(x,v).\end{aligned}$$

$$\begin{aligned}M_{\lambda_1}\psi^{m,\ell}(x,v)&=\left\{[M_{\lambda_1},(N^*_{\lambda_1})^m]\psi^{0,\ell}(x,v)+(N^*_{\lambda_1})^mM_{\lambda_1}\psi^{0,\ell}(x,v)\right\}/\sqrt{m!}\\&=m(N^*_{\lambda_1})^{m-1}\psi^{0,\ell}(x,v)/\sqrt{m!}\\&=\sqrt{m}\psi^{m-1,\ell}(x,v).\end{aligned}$$

$$\begin{aligned}M_{\lambda_2}\psi^{m,\ell}(x,v)&=\left\{[M_{\lambda_2},(N^*_{\lambda_2})^\ell]\psi^{m,0}(x,v)+(N^*_{\lambda_2})^\ell M_{\lambda_2}\psi^{m,0}(x,v)\right\}/\sqrt{\ell!}\\&=-\ell(N^*_{\lambda_2})^{\ell-1}\psi^{m,0}(x,v)/\sqrt{\ell!}\\&=-\sqrt{\ell}\psi^{m,\ell-1}(x,v).\end{aligned}$$

(2) By (1) we have

$$\begin{aligned}\frac{N^k_{\lambda_1}(M^*_{\lambda_1})^k}{k!}\psi(x,v)&=N^k_{\lambda_1}\psi_{k,0}(x,v)/\sqrt{k!}=\sqrt{k}N^{k-1}_{\lambda_1}\psi_{k-1,0}(x,v)/\sqrt{k!}\\&=N^{k-1}_{\lambda_1}\psi_{k-1,0}(x,v)/\sqrt{(k-1)!}=\frac{N^{k-1}_{\lambda_1}(M^*_{\lambda_1})^{k-1}}{(k-1)!}\psi(x,v).\end{aligned}$$

We get the assertion by the induction with respect to k. Similarly we obtain the assertion for

$$\frac{(-M_{\lambda_2})^k (N_{\lambda_2}^*)^k}{k!}\psi(x,v).$$

(3) The following equations hold:

$$\begin{aligned}
&S\Big(\psi_{m,k}(x,v)\psi^{n,\ell}(x',v')\Big)\\
&= \sqrt{(m+1)(n+1)}\psi_{m+1,k}(x,v)\psi^{n+1,\ell}(x',v') + (m+n+1)\psi_{m,k}(x,v)\psi^{n,\ell}(x',v')\\
&\quad + \sqrt{mn}\psi_{m-1,k}(x,v)\psi^{n-1,\ell}(x',v'),
\end{aligned}$$

$$\begin{aligned}
&\tilde{S}\Big(\psi_{m,k}(x,v)\psi^{n,\ell}(x',v')\Big)\\
&= \sqrt{(k+1)(\ell+1)}\psi_{m,k+1}(x,v)\psi^{n,\ell+1}(x',v') + (k+\ell+1)\psi_{m,k}(x,v)\psi^{n,\ell}(x',v')\\
&\quad + \sqrt{k\ell}\psi_{m,k-1}(x,v)\psi^{n,\ell-1}(x',v'),
\end{aligned}$$

which will be shown as follows:

We note that

$$\begin{aligned}
&(M_{\lambda_1}^* + M_{\lambda_1}')\big(\psi_{m,k}(x,v)\psi^{n,\ell}(x',v')\big)\\
&\quad = \sqrt{m+1}\psi_{m+1,k}(x,v)\psi^{n,\ell}(x',v') + \sqrt{n}\psi_{m,k}(x,v)\psi^{n-1,\ell}(x',v')
\end{aligned}$$

and

$$\begin{aligned}
&(N_{\lambda_1} + (N_{\lambda_1}')^*)\big(\psi_{m,k}(x,v)\psi^{n,\ell}(x',v')\big)\\
&\quad = \sqrt{m}\psi_{m-1,k}(x,v)\psi^{n,\ell}(x',v') + \sqrt{n+1}\psi_{m,k}(x,v)\psi^{n+1,\ell}(x',v').
\end{aligned}$$

So by the above equations the assertion for $S = (N_{\lambda_1} + (N_{\lambda_1}')^*)(M_{\lambda_1}^* + M_{\lambda_1}')$ is obtained. Similarly we can show the assertion for $\tilde{S}$.

We will show the assertion (3) by the induction with respect to ℓ. The assertion holds for $\ell = 0$ because $C_0(P) = 1$ for all P. Note that for any operator P the operators $C_j(P)$ satisfies the following equation (7.3) for which we give a proof in Proposition 7.10 below in the more general form.

$$C_{\ell+1}(P) = PC_\ell(P) - (2\ell+1)C_\ell(P) - \ell^2 C_{\ell-1}(P), (\ell \geq 0), \tag{7.3}$$

$$C_0(P) = 1. \tag{7.4}$$

Assume that (3) holds for all $\ell \leq j$. Then by the equation (7.3) we have

$$\begin{aligned}
&C_{j+1}(S)\Big(\psi_{0,m}(x,v)\psi^{0,r}(x',v')\Big)\\
&= SC_j(S)\Big(\psi_{0,m}(x,v)\psi^{0,r}(x',v')\Big) - (2j+1)C_j(S)\Big(\psi_{0,m}(x,v)\psi^{0,r}(x',v')\Big)\\
&\quad - j^2 C_{j-1}(S)\Big(\psi_{0,m}(x,v)\psi^{0,r}(x',v')\Big)\\
&= j!\Big\{S\big(\psi_{j,m}(x,v)\psi^{j,r}(x',v')\big) - (2j+1)\psi_{j,m}(x,v)\psi^{j,r}(x',v')\\
&\quad - j\psi_{j-1,m}(x,v)\psi^{j-1,r}(x',v')\Big\}.
\end{aligned}$$

On the other hand it holds by the equation for S

$$\begin{aligned} &S\big(\psi_{j,m}(x,v)\psi^{j,r}(x',v')\big) \\ &\quad = (j+1)\psi_{j+1,m}(x,v)\psi^{j+1,r}(x',v') + (2j+1)\psi_{j,m}(x,v)\psi^{j,r}(x',v') \\ &\qquad + j\psi_{j-1,m}(x,v)\psi^{j-1,r}(x',v'). \end{aligned}$$

So we get (3) for $j+1$.
(4) is proved similarly. □

Now we give a proof of Theorem 7.3.

(1) By Proposition 7.5 we obtain the following equation:

$$\begin{aligned} C_j(S)C_k(\tilde{S})\Big(\psi(x,v)\psi(x',v')\Big) &= k!C_j(S)\Big(\psi_{0,k}(x,v)\psi^{0,k}(x',v')\Big) \\ &= k!j!\psi_{j,k}(x,v)\psi^{j,k}(x',v'). \end{aligned}$$

Finally we have

$$\begin{aligned} &e(t;x,v,x',v') \\ &\quad = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{j!k!} C_j(S)C_k(\tilde{S})\Big(\psi(x,v)\psi(x',v')\Big) z_1^j z_2^k \\ &\quad = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \psi_{j,k}(x,v)\psi^{j,k}(x',v') z_1^j z_2^k \\ &\quad = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} e^{-(\lambda_1 j+\lambda_2 k)t}\psi_{j,k}(x,v)\psi^{j,k}(x',v'). \end{aligned}$$

(2) By Proposition 7.4 and Proposition 7.5 we have

$$\begin{aligned} K\psi_{m,k} &= \lambda_1\sqrt{m}M_{\lambda_1}^*\psi_{m-1,k} - \lambda_2\sqrt{k}M_{\lambda_2}^*\psi_{m,k-1} \\ &= \lambda_1 m\psi_{m,k} + \lambda_2 k\psi_{m,k} \\ &= (\lambda_1 m + \lambda_2 k)\psi_{m,k}. \end{aligned}$$

(3) Using Proposition 7.5, we have for $m \geq n$

$$\begin{aligned} \sqrt{m!n!}\langle\psi_{m,0},\psi^{n,0}\rangle &= \int \psi(x,v)\ M_{\lambda_1}^{m-n}M_{\lambda_1}^{n}(N_{\lambda_1}^*)^n\psi(x,v)dxdv \\ &= \int \psi(x,v)\ M_{\lambda_1}^{m-n}n!\psi(x,v)dxdv \\ &= n!\delta_{m,n}\int |\psi(x,v)|^2 dxdv = n!\delta_{m,n}. \end{aligned}$$

If $\ell \geq k$, we have

$$\begin{aligned}\sqrt{m!k!n!\ell!}\langle \psi_{m,k}, \psi^{n,\ell}\rangle &= \int (M^*_{\lambda_1})^m(-M^*_{\lambda_2})^k\psi(x,v)\ (N^*_{\lambda_1})^n(N^*_{\lambda_2})^\ell\psi(x,v)dxdv \\ &= \int (M^*_{\lambda_1})^m N^{\ell-k}_{\lambda_2} N^k_{\lambda_2}(-M^*_{\lambda_2})^k\psi(x,v)\ (N^*_{\lambda_1})^n\psi(x,v)dxdv \\ &= k!\delta_{k,\ell}\int (M^*_{\lambda_1})^m\psi(x,v)\ (N^*_{\lambda_1})^n\psi(x,v)dxdv \\ &= k!\sqrt{m!n!}\delta_{k,\ell}\langle \psi_{m,0}, \psi^{n,0}\rangle = k!m!\delta_{k,\ell}\delta_{m,n}.\end{aligned}$$

7.2. K in the case $\mu = \frac{1}{4}$

In the case $\mu = \frac{1}{4}$ we have by (7.1)

$$\psi(x,v) = \frac{1}{2\sqrt{\pi}}\exp\left\{-\left(\frac{v^2}{4}+\frac{(x+4a)^2}{16}\right)\right\}$$

and with $z = e^{-t/2}$ the symbol $k(t;x,v,\xi,\rho)$ is of the form

$$\begin{aligned}k(t;x,v,\xi,\rho) = \frac{1}{(1+z)^2}\exp\Bigg\{&2\left(\frac{2z}{1+z}-\frac{tz}{(1+z)^2}\right)\left(\rho^2+\frac{v^2}{4}\right) \\ &- i\frac{4tz}{(1+z)^2}\Big(\langle v,\xi\rangle - \langle x/4+a,\rho\rangle\Big) \\ &+ 2\left(\frac{2z}{1+z}+\frac{tz}{(1+z)^2}\right)\left(4\xi^2+\frac{(x+4a)^2}{16}\right)\Bigg\}.\end{aligned}$$

So we have with $r_1 = x+x', r_2 = v+v', q_1 = x-x', q_2 = v-v'$

$$\begin{aligned}&k\left(t;\frac{r_1}{2},\frac{r_2}{2},-i\partial_{q_1},-i\partial_{q_2}\right) \\ &= \frac{1}{(1+z)^2}\exp\Bigg[\frac{4z}{1+z}\left\{\left(2\partial_{q_1}+\frac{r_1}{8}+a\right)\left(-2\partial_{q_1}+\frac{r_1}{8}+a\right)\right. \\ &\quad \left.+\left(\partial_{q_2}+\frac{r_2}{4}\right)\left(-\partial_{q_2}+\frac{r_2}{4}\right)\right\} + \frac{2tz}{(1+z)^2}\left\{\left(2\partial_{q_1}+\frac{r_1}{8}+a\right)\right. \\ &\quad \left.+\left(\partial_{q_2}+\frac{r_2}{4}\right)\right\}\left\{\left(-2\partial_{q_1}+\frac{r_1}{8}+a\right)-\left(-\partial_{q_2}+\frac{r_2}{4}\right)\right\}\Bigg] \\ &= \frac{1}{(1+z)^2}\exp\Bigg[\frac{z}{1+z}\{(b_v+b^*_{v'})(b^*_v+b_{v'})+(a_x+a^*_{x'})(a^*_x+a_{x'})\} \\ &\quad -\frac{tz}{2(1+z)^2}\{(b_v+b^*_{v'}+a_x+a^*_{x'})(b^*_v+b_{v'}-a^*_x-a_{x'})\}\Bigg],\end{aligned}$$

where

$$b_v = \partial_v + \frac{v}{2},\quad a_x = 2\partial_x + \frac{x}{4}+a,\quad b_{v'} = \partial_{v'}+\frac{v'}{2},\quad a_{x'} = 2\partial_{x'}+\frac{x'}{4}+a.$$

So we have

$$e(t;x,v,x',v') = \frac{1}{(1+z)^2}\exp\Big[\frac{z}{1+z}\Big\{(b_v+b_{v'}^*)(b_v^*+b_{v'})+(a_x+a_{x'}^*)(a_x^*+a_{x'})\Big\} \\ -\frac{tz}{2(1+z)^2}(b_v+b_{v'}^*+a_x+a_{x'}^*)(b_v^*+b_{v'}-a_x^*-a_{x'})\Big]\,(\psi(x,v)\psi(x',v')) .$$

Definition 7.6.

$$M=\frac{1}{\sqrt{2}}(b_v-a_x),\ \ N=\frac{1}{\sqrt{2}}(b_v+a_x),\ \ M'=\frac{1}{\sqrt{2}}(b_{v'}-a_{x'}),\ \ N'=\frac{1}{\sqrt{2}}(b_{v'}+a_{x'}),$$

$$Q=(N+(N')^*)(M^*+M'),\quad S_M=(M+(M')^*)(M^*+M')$$
$$S_N=(N+(N')^*)(N^*+N'),\qquad S=S_M+S_N.$$

Using Definition 7.6 we can write

$$e(t;x,v,x',v')=\frac{1}{(1+z)^2}\exp\left(\frac{z}{1+z}S-\frac{tz}{(1+z)^2}Q\right)\Big(\psi(x,v)\psi(x',v')\Big).$$

It is clear that

$$[a_x,a_x^*]=1,\quad [b_v,b_v^*]=1.$$

So we have

$$[M,M^*]=[N,N^*]=1,\quad [M,N^*]=[N,M^*]=0,$$
$$[M^*+M',N+(N')^*]=[M+(M')^*,M^*+M']=[N+(N')^*,N^*+N']=0.$$

It is easy to see that S_M, S_N and Q are commuting. Then we have

$$\frac{1}{(1+z)^2}\exp\Big(\frac{z}{1+z}(S_M+S_N)\Big)\exp\Big(-\frac{tz}{(1+z)^2}Q\Big) \\ =\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\sum_{j=0}^{\infty}\sum_{\ell=0}^{j}\frac{1}{(j-\ell)!\ell!}C_\ell(S_N,k)C_{j-\ell}(S_M,k)Q^k z^{j+k},$$

where operators $C_j(P,k)$ are defined as

$$\frac{1}{(1+z)^{k+1}}\exp\Big(\frac{z}{1+z}P\Big)=\sum_{j=0}^{\infty}\frac{C_j(P,k)}{j!}z^j. \tag{7.5}$$

Definition 7.7. For a pair of nonnegative integers m and k we set

$$\psi_{m,k}(x,v)=\frac{(M^*)^m}{\sqrt{m!}}\frac{(N^*)^k}{\sqrt{k!}}\psi(x,v),$$
$$\varphi_{m,k}(x,v)=\psi_{m-k,k}(x,v),\quad (0\le k\le m).$$

We obtain the following expansion of the kernel of the fundamental solution. By the assertion (2) and (3) of Theorem 7.8 for any nonnegative integer n the set $\{\varphi_{n,k}(x,v)\}_{k=0}^{n}$ are the set of orthogonal generalized eigenfunctions whose eigenvalue is $n/2$.

Theorem 7.8.

(1) *We have the eigenfunction expansion for the kernel of the heat operator:*

$$e(t;x,v,x',v') = \sum_{n=0}^{\infty} e^{-nt/2}\left(\sum_{k=0}^{n}\frac{(-t)^k}{k!}\sum_{\ell=0}^{n-k}\sqrt{\frac{(n-\ell)!(k+\ell)!}{\ell!(n-k-\ell)!}}\varphi_{n,\ell}(x,v)\varphi_{n,k+\ell}(x',v')\right).$$

(2) *For any* $(m,k)\,(m\in\mathbb{N}_0, 0\le k\le m)$ *it holds that*

$$K\varphi_{m,k}(x,v) = (m/2)\,\varphi_{m,k}(x,v)+\sqrt{k(m-k+1)}\varphi_{m,k-1}(x,v).$$

(3) *For any* $(m,k)\,(m\in\mathbb{N}_0, 0\le k\le m)$ *and* $(n,\ell)\,(n\in\mathbb{N}_0, 0\le \ell\le n)$

$$\langle\varphi_{m,k},\varphi_{n,\ell}\rangle = \delta_{m,n}\delta_{k,\ell}.$$

For the proof of Theorem 7.8 we prepare the following two propositions.

Proposition 7.9. *K has the following representation by the operators M and N:*

$$K=\frac{1}{2}\big(M^*M+N^*N+2M^*N\big).$$

Proof.

$$\begin{aligned}K &= b_v^*b_v+\frac{1}{2}\big(b_v^*a_x-a_x^*b_v\big)\\ &= \frac{1}{4}\Big\{2\big(M^*+N^*\big)\big(M+N\big)+\big(M^*+N^*\big)\big(N-M\big)-\big(N^*-M^*\big)\big(M+N\big)\Big\}\\ &= \frac{1}{2}\big(M^*M+N^*N+2M^*N\big).\end{aligned}$$

□

Proposition 7.10.

(1) *For any* $(m,\ell)\in\mathbb{N}_0\times\mathbb{N}_0$ *we have*

$$a_x\psi(x,v)=b_v\psi(x,v)=0, M\psi(x,v)=N\psi(x,v)=0.$$

Moreover

$$M\psi_{m,\ell}(x,v)=\sqrt{m}\psi_{m-1,\ell}(x,v),\quad N\psi_{m,\ell}(x,v)=\sqrt{\ell}\psi_{m,\ell-1}(x,v)$$

(2) *For nonnegative integers k we have*

$$M^k(M^*)^k\psi(x,v)=k!\psi(x,v), N^k(N^*)^k\psi(x,v)=k!\psi(x,v)\quad (k\in\mathbb{N}_0).$$

(3) *For nonnegative integers k we have*

$$Q^k\Big(\psi(x,v)\psi(x',v')\Big)=k!\psi_{k,0}(x,v)\psi_{0,k}(x',v')\quad (k\in\mathbb{N}_0).$$

(4) *We have for any* $\ell,k,m\in\mathbb{N}_0$

$$C_\ell(S_M,k)\Big(\psi_{k,m}(x,v)\psi_{0,r}(x',v')\Big)=\sqrt{\frac{(k+\ell)!\ell!}{k!}}\psi_{k+\ell,m}(x,v)\psi_{\ell,r}(x',v').$$

(5) *We have for any* $\ell,k,m\in\mathbb{N}_0$

$$C_\ell(S_N,k)\Big(\psi_{m,0}(x,v)\psi_{r,k}(x',v')\Big)=\sqrt{\frac{(k+\ell)!\ell!}{k!}}\psi_{m,\ell}(x,v)\psi_{r,k+\ell}(x',v').$$

Proof. (1) The proof of the assertion for the operators M and N are similar.

$$M\psi_{m,\ell}(x,v) = \left\{[M,(M^*)^m]\psi_{0,\ell}(x,v) + (M^*)^m M\psi_{0,\ell}(x,v)\right\}/\sqrt{m!}$$
$$= m(M^*)^{m-1}\psi_{0,\ell}(x,v)/\sqrt{m!} = \sqrt{m}\psi_{m-1,\ell}(x,v).$$

(2) By (1) we have

$$\frac{M^k(M^*)^k}{k!}\psi(x,v) = M^k\psi_{k,0}(x,v)/\sqrt{k!} = M^{k-1}\psi_{k-1,0}(x,v)/\sqrt{(k-1)!}$$
$$= \frac{M^{k-1}(M^*)^{k-1}}{(k-1)!}\psi(x,v).$$

We get the assertion by the induction. The proof of the assertion for the operator N is similar.

(3) We will show by the induction with respect to k. We note that

$$(M^* + M')\big(\psi_{m,k}(x,v)\psi_{n,\ell}(x',v')\big)$$
$$= \sqrt{m+1}\psi_{m+1,k}(x,v)\psi_{n,\ell}(x',v') + \sqrt{n}\psi_{m,k}(x,v)\psi_{n-1,\ell}(x',v')$$

and

$$(N + (N')^*)\big(\psi_{m,k}(x,v)\psi_{n,\ell}(x',v')\big)$$
$$= \sqrt{k}\psi_{m,k-1}(x,v)\psi_{n,\ell}(x',v') + \sqrt{\ell+1}\psi_{m,k}(x,v)\psi_{n,\ell+1}(x',v').$$

Applying these indentities, we have

$$Q\big(\psi_{m,0}(x,v)\psi_{0,\ell}(x',v')\big) = \sqrt{(m+1)(\ell+1)}\psi_{m+1,0}(x,v)\psi_{0,\ell+1}(x',v').$$

So we have

$$Q^k\Big(\psi(x,v)\psi(x',v')\Big) = k!\psi_{k,0}(x,v)\psi_{0,k}(x',v').$$

(4) The following equations hold:

$$S_M\Big(\psi_{m,k}(x,v)\psi_{n,\ell}(x',v')\Big)$$
$$= \sqrt{(m+1)(n+1)}\psi_{m+1,k}(x,v)\psi_{n+1,\ell}(x',v')$$
$$+ (m+n+1)\psi_{m,k}(x,v)\psi_{n,\ell}(x',v') + \sqrt{mn}\psi_{m-1,k}(x,v)\psi_{n-1,\ell}(x',v'),$$

$$S_N\Big(\psi_{m,k}(x,v)\psi_{n,\ell}(x',v')\Big)$$
$$= \sqrt{(k+1)(\ell+1)}\psi_{m,k+1}(x,v)\psi_{n,\ell+1}(x',v')$$
$$+ (k+\ell+1)\psi_{m,k}(x,v)\psi_{n,\ell}(x',v') + \sqrt{k\ell}\psi_{m,k-1}(x,v)\psi_{n,\ell-1}(x',v'),$$

which will be shown as follows: By (1) we have

$$(M + (M')^*)\big(\psi_{m,k}(x,v)\psi_{n,\ell}(x',v')\big)$$
$$= \sqrt{m}\psi_{m-1,k}(x,v)\psi_{n,\ell}(x',v') + \sqrt{n+1}\psi_{m,k}(x,v)\psi_{n+1,\ell}(x',v').$$

The proof for $S_M = (M + (M')^*)(M^* + M')$ is complete by the above equation. Similarly we get the result for S_N.

We will show the assertion (4) by the induction with respect to ℓ. The assertion holds for $\ell = 0$ because $C_0(P, k) = 1$ for all P. Note that for any operator P the operators $C_j(P, k)$ defined by (7.5) satisfy the following equation of which we give a proof below.

$$C_{\ell+1}(P, k) = PC_\ell(P, k) - (2\ell + k + 1)C_\ell(P, k) - \ell(\ell + k)C_{\ell-1}(P, k), \tag{7.6}$$

$$C_0(P, k) = 1. \tag{7.7}$$

Assume that (4) holds for all $\ell \le j$. Then by the equation (7.6) we have

$$\begin{aligned} &C_{j+1}(S_M, k)\Big(\psi_{k,m}(x, v)\psi_{0,r}(x', v')\Big) \\ &\quad= S_M C_j(S_M, k)\Big(\psi_{k,m}(x, v)\psi_{0,r}(x', v')\Big) \\ &\qquad- (2j + k + 1)C_j(S_M, k)\Big(\psi_{k,m}(x, v)\psi_{0,r}(x', v')\Big) \\ &\qquad- j(j + k)C_{j-1}(S_M, k)\Big(\psi_{k,m}(x, v)\psi_{0,r}(x', v')\Big) \\ &\quad= \sqrt{\frac{(k + j)!j!}{k!}} S_M\left(\psi_{k+j,m}(x, v)\psi_{j,r}(x', v')\right) \\ &\qquad- (2j + k + 1)\sqrt{\frac{(k + j)!j!}{k!}}\psi_{k+j,m}(x, v)\psi_{j,r}(x', v') \\ &\qquad- j(j + k)\sqrt{\frac{(k + j - 1)!(j - 1)!}{k!}}\psi_{k+j-1,m}(x, v)\psi_{j-1,r}(x', v'). \end{aligned}$$

On the other hand it holds by the argument for S_M

$$\begin{aligned} S_M\left(\psi_{k+j,m}(x, v)\psi_{j,r}(x', v')\right) &= \sqrt{(j + k + 1)(j + 1)}\psi_{k+j+1,m}(x, v)\psi_{j+1,r}(x', v') \\ &\quad+ (2j + k + 1)\psi_{k+j,m}(x, v)\psi_{j,r}(x', v') \\ &\quad+ \sqrt{j(j + k)}\psi_{k+j-1,m}(x, v)\psi_{j-1,r}(x', v'). \end{aligned}$$

So we get (4) for $j + 1$.

We will give a proof of (7.6). Set an analytic function $f(z)$ as follows:

$$f(z) = \frac{1}{(1 + z)^{k+1}} \exp\left(\frac{z}{1 + z}P\right).$$

Then the following equations hold.

$$f'(z)(1 + z)^2 + (k + 1)(1 + z)f(z) = f(z)P, \quad f(0) = 1.$$

Differentiating the above equation ℓ times, we get

$$\begin{aligned} &(1 + z)^2 f^{(\ell+1)}(z) + 2\ell(1 + z)f^{(\ell)}(z) + \ell(\ell - 1)f^{(\ell-1)}(z) \\ &\qquad+ (k + 1)\left\{(1 + z)f^{(\ell)}(z) + \ell f^{(\ell-1)}(z)\right\} = f^{(\ell)}(z)P. \end{aligned}$$

Putting $z = 0$ we get (7.6) for $C_\ell(P, k) = f^{(\ell)}(0)$.

(5) is proved by the similar way to (4). □

Finally we will give the proof of Theorem 7.8.

(1) By Proposition 7.10 we obtain the following equation:

$$\begin{aligned} &C_\ell(S_N,k)C_{j-\ell}(S_M,k)Q^k\Big(\psi(x,v)\psi(x',v')\Big) \\ &\quad = k!C_\ell(S_N,k)C_{j-\ell}(S_M,k)\Big(\psi_{k,0}(x,v)\psi_{0,k}(x',v')\Big) \\ &\quad = k!\sqrt{\frac{(j-\ell)!(j-\ell+k)!}{k!}}C_\ell(S_N,k)\Big(\psi_{k+j-\ell,0}(x,v)\psi_{j-\ell,k}(x',v')\Big) \\ &\quad = k!\sqrt{\frac{(j-\ell)!(j-\ell+k)!}{k!}}\sqrt{\frac{\ell!(\ell+k)!}{k!}}\psi_{k+j-\ell,\ell}(x,v)\psi_{j-\ell,k+\ell}(x',v') \\ &\quad = \sqrt{(j-\ell)!(j-\ell+k)!\ell!(\ell+k)!}\psi_{k+j-\ell,\ell}(x,v)\psi_{j-\ell,k+\ell}(x',v'). \end{aligned}$$

Finally we have for $z = e^{-t/2}$

$$\begin{aligned} e(t;x,v,x',v') &= \sum_{k=0}^\infty \frac{(-tz)^k}{k!}\frac{1}{(1+z)^{2k+2}}\exp\Big(\frac{zS}{1+z}\Big)Q^k\Big(\psi(x,v)\psi(x',v')\Big) \\ &= \sum_{k=0}^\infty \frac{(-tz)^k}{k!}\sum_{j=0}^\infty z^j\sum_{\ell=0}^j \frac{1}{(j-\ell)!\ell!}C_\ell(S_N,k)C_{j-\ell}(S_M,k)Q^k\Big(\psi(x,v)\psi(x',v')\Big) \\ &= \sum_{k=0}^\infty \frac{(-tz)^k}{k!}\sum_{j=0}^\infty z^j\sum_{\ell=0}^j \sqrt{\frac{(j+k-\ell)!(\ell+k)!}{(j-\ell)!\ell!}}\psi_{k+j-\ell,\ell}(x,v)\psi_{j-\ell,k+\ell}(x',v') \\ &= \sum_{n=0}^\infty z^n\sum_{k=0}^n \frac{(-t)^k}{k!}\sum_{\ell=0}^{n-k}\sqrt{\frac{(n-\ell)!(\ell+k)!}{(n-k-\ell)!\ell!}}\psi_{n-\ell,\ell}(x,v)\psi_{n-k-\ell,k+\ell}(x',v') \\ &= \sum_{n=0}^\infty z^n\sum_{k=0}^n \frac{(-t)^k}{k!}\sum_{\ell=0}^{n-k}\sqrt{\frac{(n-\ell)!(\ell+k)!}{(n-k-\ell)!\ell!}}\varphi_{n,\ell}(x,v)\varphi_{n,k+\ell}(x',v'). \end{aligned}$$

(2) By Proposition 7.9 and Proposition 7.10 we have

$$\begin{aligned} K\psi_{m,k} &= \frac{1}{2}\big\{M^*\sqrt{m}\psi_{m-1,k} + N^*\sqrt{k}\psi_{m,k-1} + 2M^*\sqrt{k}\psi_{m,k-1}\big\} \\ &= \frac{1}{2}\big\{m\psi_{m,k} + k\psi_{m,k} + 2\sqrt{k(m+1)}\psi_{m+1,k-1}\big\} \\ &= \frac{1}{2}(m+k)\psi_{m,k} + \sqrt{k(m+1)}\psi_{m+1,k-1}. \end{aligned}$$

The above equation means

$$K\varphi_{m,k}(x,v) = \frac{m}{2}\varphi_{m,k}(x,v) + \sqrt{k(m-k+1)}\varphi_{m,k-1}(x,v).$$

(3) Using Proposition 7.10, we have for $m \geq n$

$$\begin{aligned}\sqrt{m!n!}\langle \psi_{m,0}, \psi_{n,0}\rangle &= \int \psi(x,v)\ M^{m-n}M^n(M^*)^n\psi(x,v)dxdv \\ &= \int \psi(x,v)\ M^{m-n}n!\psi(x,v)dxdv = n!\delta_{m,n}\int |\psi(x,v)|^2 dxdv = n!\delta_{m,n}.\end{aligned}$$

If $\ell \geq k$, we have

$$\begin{aligned}\sqrt{m!k!n!\ell!}\langle \psi_{m,k}, \psi_{n,\ell}\rangle &= \int (M^*)^m(N^*)^k\psi(x,v)\ (M^*)^n(N^*)^\ell\psi(x,v)dxdv \\ &= \int (M^*)^m N^{\ell-k}N^k(N^*)^k\psi(x,v)\ (M^*)^n\psi(x,v)dxdv \\ &= k!\delta_{k,\ell}\int (M^*)^m\psi(x,v)\ (M^*)^n\psi(x,v)dxdv \\ &= k!\sqrt{m!n!}\delta_{k,\ell}\langle \psi_{m,0}, \psi_{n,0}\rangle = k!n!\delta_{k,\ell}\delta_{m,n}.\end{aligned}$$

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Chisato Iwasaki
Department of Mathematics
University of Hyogo
2167 Shosha Himeji
671-2201, Japan
e-mail: iwasaki@sci.u-hyogo.ac.jp