A Kummer function based alternative Zeta function theory to solve the Riemann Hypothesis and the binary Goldbach conjecture

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1. The tool set

In order to prove the Riemann Hypothesis (RH) the Polya criterion can not be applied in combination with the Müntz formula ((TiE) 2.11). The Müntz formula is divergent in the critical stripe due to the asymptotics behavior of the exponential function. The conceptual challenge is about the not vanishing constant Fourier term of the Gaussian function and its related impact on the Poisson summation formula resp. on the corresponding Riemann duality equation ((EdH) 1.7). The proposed alternative "baseline" function is the Hilbert transform of the Gaussian function, which is the Dawson function ((GrI) 3.896, (OlF) p. 44)

$$F(x) \coloneqq e^{-x^2}\int_0^x e^{t^2}dt = \int_0^\infty e^{-t^2}\sin(2xt)dt = x\,{}_1F_1\left(1,\tfrac{3}{2};-x^2\right) = xe^{-x^2}\,{}_1F_1\left(\tfrac{1}{2},\tfrac{3}{2};x^2\right).$$

In the context of the $Li(x)$ function RH approximation criterion we note that the Dawson function $F\left(\sqrt{x}\right)$ enjoys an only polynomial asymptotics in the form $O\left(x^{-1/2}\right)$. This, as well as the asymptotics $2dF(x) \sim \frac{dx}{x} = dlogx$, is a consequence of the identity $F'(x) + 2xF(x) = 1$ ((LeN) (9.13.3)).

The Mill's ration function of the standard normal law, given by

$$M(x) \coloneqq e^{x^2}\int_x^\infty e^{-t^2}dt$$

is strictly log-convex ((BaÁ), (RuM)). It is related to the Dawson function by the solutions of the self-adjoint Whittaker operator (playing a key role in spectral and scattering theory, e.g. for the 3-D spherical, harmonic quantum oscillator, e.g. (DeJ)) by the formula (BuH) p. 209)

$$\int_0^z e^{\mp x^2}dx = z\cdot {}_1F_1\left(\tfrac{1}{2},\tfrac{3}{2};\mp z^2\right) = c\cdot e^{\mp\frac{1}{2}z^2}\cdot z^{-\frac{1}{2}}\cdot M_{\pm\frac{1}{4},\frac{1}{4}}(z^2) \quad , c \coloneqq \frac{\sqrt{\pi}}{2}$$

leading to

$$F(x) = e^{-x^2}\int_0^x e^{t^2}dt = x\cdot e^{-x^2}\cdot {}_1F_1\left(\frac{1}{2},\frac{3}{2};x^2\right) = c\cdot e^{-\frac{1}{2}x^2}\cdot x^{-\frac{1}{2}}\cdot M_{-\frac{1}{4},\frac{1}{4}}(x^2)$$

$$M(x) = e^{x^2}\int_0^x e^{-t^2}dt = x\cdot e^{x^2}\cdot {}_1F_1\left(\frac{1}{2},\frac{3}{2};-x^2\right) = c\cdot e^{\frac{1}{2}x^2}\cdot x^{-\frac{1}{2}}\cdot M_{\frac{1}{4},\frac{1}{4}}(x^2)$$

We note that the term $\Gamma\left(1+\frac{s}{2}\right)$ was the originally introduced notation by Gauss for the Gamma function, appearing *„to me much more natural and Riemann´s use of it gives me a welcome opportunity to introduce it*" (quote, H. M. Edwards in (EdH), footnote, p. 8). The Riemann (exact) „approximation" function is calculated out of the $\xi(s)$ –term $\Gamma\left(1+\frac{s}{2}\right)$ given by ((EdH) 1.16)

$$\int_x^\infty \frac{dt}{t(t^2-1)logt} = \int_x^\infty \frac{1}{tlogt}\left(\sum t^{-2n}\right)dt = \sum_{n=1}^\infty \int_x^\infty t^{-2n}\frac{dt}{tlogt} = \frac{1}{2\pi i}\frac{1}{logx}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{log\Gamma\left(1+\frac{s}{2}\right)}{s}\right]x^s ds.$$

The relationship between $\Gamma\left(1+\frac{s}{2}\right)$ and the Mellin transform of the considered Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};-x^2\right)$ is given by ((GrI) 7.612)

$$\Gamma\left(1+\tfrac{s}{2}\right) = s(1-s)\int_0^\infty x^s\,{}_1F_1\left(\tfrac{1}{2},\tfrac{3}{2};-x^2\right)\frac{dx}{x}.$$

We note that the Mellin transform

$$\int_0^\infty x^s\,{}_1F_1\left(\tfrac{1}{2},\tfrac{3}{2};-x^2\right)\frac{dx}{x} = \frac{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)}{1-s} = \frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)} \quad , \quad 0 < Re(s) < 1.$$

is mirrowing the only formally valid representation of the entire Zeta given by ((EdH) 10.3)

$$\int_0^\infty u^{-s}G(u)du = \frac{2\xi(s)}{s(s-1)}.$$

In ((BuH) p.184) the product representation of the Kummer functions

$$_1F_1(a,c,z) = \frac{1}{\Gamma(c)} e^{\frac{a}{c}z} \prod (1 - \frac{z}{\alpha_n}) e^{z/\alpha_n}$$

are provided, where α_n denotes the infinite set of zeros of $_1F_1(a,c,z)$, i.e. the Kummer functions are elements of the Laguerre-Polya class LP (CaD), which is related to the topic *„zeros of certain trigonometric integrals in the context of entire transcendental higher genus 1 functions*" (PoG2).

The fractional part function related Zeta function theory is provided in ((TiE) II). The Hilbert transform of the fractional part function is given by the $\log(sinx)$ −function. A $log(sinx)$ −function based Zeta function theory deals with a *„Zeta*" function $\zeta^*(s)$ the form

$$\zeta^*(s) = \frac{\cot(\frac{\pi}{2}s)}{1-s} \cdot \zeta(s) = \frac{\tan(\frac{\pi}{2}(1-s))}{1-s} \cdot \zeta(s) \quad .$$

The correspondingly defined distributional ("periodical") Hilbert space framework deals with the Hilbert spaces $H_0^{\#}(0,1)$ $H_{-1/2}^{\#}(0,1)$, $H_{-1}^{\#}(0,1)$, with its relationship to the Bagchi reformulation of the Nyman-Beurling RH criterion. This criterion basically becomes a *„dense embedding*" argument of $H_{-1/2}^{\#}(0,1)$ into $H_{-1}^{\#}(0,1)$, as the cot and the Zeta function on the critical line are $\in H_{-1}^{\#}(0,1)$.

The considered Hilbert space in [BaB] is about of all sequences $a = \{a_n | n \in N\}$ of complex numbers such that

$$\sum_{n=1}^{\infty} \omega_n |a_n|^2 < \infty \quad \text{with} \quad \frac{c_1}{n^2} \leq \omega_n \leq \frac{c_2}{n^2}$$

which is isomorph to the Hilbert space $H_{-1} \cong l_2^{-1}$. Let $\gamma := \{11,1,1,\ldots\ldots\}$, then it holds

$$\|\gamma\|_{-1}^2 = \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

i.e. $\gamma \in l_2^{-1}$, resp. the Zeta function on the critical line $\Xi \in H_{-1}(-\infty,\infty)$.

Theorem (Bagchi-Nyman criterion, (BaB)): Let

$$\gamma_k := \left\{ \rho(\frac{n}{k}) \middle| n = 1,2,3,\ldots. \right\} \quad \text{for} \quad k = 1,2,3,\ldots$$

and Γ_k be the closed linear span of γ_k. Then the Nyman criterion states that the following statements are equivalent:

i) The Riemann Hypothesis is true

ii) $\gamma \in \overline{\Gamma}_k$.

As $l_2^{-1/2}$ is dense in l_2^{-1} with respect to the l_2^{-1} −norm, then γ belongs to the closed linear span of a double infinite matrix $\{\gamma_k^H\}_{k \in N}$, i.e.

$$\gamma \in l_2^{-1} = l_2^{-1/2} \overset{----\|\circ\|_{-1}}{}$$

if $\gamma_k^H \in l_2^{-1/2}$.

The zeros of the considered Kummer functions (and therefore also the zeros of the solution of the corresponding Whittaker eigenvalue equations) enjoy appropriate behaviors, e.g. for the real part values ω_n of ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ it holds (SeA)

i) $n-\frac{1}{2}<\omega_n<n$, $\frac{1}{2}<\omega_1<a\coloneqq s_n\coloneqq\frac{\omega_n}{n}\to 1\ n\in N$

ii) $2n-1<2\omega_n<2n<\omega_n+\omega_{n+1}<2n+1<2\omega_{n+1}<2(n+1)$

iii) the sequences $2\omega_n$ and $\omega_n+\omega_{n+1}$ fulfill the Hadamard gap condition

$\frac{\omega_{n+1}}{\omega_n}>\frac{n+\frac{1}{2}}{n}=1+\frac{1}{2n}>q>1$ resp. $\frac{\omega_{n+1}+\omega_{n+2}}{\omega_n+\omega_{n+1}}>\frac{2n+2}{2n+1}=1+\frac{1}{2n+1}>q>1$

iv) $\theta\coloneqq\frac{1}{4}<\frac{\omega_{n+1}}{2}-\frac{\omega_n}{2}<1-\frac{1}{4}=1-\theta$.

For the related sequences $a_n\coloneqq\frac{2\omega_n}{2n}-\frac{1}{2}$, $b_n\coloneqq\frac{\omega_{n+1}+\omega_n}{2n}-\frac{1}{2}$ it therefore follows

i) $0<a_1=\omega_1-\frac{1}{2}\le a_n\to\frac{1}{2}$, $\frac{1}{2}\leftarrow b_n<b_1=\frac{\omega_1+\omega_2-1}{2}<1$

ii) $a_n,b_n-a_n\in(0,\frac{1}{2})$, $b_n,1-a_n\in(\frac{1}{2},1)$.

The sequence $s_n\coloneqq\frac{\omega_n}{n}$ fulfills the Hardy-Littlewood condition $|s_{n+1}-s_n|<\frac{K}{n}$, i.e. it has a defined Abel average ((EdH) 12.7)

$$\lim_{r\uparrow 0}\frac{s_1r+s_1r^2+s_3r^3+\cdots\ldots}{r+r^2+r^3+\cdots\ldots}=L$$

and (with respect to the Goldbach conjecture below) the following inequalities are valid

$$c_1\cdot\left[{}_1F_1\left(\frac{1}{2},c;x\right)\right]^2\le{}_1F_1(a_n,c;x)\cdot{}_1F_1(1-a_n,c;x)\le\left[{}_1F_1\left(\frac{1}{2},c;x\right)\right]^2.$$

The identities $n^2=1+3+\cdots+(2n-1)$ and $n(n+1)=2+4+\cdots 2n$ in combination with the imaginary parts of the zeros of the considered special Kummer function lead to the inequality

$$n-\frac{1}{2}<\frac{2}{n}\sum_{k=1}^{n}\omega_k<n+\frac{1}{2}$$

enabling an alternative Rogers-Ramanujan Continued Fraction with corresponding identities. We note that the integer "2" is not an element of the set $\boldsymbol{A}\coloneqq\{[2\omega_n],n\in N\}$, i.e. the Snirelmann density of $\boldsymbol{A}$ is $\le\frac{1}{2}$.

The Kummer function zeros related sequence ω_n enables the following replacements

$2n\ \to\ 2\omega_n\ \to\ \omega_n+\omega_{n+1}$ $\quad e^{i\pi(2n)x}\ \to\ e^{i\pi(2\omega_n)x}\ \to\ e^{i\pi(\omega_n+\omega_{n+1}-1)x}$ $\quad S(x)\coloneqq\sum_n e^{2\pi inx}\ \to\ S_{\square}^{(1)}(x)\coloneqq\sum_n e^{i\pi(2\omega_n)x}\ \to\ S_{\square}^{(2)}(x)\coloneqq\sum_n e^{i\pi(\omega_n+\omega_{n+1}-1)x}$

The number $N(T)$ of zeros of the Zeta function with the ordinate axis values $0<\gamma_1\le\gamma_2\le\cdots$ within $0<t\le T$ in the critical stripe fulfills $N(T)=A\cdot TlogT+O(T)$, from which it follows the

Theorem 468 ((LaE4) p. 139): the series $S(\sigma)=\sum_{n=1}^{\infty}\frac{\sin(\gamma_n\sigma)}{\gamma_n}e^{-\gamma_n\sigma}$ is absolutely convergent with $\lim_{\sigma\to 0}\frac{S(\sigma)}{log\frac{1}{\sigma}}=\alpha_{135}$.

With respect to the zeros of the Kummer function above this gives, that the series $\sum_{p\in P}^{\square}\frac{\sin(\omega_p\sigma)}{\omega_p}e^{-\omega_p\sigma}$ is absolutely convergent, as well. From (BeB) 5, Corollary 4, we recall that the representation ($(Re(s)>1)$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^s}=\prod_p(1-\frac{sin(\frac{\pi}{2}s)}{p^s})^{-1}$$

implies the convergence of the series $\sum_p p^{-1}sin(\frac{\pi}{2}p)$.

Let H and M denote the Hilbert and the Mellin transform operators. For the Gaussian function $f(x)$ it holds

$$M[f](s) = \frac{1}{2}\pi^{-s/2}\Gamma(\tfrac{s}{2}) \ , \quad M\big[-xf'(x)\big](s) = \frac{s}{2}\pi^{-s/2}\Gamma(\tfrac{s}{2}) = \frac{1}{2}\pi^{-s/2}\Pi(\tfrac{s}{2}) \ .$$

The corresponding entire Zeta function is given by ([EdH] 1.8)

$$\xi(s) := \frac{s}{2}\Gamma(\tfrac{s}{2})(s-1)\pi^{-s/2}\zeta(s) = (1-s)\cdot\zeta(s)M[-xf'(x)](s) = \xi(1-s).$$

Replacing the Gaussian function by the Dawson function (which is the Hilbert transform of the Gaussian function) leads to an alternative entire Zeta function $\xi^*(s)$ in the form

$$\xi^*(s) := \frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma(\tfrac{s}{2})\tan(\tfrac{\pi}{2}s)\cdot\zeta(s) = \zeta(s)\cdot M\left[\frac{d}{dx}[-x\cdot f_H(x)]\right](s)$$

with same zeros as $\xi(s)$, as it holds $s(1-s)\xi^*(s)\xi^*(1-s) = \pi\xi(s)\xi(1-s)$.

Replacing the Mellin transform of the Gaussian function in the definition of the $\xi(s)$ –function goes along with a replacement of the Gauss „Gamma" function definition $\Gamma\left(1+\frac{s}{2}\right)$ by

$$\rightarrow \qquad \Gamma^*\left(\frac{s}{2}\right) := \Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right) = \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{\Gamma\left(1+\frac{s-1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma\left(1+\frac{s}{2}\right)}{(k-\frac{1}{2})^2-(\frac{s}{2})^2}.$$

It results into a correspondingly modified Riemann approximation error function with now newly three summation terms governed by product fomula

$$\Gamma(1\pm s) = \prod\prod(1+\frac{1}{n})^{\pm s}(1\pm\frac{s}{n})^{-1}$$

with improved ($|li(x)-\pi(x)|$ approximation behavior. We mention the product representation ((AbM) 6.1.3)

$$\Gamma^*\left(\frac{s}{2}\right) := \tan(\tfrac{\pi}{2}s)\Gamma\left(\frac{s}{2}\right) = \frac{\pi}{2}e^{-\gamma s}\prod(1+\frac{s}{2n})^{s/2n}\frac{(1-(\frac{s}{2k})^2}{(1-(\frac{s}{(2k-1)})^2} \ .$$

As the Hilbert transform defines a convolution operator, the Dawson function approach enables the *„zeros of entire functions*" analysis techniques in the context of the Hilbert-Polya conjecture (e.g. (CaD)).

The Dawson function approach also enables an *„analysis of zeros of certain trigonometric integrals and entire higher genus-1 functions*" (PoG2), based on the identity ((GrI) 3.896) $F(x) = e^{-x^2}H(x)$ with $H(x) := x\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};x^2\right)$.

The proposed alternative entire Zeta function $\xi^*(\frac{1}{1-s})$ is also suggested to verify the corresponding Li criteron (LiX). This criterion results from a necessary and sufficient condition, that the logarithmic of the function $\xi(\frac{1}{1-s})$ be analytic in the unit disk resp. that the sequence of real constants $\sigma_n := \frac{1}{\Gamma(n)}\frac{d^n}{ds^n}[s^{n-1}log\xi(s)]_{s=1}$ are not-negative (CoM). The proof of the Li criterion is built on the two representations of the Zeta function, its (product) representation over all its nontrivial zeros ((HdE) 1.10) and Riemann's integral representation derived from the Riemann duality equation, based on the Jacobi theta function ((EdH) 1.8). In (CoF), (KeJ) corresponding Li/Keiper constants are considered.

With respect to the binary Goldbach conjecture the above enables a new circle method on the unit circle, going along with corresponding *additive* number theory problem adequate arithmetic functions.

The Landau theorem to build appropriate arithmetic functions, is a special case of the (PoG1):

Generalized Landau Theorem: Let $w(x)$ a positive, non decreasing function with $\lim_{n\to\infty}\frac{w(\beta x)}{w(\alpha x)} = 1$ with α,β positive numbers. Then

$$\lim_{n\to\infty}\frac{w(x)}{x}\sum_{n\le x} f(\frac{x}{n}) = \int_0^1 f(t)dt \ .$$

Applying this theorem in the context of the concpetual approach of this paper leads to the definitions (see also e.g. (BrK4) Notes O27/37)

$$\sigma^*_{\square}(x) \coloneqq \frac{1}{2}\left[\sigma^{(1)}_{\square}(x) + \sigma^{(2)}_{\square}(x)\right]$$

with

$$\sigma^{(1)}_{\square}(x) \coloneqq \frac{1}{x}h^{(1)}\left(\frac{1}{x}\right)\sum_{n\le x}{}_{\square}\, h^{(1)}\left(\frac{x}{2\omega_n}\right) \ , \ \sigma^{(2)}_{\square}(x) \coloneqq \frac{1}{x}h^{(2)}\left(\frac{1}{x}\right)\sum_{n\le x}{}_{\square}\, h^{(2)}\left(\frac{x}{\omega_n+\omega_{n+1}-1}\right)$$

an

$$h^{(1)}(x) \coloneqq -(\pi x)cot^{(1)}(\pi x) \ , \ h^{(2)}(x) \coloneqq -(\pi x)cot^{(2)}(\pi x).$$

This paper is concerned with completely different topics, but all of those are concerned with the very „big" and the very „small", what ever it means; the harmonic numbers $H_n = \sum_{k=1}^{n}\frac{1}{k}$ might give a frist impression from a number theory point of view: the H_n are always fractions (except for $H_1 = 1$, $H_2 = 1.5$, $H_6 = 2.45$), the series is divergent, but the number n that the sum H_n past 100 is in the size of 10^{43}, i.e. a computer which takes 10^{-9} seconds to add each new term to the sum will have been completed in not less than 10^{17} (American) billion years (HaJ). The Fourier theory of cardinal functions enables a correspondingly absolute convergent cardinal series in the form $C(x) \coloneqq \frac{\sin(\pi x)}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{H_{2n}-\frac{1}{2}H_n}{x-n}$ alternatively to $\frac{\sin(\pi x)}{\pi x}$ (WhJ2). What we propose in this paper is an alternative approximation term in the form $H_n^* \coloneqq \frac{1}{2}H_n + \frac{\omega_n}{n}$ with $\lim_{n\to\infty}\left[H_n^* - \log\sqrt{e\cdot n}\right] = \frac{\gamma+1}{2}$.

For the harmonic numbers $2h_n = \sum_{k=1}^{n}\frac{2}{2k-1} = 2H_{2n} - H_n$, $H_n = \sum_{k=1}^{n}\frac{1}{k}$ resp. $c_n \coloneqq \frac{2h_n}{n}$ it holds the Fourier series representation (ElL)

$$\frac{\pi}{2}log\left(tan\left(\frac{\pi}{2}x\right)\right) = -\sum\frac{2h_n}{n}sin(2\pi nx) = -\sum c_n\, sin(2\pi nx) \in L_2^{\#}(0,1)$$

i.e.

$$\sum_{n=1}^{\infty} c_n^2 < \infty.$$

The following two a priori estimates ((HaJ) p. 47, resp. (Goldbach conjecture) section 1g below)

$$\frac{1}{2n} < \frac{1}{2}(H_n - logn) < \frac{1}{2} \quad , \quad \frac{1}{2} - \frac{1}{2n} < a_n \coloneqq s_n - \frac{1}{2} = \frac{\omega_n}{n} - \frac{1}{2} < \frac{1}{2}$$

leads to an alternative approximation term in the form

$$\frac{1}{2} < H_n^* - \log\sqrt{e\cdot n} = \frac{1}{2}H_n + \frac{\omega_n}{n} - \log\sqrt{e\cdot n} = \frac{1}{2}(H_n - logn - 1) + \frac{\omega_n}{n} < 1$$

with

$$\lim_{n\to\infty}\left[\frac{1}{2}(H_n - logn - 1) + \frac{\omega_n}{n}\right] = \frac{1}{2}(\gamma - 1) + 1 = \frac{\gamma+1}{2} \ .$$

2. An alternative entire Zeta function $\xi^*(s)$ based on Kummer / Dawson and $\tan\left(\pi\frac{s}{2}\right)=\cot\left(\pi\frac{1-s}{2}\right)$ functions

Let H and M denote the Hilbert and the Mellin transform operators. For the Gaussian function $f(x)$ it holds

$$M[f](s)=\tfrac{1}{2}\pi^{-s/2}\Gamma(\tfrac{s}{2})\ ,\quad M\big[-xf'(x)\big](s)=\tfrac{s}{2}\pi^{-s/2}\Gamma(\tfrac{s}{2})=\tfrac{1}{2}\pi^{-s/2}\Pi(\tfrac{s}{2})\ .$$

The corresponding entire Zeta function is given by ([EdH] 1.8)

$$\xi(s):=\tfrac{s}{2}\Gamma(\tfrac{s}{2})(s-1)\pi^{-s/2}\zeta(s)=(1-s)\cdot\zeta(s)M[-xf'(x)](s)=\xi(1-s).$$

Putting

$$G(u):=\textstyle\sum_{-\infty}^{\infty}e^{-\pi n^2u^2}=\sum_{-\infty}^{\infty}f(nu)$$

Riemann's functional equation implies, that the invariant operator $g(x)\to\int_0^\infty g(ux)G(u)du$ is formally self-adjoint. A valid invariant operator would prove the Hilbert Polya conjecture. But the operator has no transform at all (due to the not vanishing constant Fourier term of the Gaussian function); that is the integral $\int_0^\infty u^{-s}G(u)du$, which is formally represented in the form ((EdH) 10.3)

$$\int_0^\infty u^{-s}G(u)du=\frac{2\xi(s)}{s(s-1)}$$

does not converge for any s. The central idea is to replace $M[-xf'(x)](s)\to M[-f_H(x)](s)$ with $f_H(x):=H[f](x)$ $(\hat{f}_H(0)=0)$ and

$$M[f_H(x)](s)=M\left[2\pi x\,{}_1F_1\left(1,\tfrac{3}{2},-\pi x^2\right)\right](s)=M\big[2\sqrt{\pi}F\big(\sqrt{\pi}x\big)\big](s)=\tfrac{1}{2}\pi^{\frac{1-s}{2}}\Gamma(\tfrac{s}{2})\,tan(\tfrac{\pi}{2}s)\ ,$$

leading to the alternative *entire* Zeta function $\xi^*(s)$ in the form

$$\xi^*(s):=\tfrac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma(\tfrac{s}{2})\,tan(\tfrac{\pi}{2}s)\cdot\zeta(s)=\zeta(s)\cdot M\left[\frac{d}{dx}[-x\cdot f_H(x)]\right](s)\ .$$

The link to the Zeta functions $\xi(s)$ resp. $\zeta(s)$ is given by the equation

$$\xi^*(s)\xi^*(1-s)=\pi^{\frac{\xi(s)\xi(1-s)}{s(1-s)}},$$

i.e. $\xi(s)$ and $\xi^*(s)$ do have the same set of zeros in the critical stripe.

The corresponding invariant operator $f(x)\to\int_0^\infty f(ux)G_H(u)du$ is built on

$$G_H(u):=\textstyle\sum_{-\infty}^{\infty}f_H(nu)$$

replacing Riemann's auxiliary function ((EdH) 10.3) $H(u):=\frac{d}{du}\left[u^2\frac{d}{du}G(u)\right]$.

The Kummer function related Mellin transforms are given by the formula (GrI) 7.612

$$\int_0^\infty x^s\,{}_1F_1(\alpha,\beta;-x)\frac{dx}{x}=\frac{\Gamma(\beta)}{\Gamma(\alpha)}\Gamma(s)\frac{\Gamma(\alpha-s)}{\Gamma(\beta-s)}\ ,\ 0<Re(s)<Re(\alpha)\ ,$$

leading to

$$\textstyle\int_0^\infty x^{s/2}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\frac{dx}{x}=\frac{\Gamma\left(\frac{s}{2}\right)}{1-s}=\frac{2}{s(1-s)}\Gamma\left(1+\frac{s}{2}\right)\ ,\ 0<Re(s)<1.$$

The link of the Hermite polynomials to the corresponding Hilbert transform of the Gaussian (the Dawson function) is given by (AbM), 7.1.15, resp. (RyG)

$$\tfrac{1}{\sqrt{\pi}}H\big[e^{-x^2}\big]=F(x)=\tfrac{1}{\sqrt{\pi}}\lim_{n\to\infty}\textstyle\sum_{k=1}^{n}\frac{H_k^{(n)}}{x-x_k^{(n)}}=\frac{1}{\sqrt{\pi}}\lim_{h\to0}\sum_{n\ odd}\frac{e^{-(x-nh)^2}}{n}\approx\frac{1}{x}.$$

where $x_k^{(n)}$ and $H_k^{(n)}$ are the zeros and weight factors of the Hermite polynomials. The operator theory associated with the Hermite polynomials does not extend to the

generalized Hermite polynomials because the even and odd polynomials satisfy different equations. Nevertheless, in (KrA) a united spectral expansion is provided. In this context we note that the two sequences $\{2\omega_n\}$, $\{\omega_n+\omega_{n+1}\}$ (with $[2\omega_n]=2n-1$, $[\omega_n+\omega_{n+1}]=2n$) of the zeros of ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ alternatively applied to the sequences of $e^{2\pi iz}$ satisfy the same differential equation.

The link to the normal distribution $\bar{F}(x)=\int_{-\infty}^{\infty}e^{-u^2}du$ resp. to the topic of „Convolution Operators and the Zeros of Entire Functions" (CaD), is given by by the formula

$$h_n(z)=\int_{-\infty}^{\infty}(z-is)^n d\bar{F}(s)=\int_{-\infty}^{\infty}(z-is)^n e^{-s^2}ds=2^{-n}H_n(z).$$

Riemann inverted the formula ((EdH), 1.13)

$$J(x)=\sum_{n\le x}\Lambda(n)\log n=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^s\frac{ds}{s}=Li(x)-\sum_\rho Li(x^\rho)-\log 2+\int_x^\infty\frac{1}{t(t^2-1)}\frac{dt}{\log t}$$

by means of the Möbius inversion formula, getting ((EdH), 1.17)

$$\pi(x)=Li(x)-\sum_{n=2}^{\infty}Li\left(x^{1/n}\right)\ ,\qquad Li(x):=\int_2^x\frac{dt}{\log t}-Li(2),\ Li(2)\sim 1.04\ldots\ .$$

He suggested a more natural and better approximation in the form

$$\pi(x)\sim R(x):=Li(x)-\sum_{n=2}^{\infty}\frac{\mu(n)}{n}Li\left(x^{1/n}\right)\ .$$

However, he was aware of the defects of this approximation and his analysis of it, which is basically due to fact, that he has no estimate at all of the size of the „periodic" terms $\sum_{n=1}^{\infty}\sum_\rho Li(x^{\rho/n})$ ((EdH), 1.17). The $Li(x)$ function RH approximation criterion is given by

$$|Li(x)-\pi(x)|=O\left(\sqrt{x}\log x\right)=O\left(x^{\frac{1}{2}+\varepsilon}\right);\varepsilon>0.$$

The challenge to prove the $Li(x)$ function approximation criterion is about the (exponential) asymptotics of the Gaussian function (which we propose to be replaced by the Dawson function). Based on the product representations of the Gamma function

$$\Gamma(1+s)=\prod\frac{\left(1+\frac{1}{n}\right)^s}{\left(1+\frac{s}{n}\right)}\ \ ,\ \ \Gamma(1-s)=\prod\frac{\left(1+\frac{1}{n}\right)^{-s}}{\left(1-\frac{s}{n}\right)}\ \ ,\ \ \Gamma\left(1+\frac{s}{2}\right)=\prod\frac{\left(1+\frac{1}{n}\right)^{s/2}}{\left(1+\frac{s}{2n}\right)}$$

the Riemann approximation error function ((EdH) 1.16)

$$\int_x^\infty\frac{dt}{t(t^2-1)\log t}=\int_x^\infty\frac{1}{t\log t}\left(\sum t^{-2n}\right)dt=\sum_{n=1}^{\infty}\int_x^\infty t^{-2n}\frac{dt}{t\log t}=\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{\mathrm{d}}{\mathrm{ds}}\left[\frac{\log\Gamma\left(1+\frac{s}{2}\right)}{s}\right]x^s ds$$

is calculated from the terms

$$\frac{d}{ds}\left[\frac{\log\Gamma\left(1+\frac{s}{2}\right)}{s}\right]\ \text{ and }\ H(\beta):=\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log\left(1-\frac{s}{\beta}\right)}{s}\right]x^s ds\ ,\ (H(1)=Li(x)-i\pi)\ .$$

As the alternative entire $\xi^*(s)$ function is going along with the replacement (*) above, this results into a correspondingly modified Riemann approximation error function with now newly three summation terms with improved ($|li(x)-\pi(x)|$ error) convergence behavior.

We mention the Kummer function based representation of the $li(x)$-function in the form $li(x)=x\,{}_1F_1(1,1;-\log x)$ ((LeN) (9.13.7)) with the asymptotics

$$li(x)=-x\,{}_1F_1(1;1,-\log x)=Ei(\log x)\approx\log^{-1}(x).$$

From (ObF), 2.4, we recall the following inverse Mellin transforms:

$a^z\cot(\pi z)$, $-n<Re(z)<1-n, n=0,\pm1,\pm2,\ldots.$ Principle value $\rightarrow$ $\pi^{-1}\left(\frac{x}{a}\right)^n\left(1-\left(\frac{x}{a}\right)\right)^{-1}$

$a^z\tan(\pi z)$, $-n-1/2<Re(z)<1/2-n$, $n=0,\pm1,\pm2,\ldots.$ Principle value $\rightarrow$ $-\pi^{-1}\left(\frac{x}{a}\right)^{n+1/2}\left(1-\left(\frac{x}{a}\right)\right)^{-1}$

3. An alternative Zeta function $\zeta^*(s)$ based on the $\cot(\pi s) = \tan\left(\frac{\pi}{2}(1-2s)\right)$ function

The fractional part function related Zeta function theory is provided in ((TiE) II). The Hilbert transform of the fractional part function is given by the $\log(sinx)$ −function. The correspondingly defined distributional ("periodical") Hilbert space framework enables the Bagchi reformulation of the Nyman-Beurling RH criterion, which then becomes basically a *„being dense embedded"* argument of $H^{\#}_{-1/2}(0,1)$ into $H^{\#}_{-1}(0,1)$. The $H^{\#}_{-1}(0,1)$ related counterpart of the $\log(sinx)$ −function is given by the Clausen integral (ChJ) define by

$$Cl_2(x) := \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} = -\int_0^x log\left[2sin\frac{y}{2}\right] dy \ , \ x \in R$$

with the periodic properties $Cl_2(2n\pi \pm \vartheta) = Cl_2(\pm\vartheta) = \pm Cl_2(\vartheta)$ and $Cl_2(\pi + \vartheta) = -Cl_2(\pi - \vartheta)$.

One proof of the Riemann functional equation is based on the fractional part function $\rho(x)$, whereby the zeta function $\zeta(s)$ in the critical stripe is given by the Mellin transform

$$\zeta(1-s) = M\left[-x \cdot \rho'(x)\right](s-1) \ , \text{ (TiE) (2.1.5).}$$

The functional equation is given by ((TiE) (2.1.12)

$$\zeta(s) = \chi(s)\zeta(1-s) = \pi^{s-1/2}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s) \ .$$

The Hilbert transform of the fractional part function $\rho(x) = x - [x] = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{sin(2\pi nx)}{\pi n} \in L^{\#}_2(0,1)$ is given by $\rho_H(x) = \sum_{n=1}^{\infty}\frac{cos(2\pi nx)}{\pi n} = -\frac{1}{\pi} log\,(2\,sin(\pi x)) \in L^{\#}_2(0,1)$, resulting into $\rho'_H(x) = -cot(\pi x) = -2\sum_{n=1}^{\infty} sin(2\pi nx) \in H^{\#}_{-1}(0,1)$. We note that the $\cot(\pi x)$ series representation is Cesàro summable (mean of order 1) (BrK4) Note O25, (ZyA) VI-3, VII-1).

The corresponding alternative $\zeta^*(s)$ function is given by ((BrK4) lemma 1.4, lemma 3.1 (GrI) 1.441, 3.761, 8.334, 8.335)

$$\zeta^*(1-s) = \frac{\tan(\frac{\pi}{2}s)}{s}\cdot\zeta(1-s) \quad \text{resp.} \qquad \zeta^*(s) = \frac{\cot(\frac{\pi}{2}s)}{1-s}\cdot\zeta(s) = \frac{\tan(\frac{\pi}{2}(1-s))}{1-s}\cdot\zeta(s)$$

resp.

$$log\zeta^*(s) = \log\left(\tan\frac{\pi}{2}(1-s)\right) + log\left(\frac{1}{1-s}\right) + log\zeta(s).$$

We note the series representation (TiE) 4.14,

$$\sum_{n>x}\frac{1}{n^s} = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} z^{1-s}(-\pi\, cot(\pi z)\frac{dz}{z} \quad .$$

The functions $\log(2\sin(\pi x))$, $cot(\pi x)$, $sin^{-2}(\pi x)$ are also related to the convolution kernels of the model adequate Pseudo-Differential operators in the context of the proposed new ground state energy model of the harmonic quantum oscillator model problem.

We recall the following inverse Mellin transform formulas ((NiN) §91/92):

$$\log\left(1+\frac{1}{x}\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{-s}\frac{\pi}{\sin(\pi s)}\frac{ds}{s} \quad ,$$

$$\frac{\log(1+x)+\gamma}{1+x} = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\psi(s)x^{1-s}\frac{\pi}{\sin(\pi s)}\frac{ds}{s}$$

with $\psi(x) := log'\Gamma(x) = \Gamma'(x)/\Gamma(x)$ and $0 < a < 1$.

We further mention the harmonic number related *„Elementary RH criterion"* (LaJ):

The RH holds true iff for each $n \geq 1$ $\sum_{d|n} d \leq H_n + e^{H_n}\log(H_n)$.

The function $\pi\, cot(\pi x)$ is holomorphic except the pole $z = 1$ and the following series representation is valid ((GrI) 1.442)

$$log\big(cot(\pi x)\big) = \sum \frac{2\cos{(2k-1)\pi x}}{(2k-1)}\ ,\ 0 < x < 1\ \ ,\ \ \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2}{2k-1} = \frac{\pi}{2}.$$

Replacing the harmonic numbers $H_n = \sum_{k=1}^{n}\frac{1}{k}$ by the „alternative" harmonic numbers

$$2h_n := \sum_{k=1}^{n}\frac{2}{2k-1} = 2H_{2n} - H_n$$

and avering of the the two series

$$-\log{(1-x)} = \int_0^1 \frac{dt}{1-t} = \sum_{k=1}^{\infty}\frac{x^k}{k}\ \ ,\ \ +\log{(1+x)} = \int_0^1 \frac{dt}{1-t} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$$

results into the representation (ChH)

$$\frac{1}{2}\log{\left(\frac{1+x}{1-x}\right)} = \sum_{k=1}^{\infty}\frac{1}{2k-1}x^{2k-1} \quad \text{resp.} \quad \frac{1}{4}log^2\left(\frac{1+x}{1-x}\right) = \sum_{k=1}^{\infty}\frac{h_k}{k}x^{2k}.$$

Based on this results in (EIL) the following series representation and corresponding regularity results are derived ($c_n := \frac{2h_n}{n}$)

$$T(x) := -\frac{\pi}{2}log\left(tan\left(\pi\frac{x}{2}\right)\right) = -\frac{\pi}{2}log\left(cot\left(\pi\frac{1-x}{2}\right)\right) = \sum\frac{2h_n}{n}sin\,(\pi(2n)x) = \sum c_n\, sin\,(2\pi nx)$$

with

$$T(x) \in L_2^{\#}(0,1) \ \text{ i.e. } \ \sum_{n=1}^{\infty} c_n^2 < \infty\ .$$

While $T(x)$ is of same regularity as the fractional part function, its first derivative

$$log'\left(tan\left(\pi\frac{x}{2}\right)\right) = \frac{\pi}{sin\,(\pi s)} \in H_{-1}^{\#}(0,1)$$

is of same regularity as the cot − and the Zeta function on the critical line.

We note that the analysis in (EIL) is based on the („orthogonality") formula

$$\int_0^1 log\ (tan(\frac{\pi}{2}x))\ cos(k\pi x)\,dx = \begin{cases} -1/k & \square\ k\ odd \\ 0 & \square\ k\ even \end{cases}.$$

To take advantage out of the convergent series $\sum_{n=1}^{\infty} c_n^2 < \infty$, we recall the

Lemma (KaM1): Let $\{n_k\}$ be a sequence of integers satisfying the "Hadamard" gap" condition, i.e. $\frac{n_{k+1}}{n_k} > q > 1$. Then the trigonometric gap series $\sum_{k=1}^{\infty} c_k \sin{(2\pi n_k x)}$ converges almost everywhere, if and only if, $\sum_{k=1}^{\infty} c_k^2 < \infty$

from which it follows the

Corollary: the series $\sum\frac{2h_k}{k}sin\,(2\pi\omega_k x)$ and $\sum\frac{2h_k}{k}sin\,(\pi(\omega_k + \omega_{k+1})x)$ converge almost everywhere.

The generalization of H_n to the real variable is given by

$$\psi(x) := log'\Gamma(x) = \Gamma'(x)/\Gamma(x)$$

since $\psi(n+1) = H_n - \gamma$ ((AbM) (6.3.2)).

For the alternatively proposed $\psi^*(x) := \psi(x) + \frac{\pi}{\sin{(\pi x)}}$ we note the corresponding fomulas

$$\psi(1+z) = \psi(z) + \tfrac{1}{z} \qquad \rightarrow \quad \psi^*(1+z) = \psi(z) + \tfrac{1}{z} - \tfrac{\pi}{sin\,(\pi x)}$$

$$\psi(1-z) = \psi(z) + \pi cot\,(\pi z) \quad \rightarrow \quad \psi^*(1-z) = \psi(1-z) + \pi cot\,(\pi z) + \tfrac{\pi}{sin\,(\pi x)}.$$

For any natural number $n \geq 1$ the following inequality is valid (ViM):

$$\frac{1}{2n-\frac{1}{1-\gamma}-2} \leq H_n - logn \leq \frac{1}{2n+\frac{1}{3}}$$

i.e. the approximation formula $H_n \approx logn + \gamma + \frac{1}{2n}$ is overestimating, which indicates a correction term in the form $H_n^{\square} - \frac{\pi}{\sin(\pi\omega_n)}$.

The following two a priori estimates ((HaJ) p. 47, resp. (Goldbach conjecture) section 1g below)

$$\frac{1}{2n} < \frac{1}{2}(H_n - logn) < \frac{1}{2} \quad , \quad \frac{1}{2} - \frac{1}{2n} < a_n \coloneqq s_n - \frac{1}{2} = \frac{\omega_n}{n} - \frac{1}{2} < \frac{1}{2}$$

leads to an alternative approximation term in the form

$$\frac{1}{2} < H_n^* - \log\sqrt{e \cdot n} = \frac{1}{2}H_n + \frac{\omega_n}{n} - \log\sqrt{e \cdot n} = \frac{1}{2}(H_n - logn - 1) + \frac{\omega_n}{n} < 1$$

with

$$\lim_{n\to\infty}\left[\frac{1}{2}(H_n - logn - 1) + \frac{\omega_n}{n}\right] = \frac{1}{2}(\gamma - 1) + 1 = \frac{\gamma+1}{2}\,.$$

In the context of arithmetical functions, q-series and Ramanujan's theta function we note the following (BeB1):

The Rogers-Ramanujan Continued Fraction $R(q)$ ((BeB1) (7.1.6)) is connected with the Rogers-Ramanujan functions

$$G(q) \coloneqq \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \;,\; H(q) \coloneqq \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

by

$$R(q) = q^{1/5}\frac{H(q)}{G(q)},$$

whereby

$$(a;q^j)_n \coloneqq \prod_{k=0,\dots(n-1)}(1 - aq^{jk}).$$

The Rogers-Ramanujan identities have "beautiful combinatorical interpretations, if in the definition of $G(q)$ write $n^2 = 1 + 3 + \cdots + (2n-1)$. Then the first identity is equivalent to the assertion that the number of partitions of a positive number N into distinct parts with differences at least 2 equals the number of partitions of N into parts congruent to either 1 or 4 modulo 5. For the second identity write $n(n+1) = 2 + 4 + \cdots 2n$. Then this identity is an analytic statement of the fact that the number of partitions of N into distinct parts with differences at least 2 and with no 1's is equal to the number of partitions of N ito parts congruent to either 2 or 3 modulo 5 ((BeB) 7.6).

The identities $n^2 = 1 + 3 + \cdots + (2n-1)$ and $n(n+1) = 2 + 4 + \cdots 2n$ in combination with the imaginary parts of the zeros of the considered special Kummer function lead to the inequality

$$n - \frac{1}{2} < \frac{2}{n}\sum_{k=1}^{n}\omega_k < n + \frac{1}{2}$$

enabling an alternative Rogers-Ramanujan Continued Fraction with corresponding identities.

4. Alternative arithmetical functions based on the Kummer function zeros

The sequences ω_n (the imaginary parts of the Kummer function zeros) enjoy the following properties ((KaM1), (KoA) (ZyA), see also (BrK4) Notes S43-49, O5-7, O24-25, O27/37, Notes O5-O7, O13, O15-O17)

i) $n-\frac{1}{2}<\omega_n<n$, $n+\frac{1}{2}<\omega_{n+1}<n+1$, $\frac{1}{2}<\omega_1<a:=s_n:=\frac{\omega_n}{n}\to 1\ n\in N$

ii) $2n-1<2\omega_n<2n<\omega_n+\omega_{n+1}<2n+1<2\omega_{n+1}<2(n+1)$

iii) the sequences $2\omega_n$ and $\omega_n+\omega_{n+1}$ fulfill the Hadamard gap condition

$\frac{\omega_{n+1}}{\omega_n}>\frac{n+\frac{1}{2}}{n}=1+\frac{1}{2n}>q>1$ resp. $\frac{\omega_{n+1}+\omega_{n+2}}{\omega_n+\omega_{n+1}}>\frac{2n+2}{2n+1}=1+\frac{1}{2n+1}>q>1$

iv) $\theta:=\frac{1}{4}<\frac{\omega_{n+1}}{2}-\frac{\omega_n}{2}<1-\frac{1}{4}=1-\theta$.

For the related sequences $a_n:=\frac{2\omega_n}{2n}-\frac{1}{2}$, $b_n:=\frac{\omega_{n+1}+\omega_n}{2n}-\frac{1}{2}$ it therefore follows

iii) $0<a_1=\omega_1-\frac{1}{2}\le a_n\to\frac{1}{2}$, $\frac{1}{2}\leftarrow b_n<b_1=\frac{\omega_1+\omega_2-1}{2}<1$

iv) $a_n, b_n-a_n\in(0,\frac{1}{2})$, $b_n, 1-a_n\in(\frac{1}{2},1)$.

We note that the integer "2" is not an element of the set $\boldsymbol{A}:=\{[2\omega_n], n\in N\}$, i.e. the Snirelmann density of $\boldsymbol{A}$ is $\le\frac{1}{2}$.

We further note the Mellin transform of the related Kummer function ((GrI) 7.612)

$$\int_0^\infty x^s\,{}_1F_1(a_n,a_n+1;-x)\frac{dx}{x}=\frac{a_n}{a_n-s}\,\Gamma(s)\quad,\quad 0<Re(s)<a_n$$

i.e.

$$\int_0^\infty x^s\,{}_1F_1(a_n,a_n+1;-x)\frac{dx}{x}\ \to\ \int_0^\infty x^{s/2}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\frac{dx}{x}=\frac{\Gamma\left(\frac{s}{2}\right)}{1-s}=\frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)}\quad,\quad 0<Re(s)<1.$$

The sequence $s_n:=\frac{\omega_n}{n}$ fulfills the Hardy-Littlewood condition $|s_{n+1}-s_n|<\frac{K}{n}$, i.e. it has a defined Abel average ((EdH) 12.7)

$$\lim_{r\uparrow 0}\frac{s_1r+s_1r^2+s_3r^3+\cdots\ldots}{r+r^2+r^3+\cdots\ldots}=L\quad .$$

As a consequence, the sequence s_n enables a point measure, which is s_n at n and zero elsewhere, i.e.

$$d\sigma(x)=d\sigma_n(x)=\begin{cases}s_n & x=n\\ \ldots & else\\ 0 & \end{cases}$$

enjoying the identities

$$\lim_{r\uparrow 1}\frac{\int_0^\infty r^x d\sigma(x)}{\int_0^\infty r^x dx}=L\ ,\ \lim_{A\to\infty}\frac{\int_0^A d\sigma(x)}{\int_0^A dx}=L\ .$$

which is a restated Abel average representation of the sequence s_n. With respect to the relationship of series with Hadamard gaps, launary sequences, Abel summability and Tauberian theorems for distributional point values to local behavior of distributions we also refer to ((ViJ1) pp. 98, 119, 123, 125).

The (almost everywhere) convergence of the series $\sum\frac{2h_k}{k}sin\,(2\pi\omega_k x)$ and $\sum\frac{2h_k}{k}sin\,(\pi(\omega_k+\omega_{k+1})x)$ indicate the following replacements

1. $cot(\pi x)=2\sum_{n=1}^{\infty}sin(2\pi nx))\in H_{-1}^{\#}(0,1)$ *Cesàro summable (ZyA) VI-3, VII-1)*

→ $cot^{(1)}(\pi x):=\sum sin(\pi(2\omega_n)x)\in H_0^{\#}(0,1)$ *Abel summable*

→ $cot^{(2)}(\pi x):=\sum sin\,(\pi(\omega_n+\omega_{n+1})x\in H_0^{\#}(0,1)$ *Abel summable*

2. $-\frac{\pi}{2}\log\left(tan\left(\frac{\pi}{2}x\right)\right)=\sum\frac{2h_n}{n}\sin\,(2\pi nx))\in L_2^{\#}(0,1)$ →

→ $-\frac{\pi}{2}\log\left(tan^{(1)}\left(\frac{\pi}{2}x\right)\right):=\sum\frac{h_n}{n}\sin\,(\pi(2\omega_n))\in H_1^{\#}(0,1)$

→ $-\frac{\pi}{2}log\left(tan^{(2)}\left(\frac{\pi}{2}x\right)\right):=\sum\frac{h_n}{n}sin\,(\pi(\omega_n+(\omega_{n+1}))\in H_1^{\#}(0,1)$

3.

→ $\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}z^{1-s}\left(-\pi\,cot^{(1)}(\pi z)\right)\frac{dz}{z}$ resp. $\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}z^{1-s}\left(-\pi\,cot^{(2)}(\pi z)\right)\frac{dz}{z}$.

(EdH) 12.7: *„The PNT is about the asymptotics equivalence of $\psi(x)$~x, which is equivalent to the statement that $d\psi(x)$~dx as a Cesaro average in the context of Tauberian theorems. Hardy-Littlewood were able to prove the PNT by showing $d\psi(x)$~dx as an Abel average, where a significant amount of work is done by a Tauberian theorem."*

The Mangoldt resp. Landau density functions are given by

$$\psi(x)=\sum_{n<x}\Lambda(n)\quad\text{resp.}\quad\vartheta(x)=\sum_{n=1}^{\infty}\Lambda(n)log\,(\tfrac{x}{n})\ .$$

We propose an alternative density function (based on the sequence s_n) to enable a proof in which all of the work is done by a Tauberian theorem:

What cannot derived from the PNT is the convergence of the series

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}log(\tfrac{1}{n})=1\ .$$

„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived" ((LaE) §160). In order to anticipate this finding we suggest to apply the point measures $d\sigma_n(x)$ and

$$\vartheta^{*}(x)=\sum_{n\leq x}\frac{\Lambda(n)}{logn}\log(\tfrac{x}{\omega_n})\quad,\quad\vartheta^{**}(x)=\sum_{n\leq x}\frac{\mu(n)}{n}\log(\tfrac{x}{\omega_n})\qquad(\tfrac{x}{n}<\tfrac{x}{\omega_n}=\tfrac{x}{n}\tfrac{1}{s_n}<2\tfrac{x}{n})$$

(whereby $d\vartheta=d\vartheta^{*}=d\vartheta^{**}$), in combination with the Ikehara theorem ((EdH) 12.7).
The convergence of the series

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}log(\tfrac{1}{n})=1\ .$$

is related to the proposed Hilbert space $H_{-\frac{1}{2}}$ by the identity

$$1=\sum_{n=1}^{\infty}\frac{1}{n}\mu(n)\,log(\frac{1}{n})=\sum_{1}^{\infty}\frac{1}{n}a_nb_n=:\left((u,v)\right)_{-\frac{1}{2}}:=\lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega}u\left(\frac{1}{2}+it\right)v\left(\frac{1}{2}-it\right)dt$$

where

$$u(t):=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\in H_{-\frac{1}{2}}\quad,\quad v(t):=\sum_{n=1}^{\infty}\frac{log(1/n)}{n^s}\in H_{-\frac{1}{2}}\quad,\quad s=\frac{1}{2}+it\ .$$

Property v) in combination with the theorem of E. Landau (LaE3) applied for $\alpha_n,\varphi_n:=e^{2\pi i\alpha_n}=e^{\pi i\omega_n}$ leads to the estimate

$$\forall\varepsilon>0\quad cot\left(\frac{\pi}{2}\theta\right)-\varepsilon=cot\left(\frac{\pi}{8}\right)-\varepsilon<S_m:=|\textstyle\sum_{n=1}^{m}\varphi_n|=\left|\sum_{n=1}^{m}e^{\pi i\omega_n}\right|<cot\left(\frac{\pi}{8}\right)=cot\left(\frac{\pi}{2}\theta\right)\ .$$

5. New arithmetical functions for additive number theory

From the section above we recall $2n-1<2\omega_n,\omega_n+\omega_{n+1}-1<2n$, i.e.

$$g_n:=\frac{1}{2\omega_n},\ f_n:=\frac{1}{\omega_n+\omega_{n+1}-1}\in\left(\frac{1}{2n},\frac{1}{2n-1}\right)$$

indicating a replacement of

$$\beta(x):=\sum_{n<x}a_n log\left(\frac{x}{n}\right)\ \rightarrow\quad \beta^*_{\square}(x):=\frac{1}{2}\left[\sum_{\substack{n<x\\ n\ odd}}a_n log\left(\frac{x}{2\omega_n}\right)+\sum_{\substack{n<x\\ n\ even}}a_n\, log\left(\frac{x}{\omega_n+\omega_{n+1}-1}\right)\right]$$

with $d\beta=\frac{dx}{x}=d\beta^*$ for a proper addtive arithmetic function definition in line with an integer subset $\boldsymbol{A}^*:=\{0\}\cup\{1\}\cup\{2\omega_n,\omega_n+\omega_{n+1}-1\}$ with Snirelmann density $\frac{1}{2}$.

We note the inequality

$$4n-1=\left(2n-\tfrac{1}{2}\right)+\left(2n+\tfrac{1}{2}\right)-1<\{2\omega_n\}+\{\omega_n+\omega_{n+1}-1\}=\frac{1}{g_n}+\frac{1}{f_n}<\left(2n+\tfrac{1}{2}\right)+\left(2n+\tfrac{1}{2}\right)-1=4n.$$

For the fractional part function $\rho(x)=x-[x]=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{sin(2\pi n)}{\pi n}\in L_2^{\#}(0,1)$ we recall the (PoG1):

Landau theorem: Let q_n denote a divergent sequence of positive numbers $0<q_1\le q_2\le q_3\le\cdots\quad \lim_{n\to\infty}q_n=\infty$, $\tau(x)$ the corresponding counting function of the numbers of q_n less than $\le x$ and $w(x)$ a positive, non decreasing function with

$$\lim_{n\to\infty}\frac{w(2x)}{w(x)}=1,\ \text{and}\ \lim_{n\to\infty}\frac{\tau(x)w(x)}{x}=1.$$

Then

$$\lim_{n\to\infty}\frac{1}{\tau(x)}\sum_{q\le x}\rho\left(\frac{x}{q}\right)=1-\gamma\ .$$

We mention that a consequence of the condition $\lim_{n\to\infty}\frac{w(2x)}{w(x)}=1$ is $\lim_{n\to\infty}\frac{w(bx)}{w(ax)}=1$ for any positive $a,b>0$ and the formulas $\frac{\sin(2x)}{sinx}=2cosx$ resp. $\frac{\cot(2x)}{cotx}=\frac{1}{2}\frac{\cos(2x)}{sin^2(x)}$.

The Landau theorem above is a special case of the (PoG1):

Generalized Landau Theorem: Let $w(x)$ a positive, non decreasing function with $\lim_{n\to\infty}\frac{w(\beta x)}{w(\alpha x)}=1$ with α,β positive numbers. Then

$$\lim_{n\to\infty}\frac{w(x)}{x}\sum_{n\le x}f\left(\frac{x}{n}\right)=\int_0^1 f(t)dt\ .$$

For $h(x):=(\pi x)cot(\pi x)$ it holds $h(x)\rightarrow_{x\to 0}-1$ and $\int_0^1 h\left(\frac{x}{2}\right)dx=log2$ ((GrI) 3.747). Then for $w(x):=h\left(\frac{1}{x}\right)$ and $f(x):=h\left(\frac{x}{2}\right)$ one gets

$$\frac{w(x)}{x}f\left(\frac{x}{n}\right)=\frac{1}{x}h\left(\frac{1}{x}\right)h\left(\frac{x}{2n}\right)\quad\text{and}\quad\int_0^1 f(t)dt=log2=-log\xi(0).$$

Putting $h^{(1)}(x):=-(\pi x)cot^{(1)}(\pi x)$, $h^{(2)}(x):=-(\pi x)cot^{(2)}(\pi x)$ this enables the following arithmetic function definitions

$$\sigma^{\square}(x):=\frac{1}{x}h\left(\frac{1}{x}\right)\sum_{n\le x}h\left(\frac{x}{2n}\right)\quad\sigma^{(1)}_{\square}(x):=\frac{1}{x}h^{(1)}\left(\frac{1}{x}\right)\sum_{n\le x}h^{(1)}\left(\frac{x}{2\omega_n}\right)\ ,\ \sigma^{2}_{\square}(x):=\frac{1}{x}h^{(2)}\left(\frac{1}{x}\right)\sum_{n\le x}h^{(2)}\left(\frac{x}{\omega_n+\omega_{n+1}-1}\right).$$

With respect to the additive number theory (and to be considered prime pairs (p,q)) we propose to deal with

$$\sigma^*_{\square}(x):=\frac{1}{2}\left[\sigma^{(1)}_{\square}(x)+\sigma^{(2)}_{\square}(x)\right].$$

The regular varying function concept was introduced by Karamata (SeE).

Lemma ((VlV), I.3): if $v(x)$ is differentiable for $x \geq x_0$ and there is a limit $\lim_{x\to\infty} \frac{xu'(x)}{u(x)} \to \alpha$, then $v(x)$ is automodel (or regular varying) of order α, i.e. $\lim_{x\to\infty} \frac{u(bx)}{u(x)} = b^{\alpha}$.

Therefore, a further candidate for $w(x)$ is given by $k(x) \coloneqq -cot\left(\pi\frac{x}{2}\right)$, which is a slow regular varying (automodel) of order zero ((EsR) 3.9.8). The above criterion is fulfilled, as $\frac{xk'(x)}{k(x)} = \frac{\pi x}{\sin(\pi x)}$.

Further candidates for $f(x)$ are e.g.

i) $f_1(x) \coloneqq -\log(sin\pi x)$ with $\int_0^1 f_1(t)dt = log2$ (GrI) 4.384

ii) $f_2(x) \coloneqq -\mathrm{h}\left(\frac{x}{2}\right) = -h(x) \coloneqq \frac{1}{x} - \left(\frac{\pi}{2}x\right)cot\left(\frac{\pi}{2}x\right)$ with $\int_0^1 f_2(t)dt = \log\left(\frac{\pi}{2}\right)$ (GrI) 3.788

iii) $f_3(x) \coloneqq log\left(a \cdot tan\left(\pi\frac{x}{2}\right)\right)$ with $\int_0^1 f_3(t)dt = \log\, a\ ,\ \ a > 0$ (GrI) 4.227.

Typical examples of slow varying functions are positive constants or functions converging to a positive constant, logarithms and iterated logarithms. Specifically the function $-logx$ is slowly varying at $x = 0^+$ ((SeE) p. 47), which will be applied in the following section.

Every regular varying function f of order α has a representation $f(x) = x^{\alpha}L(x)$, where L is some slow varying function (MiT).

A general representation of slow regular varying functions is given by the

Theorem 1.2 (SeE): A positive measurable function L on $[x_0, \infty[$ is a slow varying function if and only if it can be written in the form

$$L(x) = e^{v(x)+\int_{x_0}^{x}\epsilon(y)\frac{dy}{y}}$$

where $v(\cdot)$ is a measurable bounded function, such that $\lim_{x\to\infty} v(x) = c$ $(|c| < \infty)$ and $\varepsilon(x) \to 0$ as $x \to \infty$.

Comparison Tauberian theorems are about the asymptotics behavior of the ratio of some integral transforms of two functions (distributions), if the asymptotic behavior of the ratio of some other integral transform is given ((EsR), (VlV)). In ((VlV) the Abel and Cesaro series summation with respect to an automodel weight, as well as asymptotic properties of solutions of convolution equations are considered.

Karamata's Tauberian theorem involving regular variations are provided in ((SeE) 2.2)

Theorem 2.3: Let $U(x)$ be a monotone non-decreasing function on $[0, \infty[$ such that $w(x) = \int_{0^+}^{\infty} e^{-xu}dU(u)$ is finite for all $x > 0$. Then, if $\rho \geq 0$, and L is a slowly varing function,

i) if $w(x) = x^{-\rho}L(\frac{1}{x})$ as $x \to 0^+$ then $U(x) \sim x^{\rho}\frac{L(x)}{\Gamma(\rho+1)}$ all $x \to \infty$

ii) if $w(x) = x^{-\rho}L(x)$ as $x \to \infty$ then $U(x) \sim x^{\rho}\frac{L(\frac{1}{x})}{\Gamma(\rho+1)}$ all $x \to 0^+$

The corresponding „density" extension of this theorem is provided in

Theorem 2.4: Let $U(x)$, defined and positive on $[A, \infty[$ for some given A sufficiently large, be given by $U(x) = \int_{\square}^{\infty} u(y)dy$, where $u(y)$ is ultimately monotone. Then for $\rho \geq 0$, if $U(x) = x^{\rho}L(x)$ then $\frac{xu(x)}{U(x)} \to \rho$ as $x \to \infty$.

Another such theorem, using both parts i) and ii) of theorem 2.3 above is provided in theorem 2.5, which is about the asymptotics $\int_{0^-}^{\infty} \frac{dA(t)}{(t+x)^{\rho}} \sim x^{\rho}L(x)$.

6. Dirichlet series and the (distributional) Hilbert space $H^{\#}_{-1/2} \cong l_2^{-1/2}$

In this section we are concerned with Hilbert scales $H^{\#}_{\alpha} \cong l_2^{\alpha}$, $\alpha \in R$, which is built on the 2π −periodic Hilbert space $L^{\#}_2(\Gamma)$ with $\Gamma \coloneqq S^1(R^2)$, i.e. Γ is the boundary of the unit circle sphere. Then for $u \in L^{\#}_2(\Gamma)$ and for real $\beta \in \mathrm{R}$, $\mathrm{n} \in \mathrm{Z}$ the Fourier coefficients

$$u_n \coloneqq \frac{1}{2\pi}\oint u(x)e^{inx}dx$$

enable the definition of the norms ((BrK), (BrK3,6,7))

$$\|u\|^2_{\beta} \coloneqq \sum_{n=-\infty}^{\infty}|n|^{2\beta}|u_n|^2.$$

With the notation of [LaE] §227, Satz 40, the for $s > 0$ convergent Dirichlet series (in a classical L_∞ resp. C^0 sense, where H_k is a subset of C^0 for $k \geq \frac{1}{2} + \varepsilon, \varepsilon > 0$)

$$f(s) \coloneqq \sum_{1}^{\infty} a_n e^{-s\log n} \quad g(s) \coloneqq \sum_{1}^{\infty} b_n e^{-s\log n}$$

are linked to the (distributional) Hilbert space $H^{\#}_{-1/2} \cong l_2^{-1/2}$ by ((EdH) 9.8, (NaS))

$$((f,g))_{-1/2} \coloneqq \lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega} f(1/2+it)g(1/2-it)dt = \sum_{1}^{\infty}\frac{1}{n}a_n b_n \,.$$

Let $s = \sigma + it$ and $\mu_n = log\lambda_n$ and the series $\sum_{n=1}^{\infty} a_n e^{-\mu_n s}$ convergent; then if $\sigma > 0$ the following definite integral representation is given (HaG)

$$\sum_{n=1}^{\infty} a_n e^{-\mu_n s} = \frac{1}{\Gamma(s)}\int_0^{\infty} x^s \left[a_n e^{-\lambda_n s}\right]\frac{dx}{x}$$

leading to the

Perron theorem ((HaG) theorem 13): if the series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ is convergent for $s = \beta + i\gamma$ and $c > 0$, $c > \beta$, $\lambda_n < \omega < \lambda_{n+1}$ then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} f(s)\, e^{\omega s}\frac{ds}{s} = \sum_{k=1}^{n} a_k,$$

the path of integration being the line $\sigma = c$. At a point of discontinuity $\omega = \lambda_n$ the integral has a value hlaf-way between ist limits on either side, but in this case the integral must regarded as being defined by ist principal value.

For $\Gamma \coloneqq S^1(R^2)$, the operators defined by the the single layer potential, the normal derivative of the double layer potential and the Hilbert transform ((KrR) theorems 8.20, 8.21, (LiI), $\alpha = -\frac{1}{2}$) leads to the following representations ($\alpha \in \mathrm{R}$) (LiI1):

$$S_{-1}\colon H_\alpha^\# \to H_{\alpha+1}^\# \qquad S_{-1}[u](x) \coloneqq 2\int_0^1 \log\left|\frac{1}{2\sin(\pi(x-y))}\right| u(y)dy \ , \ -log2\sin(\pi x) = \sum_{n=1}^\infty \frac{\cos(2n\pi x)}{n}$$

$$S_1\colon H_{\alpha+1}^\# \to H_\alpha^\# \qquad S_1[u](x) \coloneqq \frac{1}{2}\int_0^1 \frac{u(y)}{sin^2(\pi(x-y))}dy \ , \quad \left[\frac{\pi}{\sin(\pi x)}\right]^2 = \sum_{-\infty}^\infty \frac{1}{(x-n)^2}$$

$$S_0\colon H_{\alpha+1/2}^\# \to H_{\alpha+1/2}^\# \qquad S_0[u](x) \coloneqq \int_0^1 cot(\pi(x-y))u(y)dy, \quad \pi cot(\pi x) = \frac{1}{x} + \sum_{\substack{-\infty \\ n\neq 0}}^\infty \left[\frac{1}{x-n} + \frac{1}{n}\right].$$

For the two self-adjoint operators S_{-1}, S_1 it holds $(S_{-1}[u], v)_\beta \cong (u, v)_{\beta-1/2}$, $(S_1[u], v)_\beta \cong (u, v)_{\beta+1/2}$, for the Hilbert transform operator it holds $(S_0[u], v)_\beta = -(u, S_0[v])_\beta$, i.e. the three convolutions integrals are Pseudo-Differential operators of order -1, 1, 0, defining corresponding isomorphisms between the corresponding domains and ranges.

As the function $log\frac{1}{x}$ is slowly varying at $x = 0^+$ ((SeE) p. 47), the kernel function of $S_{-1}[u]$ is slowly varying at

$$x = \begin{cases} 2\mu - 1 & \square + \mu \in N \\ 2\mu & \square - \mu \in N \end{cases}$$

With respect to the kernel functions of $S_1[u]$, $S_0[u]$ we note the relationships

$$\frac{d^3}{dx^3}[-\log(sinx)] = \frac{d^2}{dx^2}[-\mathrm{cotx}] = \frac{d}{dx}\left[\frac{1}{sin^2x}\right] = (-cotx)\frac{1}{sin^2x} = (-cotx)\frac{d}{dx}[-cotx] = \frac{1}{2}\frac{d}{dx}[cot^2x]$$

with its relationship to the Claussen function ((AbM) 27.8, (BrK4) Notes O27,28).

For the (transformed) Zeta function on the critical line Ξ it holds $\Xi \in H_{-1}^\#$, i.e. there exists a *convolution* integral representation of the Zeta function by an unique $\omega \in H_0^\#$ with $S_1[\omega](x) \coloneqq \Xi(x)$ (CaD). It leads to the corresponding variational representation in the form

$$(S_1[\omega], v)_{-1} = (\omega, v)_{-1/2} = (\Xi, v)_{-1} \ , \ \forall v \in H_{-1}^\#$$

The distributional $\mathrm{H}_{-1/2}$ Hilbert space framework enables the Bagchi RH criterion, which is a reformulation of the Nyman-Beurling RH criterion (BaB). It is basically a standard density argument of the Hilbert sub-space $H_{-1/2}^\#$, which is densely embedded into $H_{-1}^\#$ with respect to the $H_{-1}^\#$ – norm ((BrK4) remark 3.5, notes S21 & 24, see also „Riesz theory" ((KrR) chapter 3). The corresponding approximating sequence ${\Xi_n}^*$ of the Zeta function Ξ is defined by

$$(\omega_n^*, v)_{-1/2} = ({\Xi_n}^*, v)_{-1} \ , \ \forall v \in H_{-1/2}^\# \ .$$

From (LiI) 1.2.34, we mention

$$S_1[a_n\cos(2\pi nx) + b_n\sin(2\pi nx)](x) = -2n[a_n\cos(2\pi nx) + b_n\sin(2\pi nx)]$$

With its relationship to the concept of logarithmic capacity of sets and convergence of Fourier series of functions fulfilling $\sum_1^\infty n[a_n^2 + b_n^2]$ and harmonic analysis ((ZyA) V-11, (BrK4) remarks 4.1, 4.2, Notes S37/38).

From the representation (ElL)

$$\frac{\pi}{2} log\left(tan\left(\pi\frac{x}{2}\right)\right) = \frac{\pi}{2} log\left(cot\left(\pi\frac{1-x}{2}\right)\right) = -\sum\frac{2h_n}{n} sin\,(\pi(2n)x) \in H_0^{\#}(0,1)$$

with the harmonic numbers $2h_n := \sum_{k=1}^{n}\frac{2}{2k-1} = 2H_{2n} - H_n$ and $H_n = \sum_{k=1}^{n}\frac{1}{k}$ it follows

$$\frac{\pi}{2} log'\left(tan\left(\pi\frac{x}{2}\right)\right) = -2\pi\sum 2h_n\, cos\,(\pi(2n)x) \in H_{-1}^{\#}(0,1).$$

From the equations $log'\left(tan\left(\frac{x}{2}\right)\right) = \frac{1}{sinx}$, $log'\big(2sin(x)\big) = cotx$, $\frac{1-cosx}{sinx} = tan\,(\frac{\pi}{2}x)$ one gets

$$2\frac{\pi}{2} log'\left(tan\left(\frac{\pi}{2}x\right)\right) = -\pi\, log'\left(cot\left(\frac{\pi}{2}x\right)\right) = \frac{\pi}{\sin\,(\pi x)}\ ,\quad -log'\big(sin(\pi x)\big) = -\pi\cot\,(\pi x)$$

and

$$\pi tan\left(\frac{\pi}{2}x\right) = 2\frac{\pi}{2} log'\left(tan\left(\frac{\pi}{2}x\right)\right) - log'\big(sin(\pi x)\big) = \frac{\pi}{\sin\,(\pi x)} - \pi\,\cot(\pi x) = \pi\frac{1-\cos(\pi x)}{\sin(\pi x)}.$$

With

$$-log'\big(sin(\pi x)\big) = \sum sin\,(\pi(2n)x) \in H_{-1}^{\#}(0,1)$$

one gets the following Fourier series representation

$$tan\left(\frac{\pi}{2}x\right) = -\sum\{8h_n\cos(2\pi nx) - \sin\,((2\pi nx))\} \in H_{-1}^{\#}(0,1)\ .$$

With respect to the Bagchi criterion and its related Hilbert space $H_{-1}^{\#}$ we further mention the relationship to the Brownian motion $B(t) = \int_0^t dB(\tau)$, which is obtained as the integral of the white noise signal $dB(t)$, which is distribution. Its spectral density $E_0 = \left|Fourier\big[B^{'}\big](\omega)\right|^2 = constant$ is flat. Therefore the energy spectrum of the Brownian motion is given by $E(\omega) = |Fourier[B](\omega)|^2 = \frac{E_0}{\omega^2}$.

The Wiener-Ikehara theorem was devised to obtain a simple proof of the PNT. In this theorem the boundary behavior of a Laplace transform in the complex plane plays a crucial role. The distributional version of this theorem shows that local pseudofunction boundary behavior, which allows mild singularities, is necessary and sufficient for the desired asymptotic relation. It follows that the twin-prime conjecture is equivalent to a pseudofunction boundary behavior of a certain analytic function (KoJ).

In (ViJ), (ViJ1) a proof of the PNT is provided, based on the Dirac function $\delta \in H_{-1/2-\varepsilon}$ in combination with the concept of quasi-asymptotically bounded distributions defined by

$$\langle\tfrac{f(\vartheta\cdot)}{\rho(\vartheta)}, \varphi(\cdot)\rangle = O(1) \text{ for } \vartheta\to\infty\ .$$

The density of the Mangoldt function $\psi(x) = \sum_{n<x}\Lambda(n)$ is then given by ((BrK4) Note S17)

$$\psi^{'}(x) = \sum_{n<x}\Lambda(n)\delta(x-n) \in H_{\frac{1}{2}-\varepsilon}$$

We propose an alternative quasi-asymptotically bounded $H_{-1/2}$ distributions concept defined by

$$\lim_{\vartheta\to\infty}\left(\tfrac{f(\vartheta\cdot)}{\rho(\vartheta)}, \varphi(\cdot)\right)_{-1/2} = O(1)\ ,\quad \forall\varphi \in H_{-1/2} \to \infty$$

leading to a replacement in the form ((BrK4) Note S19)

$$\psi^{'} \in H_{-\frac{1}{2}-\varepsilon} \quad\to\quad \widetilde{S_1}[\psi] \in H^{\square}_{-\frac{1}{2}}$$

defined by

$$\big(\widetilde{S_1}[\psi], v\big)_{-1/2} \cong (\psi, v)_0 \quad \forall v \in H^{\square}_{-1/2}.$$

The approximation by polynomials in a complex domain leads to several notions and theorems of convergence related to Newton-Gaussian and cardinal series. The latter one are closely connected with certain aspects of the theory of Fourier series and integrals. Under sufficiently strong conditions the cardinal function can be resolved by Fourier's integral. Those conditions can be considerably relaxed by introducing Stieltjes integrals resulting in (C,1) summable series ((WhJ1) theorems 16 & 17, (BrK4) Remarks 3.6/3.7). With respect to interpolations theory with points of sequences c_n (with respect to the Newton-Gauss and cardinal series), the corresponding cardinal series theory with certain aspects of the theory of Fourier series and integrals, especially Fourier-Stieltjes series and related convergent series in the form

$$\sum_{n=1}^{\infty}\frac{1}{n}(|a_n|+|a_{-n}|)<\infty$$

we refer to ((WhJ1), (WhJ2), (BrK4) Remarks 3.6 & 3.7, pp 105ff).

We recall the very „slow" divergence of the monoton increasing (harmonic numbers) sequence $H_n=\sum_{k=1}^{n}\frac{1}{k}$, e.g. $[H_n]=1$ for $n=1{,}2$, $[H_n]=2$ for $n=6$, $[H_n]=5$ for $n\sim 10^2$, $[H_n]=14$ for $n\sim 10^4$, $[H_n]=100$ for $n\sim 43$ (HaJ).

Putting $a_n:=h_n:=\sum_{k=1}^{n}\frac{1}{2k-1}=H_{2n}-\frac{1}{2}H_n$, $a_{-n}:=-h_n$ the theorem 2 in (WhJ2), then leads to the

Lemma: the cardinal series

$$C(x):=\frac{\sin(\pi x)}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{h_n}{x-n}$$

is absolutely convergent, and its sum is of the form

$$\int_0^1\{\cos(\pi xt)\,d\varphi(t)+\sin(\pi xt)\,d\omega(t)\},\quad \varphi,\omega \text{ continuous functions.}$$

Given any function $f(x)$ of this form the series

$$\frac{\sin(\pi x)}{\pi}\cdot\left[\frac{f(0)}{x}+\sum_{n=1}^{\infty}(-1)^n\left\{\frac{f(n)}{x-n}+\frac{f(-n)}{x+n}\right\}\right]$$

is $(C,1)-$ summable to $f(x)$.

We propose to apply $C(x):=\frac{\sin(\pi x)}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{h_n}{x-n}$ alternatively to $\frac{\sin(\pi x)}{\pi x}$. In the context of the Zeta function related Ramanuja formulawe refer to (EdH) 10.10.

In the context of representations of coefficient sums by integrals we recall from (LaE2), §86 the

Lemma: Let $D(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s}$ denote an absolute convergent Dirichlet series ($a:=Re(s)$) and

$$f(x):=\begin{cases}\sum_{n=1}^{x}b_n & \text{for } x \text{ not integer}\\ \sum_{n=1}^{x}b_n-b_x/2 & \text{for } x \text{ integer}\end{cases}$$

Then

$$f(x)=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{a-it}^{a+it}D(s)x^s\frac{ds}{s}\quad \text{resp.}\quad \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{D(s)}{s}x^s\frac{ds}{s}=\sum_{n=1}^{x}b_n\log\left(\frac{x}{n}\right)\ .$$

With respect to the physical aspects we refer to (NaS), where the $\mathrm{H}_{-1/2}$ dual space of $\mathrm{H}_{-1/2}$ on the circle (with its inner product defined by a Stieltjes integral) is considered in the context of Teichmüller theory and the universal period mapping via quantum calculus. For the corresponding Fourier series analysis we refer to ((ZyA) XIII, 11).

The common denominator of the alternatively proposed Hilbert space framework $\mathrm{H}_{-1/2}$ goes along with the definition of a correspondingly defined "momentum" operator (of order 1) P: $\mathrm{H}_{1/2} \to \mathrm{H}_{-1/2}$ defined in a variational form.

In the one-dimensional case the Hilbert transform H (in the n>1 case the Riesz operators R) is linked to such an operator given by $(Pu, v)_{-1/2)} = (\mathrm{Hu}, \mathrm{v})$. With respect to quantum theory this indicates an alternative Schrödinger momentum operator (where the gradient operator "grad" is basically replaced by the Hilbert transformed gradient, i.e. **P**:=-i***R**(grad) and a corresponding alternative commutator representation Q**P**-**P**Q in a weak $\mathrm{H}_{-1/2}$ form.

We note that the Riesz operators **R** commute with translations and homothesis and enjoy nice properties relative to rotations.

The theory of spectral expansions of non-bounded self-adjoint operator is connected with the notions "Lebesgue-Stieltjes integral" and "functional Hilbert equation for resolvents ((LuL) (7.8). The corresponding Hilbert scale framework plays also a key role on the inverse problem for the double layer potential. The corresponding model problem (w/o any compact disturbance operator) with the Newton kernel enjoys a double layer potential integral operator with the eigenvalue 1/2 (EbP).

The incomplete Gamma function play a key role to compute the action of the Leray projection operator on the Gaussian functions (LeN1). Those action formulas can be applied to derive in the context of the well-posedness topic of the NSE and related (based on tempered distribution and a Carleson measure characterization of the BMO space) estimates ((LeN1), (KoH), theorems 1 and 2, see also (BrK4) pp. 26, 58, 64, 99, 121).

For the related equations with respect to the incomplete Gamma function we refer to (OlF1) 7.2.2, 8.4.15, (AbM) 6.5.12, 13.6.10).

The RH is connected to the quantum theory via the Hilbert-Polya conjecture resp. the Berry-Keating conjecture. It is about the hypothesis, that the imaginary parts t of the zeros $1/2 + it$ of the Zeta function $Z(t)$ corresponds to eigenvalues of an unbounded self-adjoint operator, which is an appropriate Hermitian operator basically defined by $QP + PQ$, whereby Q denotes the location, and P denotes the (Schrödinger) momentum operator. In (BrK3) the corresponding model (convolution integral) operator S_1 (of order 1 with "density" $dcot$ for the one-dimensional harmonic quantum oscillator model is provided.

In the context of the Berry-Keating conjecture the Gaussian function $f(x)$ can be characterized as "minimal function" for the Heissenberg uncertainty inequality. Applying the same solution concept as above then leads to an alternative Hilbert operator based representation in $\mathrm{H}_{-1/2}$, resp. to a H_{-1} based definition of the commutator operator with extended domain.

7. A $H^{\#}_{-1/2} \cong l_2^{-1/2}$ based alternative circle method on the unit circle and a Kummer function zero based approach to prove the binary Goldbach conjecture

The current tool trying to prove the tertiary and binary Goldbach conjecture is about the (Hardy/Littlewood) circle method. Vinogradov's *„basic intervals"* correspond in principle to Hardy and Littlewood's *„major arcs"* ((ViI) p.61). Hardy and Littlewood dissected the circle $x = e^{2\pi i\alpha}$, or rather a smaller concentric circle, into „Farey arcs". The major acrs, or basic intervals, provide the main term in the asymptotic formula for the number of representations. Their treatment does not give rise to any very serious difficulties compared to the problems presented by the *„minor arcs"*, or *„supplementary intervals"*. The latter ones are analyzed by the Weyl (trigonometrical) sums

$$S(x) \coloneqq \sum_n e^{2\pi inx}.$$

On the one hand side, the probability that the binary Goldbach conjecture is true, is 100%, while on the other side the current (circle method) tool failed, because of not sufficient (already optimal) Weyl estimates. As those Weyl estimates are w/o any information (i.e. relevance) regarding the underlying (Goldbach) problem, the probability, that the circle method is not adequate is also 100%.

The natural Hilbert space framework (related to generalized Fourier analysis techniques and Dirichlet series, but also related to convergent Weyl sums series) is $\in H^{\#}_{-1/2}(0,1)$ (e.g. BrK9). The Cesàro summable $\cot(\pi x) = 2\sum_{n=1}^{\infty}\sin(2\pi nx)) \in H^{\#}_{-1}(0,1)$ (ZyA) VI-3, VII-1), is related to the eigenfunctions $e^{2\pi inx} = e^{i\pi(2n)x}$, while the proposed alternative Abel summable functions

$$\rightarrow \qquad cot^{(1)}(\pi x) \coloneqq \sum sin(\pi(2\omega_n)x) \in H^{\#}_0(0,1)$$

$$\rightarrow \qquad cot^{(2)}(\pi x) \coloneqq \sum sin\,(\pi(\omega_n + \omega_{n+1} - 1))x \in H^{\#}_0(0,1)$$

are related to the eigenfunctions pair $e^{i\pi(2\omega_n)x}$ and $e^{i\pi(\omega_n+\omega_{n+1})x}$ resp. to the alternative Weyl sums

$$S_1^*(x) \coloneqq \sum_n e^{i\pi(2\omega_n)x}\ ,\ \ S_1^*(x) \coloneqq \sum_n e^{i\pi(\omega_n+\omega_{n+1}-1)x}.$$

With the notation of (LaE1) the prime pair (p,q) counting function $H(x)$ with the condition $p+q \le x$ is given by

$$H(x) = \sum_{n=1}^{x} G_n = \sum_{p\le x}\pi(x-p) \sim \frac{x}{logx}\int_0^{x/2}\frac{dt}{logt} \sim \frac{1}{2}\left(\frac{x}{logx}\right)^2$$

The (improved) Stäckel formula (based on the Euler $\varphi(n)$ –function) shows the asymptotics in the form

$$\tilde{G}_{2n} \sim \frac{1}{2}\frac{105\cdot\zeta(3)}{\pi^4}\frac{n}{logn}\cdot\frac{n}{logn}\cdot\frac{1}{\varphi(n)} \sim 0{,}648\ldots\cdot\frac{1}{\varphi(n)}\cdot\frac{n}{logn}\cdot\frac{n}{logn}.$$

Therefore, Landau predicted a proof of the binary Goldbach conjecture „with high probability" (LaE1).

Related to the Stäckel formula we recall from (ApM) p. 71, the following estimates

$$\frac{\sigma(n)}{n^2} \le \frac{1}{\varphi(n)} \le \frac{\pi^2}{6}\frac{\sigma(n)}{n^2} = \zeta(2)\frac{\sigma(n)}{n^2} \quad ,\ n \ge 2$$

whereby $\sigma(n) = \sigma_1(n)$ denotes the sum of the divisors of n ((ApM) p. 38).

With respect to the factor $\zeta(2)$ we recall the related $cot\left(\frac{\pi}{2}x\right)$ estimate from the previous section

$$\left|\frac{\pi}{2}cot\left(\frac{\pi}{2}x\right)-\frac{1}{x}\right| \leq \frac{\pi^2}{6}=\zeta(2)\ , \quad |x|\leq 1\ .$$

With respect to the Zeta function itself $\zeta(s)$ we recall the related $cot(\pi x)$ representation from the previous section (TiE)

$$\sum_{n>x}\frac{1}{n^s}=\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} z^{1-s}(-\pi\, cot(\pi z)\frac{dz}{z}\quad , s=\frac{1}{2}+ix$$

and its link to the Zeta function is given by

$$(1-s)\cdot\zeta^*(s)=cot(\frac{\pi}{2}s)\cdot\zeta(s)=tan(\frac{\pi}{(2}(1-s))\cdot\zeta(s)$$

whereby it holds

$$\pi\, cot(\pi x)=\frac{\pi}{2}\left[cot\left(\frac{\pi}{2}x\right)-\frac{1}{2}cot(\frac{\pi}{2}(1-x))\right]=\frac{\pi}{2}\left[cot\left(\frac{\pi}{2}x\right)-\frac{1}{2}tan(\frac{\pi}{2}x)\right].$$

and

$$log'(tan\left(\frac{\pi}{2}s\right)=-log'(cot\left(\frac{\pi}{2}s\right)=\frac{\pi}{sin\,(\pi s)}\ .$$

In the following we shall deal with the special Kummer functions

$$K_a(z)\coloneqq {}_1K_1(a,a+1;z) \text{ with } 0<a\leq 1.$$

The asymptotics of ${}_1K_1(a,a+1;z)$ is given by ((OlF) 10.3, (AbM) 13.5.1.)

$${}_1K_1(a,a+1;z)\sim\frac{1}{\Gamma(a)}\frac{e^z}{z}\ , z\to\infty, ph(z)=0\ ,\ {}_1K_1(a,a+1;z)\sim\frac{1}{(-z)^a}\ , z\to-\infty, ph(-z)=0$$

where $ph(z)$ denotes the phase or the argument of z. For the real case ($x\in R$) we deal with the function $F_a(x)\coloneqq c\cdot {}_1F_1(a,a+1;x)$ with a given constant c fulfilling the following properties

i) ${}_1F_1(a,c;x)\sim\frac{1}{\Gamma(a)}\frac{e^x}{x^{c-a}}$ resp. ${}_1F_1(a,a+1;x)\sim\frac{1}{\Gamma(a)}\frac{e^x}{x}$, $x\to\infty$

ii) $\frac{d}{dx}F_a(x)=\frac{a}{a+1}F_{a+1}(x)$

iii) $F_a(x)\sim\frac{c}{\Gamma(a)}\frac{e^x}{x}$, $x\to\infty$ ((OlF), 7 §10.1, (AbM) 13.5.1.)

iv) $\frac{1}{2}\frac{d}{dx}F_a^2(x)=\frac{a}{a+1}F_a(x)\cdot F_{a+1}(x)\sim\frac{1}{a+1}\frac{1}{\Gamma^2(a)}(\frac{e^x}{x})^2$.

From (AbM) 13.2.9, we mention the Barnes-type contour integral representation

$${}_1F_1(a,a+1;z)=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a}{a+s}\Gamma(1-s)(-z)^s\frac{ds}{s},\ \ |\arg(-z)|<\frac{\pi}{2},\ a\neq 0,-1,-2,\ldots$$

where the contour must separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$.

The special Kummer functions $K_a(z)\coloneqq {}_1K_1(a,a+1;z)$ go along with the Hurwitz generalization of the Zeta function ((TiE) 2.17)

$$\zeta(s,a)=\sum_{n=1}^{\infty}\frac{1}{(n+a)^s}\ , Re(s)>1\ , 0<a\leq 1$$

resp. the corresponding generalized Dirchlet series.

In the following we first shall deal with the choice

$$\frac{1}{2} \le a \coloneqq \tilde{a}_n \coloneqq \frac{2n-1}{2n} \to 1$$

and

$$a \coloneqq \frac{2n-1}{2n}\ ,\ \tilde{a}_n \coloneqq \frac{1}{a+1} = \frac{2n}{4n-1}\ ,\ \tilde{b}_n \coloneqq \frac{1}{\Gamma^2(a)} = \frac{1}{\Gamma^2\left(1-\frac{1}{2n}\right)}\ ,\ c \coloneqq \sqrt{2/3}.$$

The corresponding definition of $H^*_{\square}(x)$ is motivated by the replacement

$$H(x) \sim \frac{x}{logx}\int_2^{\frac{x}{2}} \frac{dt}{logt}\,\frac{d}{dx} \ \rightarrow\ H^*(x) \sim \frac{d}{dx}F^{\square}_{\tilde{a}_n}(logx)\cdot\int_{log2}^{\log\left(\frac{x}{2}\right)}\frac{d}{dt}F^{\square}_{\tilde{a}_n}(t)dt\ .$$

fulfilling the

Lemma („Chebychev" inequality):

$$\frac{1}{3\pi}\left(\frac{x}{logx}\right)^2 \le H^*_n(x) \le \frac{1}{2}\left(\frac{x}{logx}\right)^2.$$

Proof: It holds $\tilde{a}_1 = \frac{3}{2}$, $\tilde{a}_\infty = \frac{1}{2}$, $\tilde{b}_1 = \frac{1}{\pi}$, $\tilde{b}_\infty = 1$ and therefore

$$\frac{1}{2\pi} \le \tilde{a}_n \cdot \tilde{b}_n \le \frac{3}{2} \text{ resp. } \frac{1}{3\pi} \le \widetilde{\frac{2}{3}a_n} \cdot \tilde{b}_n \le 1.$$

Chosing $c \coloneqq \sqrt{2/3}$ then proves the lemma.

We note that for $n \ge 4$ it holds $1-\frac{1}{8} < 1-\frac{1}{2n}$, $1+\frac{1}{2n} < 1-\frac{1}{8}$, with ist relationship to the Chebychev inequality

$$1-\frac{1}{8} < \frac{\pi(x)}{\frac{x}{logx}} < 1+\frac{1}{8}\ .$$

The Kummer function zeros related inequality $2n-1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n+1$ indicates a replacement

$$\frac{1}{2} \le \tilde{a}_n \le 1 \qquad \rightarrow \qquad \frac{1}{2} \le \frac{2n-1}{2n} < d_n \coloneqq \frac{\omega_n+\omega_n}{\omega_n+\omega_{n+1}} < \frac{2n}{2n+1} \le 1$$

leading to

$$H^{**}_n(x) \coloneqq \frac{1}{2}\sum_{n<x}\frac{d}{dx}F^2_{d_n}(logx)$$

With respect to a Hilbert space framework $H_{-1/2}$ (see also below) we note the identities $((dK^2, v))_{\alpha-1/2} = (K^2, v)_\alpha$ for $K^2, v \in H_\alpha$.

Concering the below with respect to the hypergeometric (Kummer) functions and their relationship to the Gamma function and the underlying Euler constant we recall from (BrR) the formula

$$\int_0^\infty x^s[e^{-x} - J_0(2x)]\frac{dx}{x} = \Gamma(s) - \frac{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{1}{s}\left[(1-\gamma s) - \frac{1-\gamma\frac{s}{2}}{1+\gamma\frac{s}{2}} + O(s^2)\right] \to 0 \quad s \to 0^+.$$

Analogue it holds

$$\int_0^\infty x^s\left[e^{-x} - J_0(2\sqrt{x})\right]\frac{dx}{x} = \Gamma(s) - \frac{\Gamma(s)}{\Gamma(1-s)} = \frac{1}{s}\left[(1-\gamma s) - \frac{1-\gamma s}{1+\gamma s} + O(s^2)\right] = \gamma\frac{1-\gamma s}{1+\gamma s} + O(s) \to \gamma \quad s \to 0^+.$$

From section 1d. we recall the following properties of the sequences ω_n (the imaginary parts of the Kummer function zeros):

v) $n-\frac{1}{2}<\omega_n<n$, $n+\frac{1}{2}<\omega_{n+1}<n+1$, $\frac{1}{2}<\omega_1<a:=s_n:=\frac{\omega_n}{n}\to 1\ n\in N$

vi) $2n-1<2\omega_n<2n<\omega_n+\omega_{n+1}<2n+1<2\omega_{n+1}<2(n+1)$

vii) the sequences $2\omega_n$ and $\omega_n+\omega_{n+1}$ fulfill the Hadamard gap condition

$\frac{\omega_{n+1}}{\omega_n}>\frac{n+\frac{1}{2}}{n}=1+\frac{1}{2n}>q>1$ resp. $\frac{\omega_{n+1}+\omega_{n+2}}{\omega_n+\omega_{n+1}}>\frac{2n+2}{2n+1}=1+\frac{1}{2n+1}>q>1$

viii) $\theta:=\frac{1}{4}<\frac{\omega_{n+1}}{2}-\frac{\omega_n}{2}<1-\frac{1}{4}=1-\theta$.

For the related sequences $a_n:=\frac{2\omega_n}{2n}-\frac{1}{2}$, $b_n:=\frac{\omega_{n+1}+\omega_n}{2n}-\frac{1}{2}$ it therefore follows

v) $0<a_1=\omega_1-\frac{1}{2}\le a_n\to\frac{1}{2}$, $\frac{1}{2}\leftarrow b_n<b_1=\frac{\omega_1+\omega_2-1}{2}<1$

vi) $a_n, b_n-a_n\in(0,\frac{1}{2})$, $b_n, 1-a_n\in(\frac{1}{2},1)$.

We note the Mellin transform of the related Kummer function ((GrI) 7.612)

$$\int_0^\infty x^s\,{}_1F_1(a_n,a_n+1;-x)\frac{dx}{x}=\frac{a_n}{a_n-s}\,\Gamma(s)\quad,\quad 0<Re(s)<a_n$$

$$\int_0^\infty x^s\,{}_1F_1(a_n,a_n+1;-x)\frac{dx}{x}\ \to\ \int_0^\infty x^{s/2}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\frac{dx}{x}=\frac{\Gamma\left(\frac{s}{2}\right)}{1-s}=2\frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)}\quad,\quad 0<Re(s)<1.$$

For

$$g_n^{(1)}(x):={}_1F_1(a_n,b_n;x),\ \ g_n^{(2)}(x):={}_1F_1(1-a_n,b_n;x),\ \varphi_n^{(k)}(x):=xg_n^{(k)}(x)\ \ (k=1,2)$$

it holds

$$g_n^{(k)}(x)\to{}_1F_1\left(\tfrac{1}{2},\tfrac{1}{2};x\right)=e^x\ \text{ resp. }\ \varphi_n^{(k)}(-logx)\to\frac{logx}{x}\ \text{ as } n\to\infty\ (k=1,2).$$

From ((LeN) (9.13.7)) we recall the Kummer function based representation of the $li(x)$-function in the form $li(x)=-x\,{}_1F_1(1,1;-logx)=Ei(logx)=\int_0^x\frac{dt}{logt}=\int_{-\infty}^{logx}\frac{e^t}{t}dt$ with the asymptotics $li(x)\approx log^{-1}(x)$, i.e. it holds $-x\,{}_1F_1\left(a_n+\frac{1}{2},b_n+\frac{1}{2};-logx\right)\to-x\,{}_1F_1(1,1;-logx)=li(x)$.

The properties of the sequences above are proposed to build two different prime density functions for a prime pair in combination with the following two Kummer function related properties:

i) $c_1\cdot\left[{}_1F_1(a,c;x)\right]^2\le{}_1F_1(a-\mu,c;x)\cdot{}_1F_1(a+\mu,c;x)\le\left[{}_1F_1(a,c;x)\right]^2$, $a>0$, $c>a\ge\mu-1$, $x\in R$

ii) $\int_0^\infty x^s\,{}_1F_1(\alpha,\beta;-x)\frac{dx}{x}=\frac{\Gamma(\beta)}{\Gamma(\alpha)}\Gamma(s)\frac{\Gamma(\alpha-s)}{\Gamma(\beta-s)}$, $0<Re(s)<Re(\alpha)$, (GrI) 7.612 .

The second inequality of property i) for positiv integer μ is provided in (BaR) and for non-integer positive μ, complemented with a reverse inequality, in (KaD) (see also appendix).

Putting e.g. $a:=\frac{1}{2}$, $\mu:=\frac{1}{2}-a_n$ it holds $a-\mu=a_n;\ a+\mu=1-a_n$ resulting into the inequalities

$$c_1\cdot\left[{}_1F_1\left(\tfrac{1}{2},c;x\right)\right]^2\le{}_1F_1(a_n,c;x)\cdot{}_1F_1(1-a_n,c;x)\le\left[{}_1F_1\left(\tfrac{1}{2},c;x\right)\right]^2.$$

From the Kummer function Mellin transform property ii) above we get

i) $\frac{1}{2}\int_0^\infty x^{s/2}\,{}_1F_1(a_n,b_n;-x)\frac{dx}{x}=\frac{\Gamma(b_n)}{\Gamma(a_n)}\Gamma(s)\frac{\Gamma(a_n-s)}{\Gamma(b_n-s)}\to\frac{1}{2}\int_0^\infty x^{s/2}e^{-x}\frac{dx}{x}=\frac{1}{2}\Gamma\left(\frac{s}{2}\right)$ $\quad 0<Re(s)<1$

ii) $\frac{1}{2}\int_0^\infty x^{s/2}\,{}_1F_1(a_n,b_n+1;-x)\frac{dx}{x}\to\frac{1}{2}\int_0^\infty x^{s/2}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\frac{dx}{x}=\frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)}$ $\quad 0<Re(s)<1$.

The circle method (based on the interior of the unit disk) is concerned with complex-valued functions $f(z) = f(r \cdot e(\varphi))$ $(e(\varphi) \coloneqq e^{2\pi i\varphi}$, $\varphi \in (0,1))$ resp. $e_n(\varphi) \coloneqq e(n\varphi)$ fulfilling

$$\int_0^1 e_n(\varphi)d\varphi = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases} \quad \text{and} \quad |1 - e(\varphi)|^2 = 4sin^2(\pi\varphi)$$

It is about Fourier analysis of complex-valued power series functions

$$f(x) = \sum_{n=0}^{\infty} a_n z^n \text{ , } |z| < 1$$

i.e. defined for the interior unit circle domain. The underlying mathematical tool set is based on the formula ((ViI1) chapter I, lemma 4, Notes)

$$r^n a_n = \int_0^1 f(re^{2\pi it})e^{-2\pi int}dt \text{ , } 0 < r < 1.$$

The corresponding (convergence) requirements are ensured by the

Theorem ((OlF) 3.3): let $\sum_{n=0}^{\infty} a_n z^n$ converge when$|z| < r$. Then for fixed k

$$\sum_{n=k}^{\infty} a_n z^n = O(z^k) \quad \text{in any disk } |z| \le \rho \text{ such that } \rho < r.$$

In line with the concepts of the previous section we propose a correspondingly modified circle method, which is about a Fourier analysis of complex-valued (generalized) Fourier series representations on the unit circle, based on a corresponding mathematical tool set enabled by the formula

$$-nf_n(x) = \frac{1}{2}\int_0^1 g(x-t)f_n(t)dt$$

with $f_n(t) = a_n \cos(2\pi nt) + b_n \sin(2\pi nt)$ and $g(x) \coloneqq sin^{-2}(\pi x)$. In the corresponding Hilbert scale framework $H_\alpha^\#(0,1)$ the mapping formula

$$v = S[u] \text{ with } v(x) \coloneqq \frac{1}{2}\int_0^1 g(x-t)u(t)dt$$

defines an (Pseudo-Differential) integral convolution operator $S\colon H_\alpha^\#(0,1) \to H_{\alpha-1}^\#(0,1)$ of order 1, where the Fourier transform of the kernel function is given by $\hat{g}(\omega) = -\omega$. It leads to a variational representation in the form

$$(S[v], w)_{\alpha-1/2} = (v, w)_\alpha, \ \forall v, w \in H_\alpha^\#(0,1).$$

The corresponding convolution kernel of the inverse operator $S_{-1}\colon H_{\alpha-1}^\#(0,1) \to H_\alpha^\#(0,1)$ is given by $k(x) \coloneqq -log\,|2sin\,(\pi x)|$ ((BrK3), (BrK4) remarks 3.6, 3.9, Notes S38, S48, O23-30).

Regarding orthogonal polynomials on the unit circle, built on a non-negative, integrable (in a Lebesgue-sense) function $f(\theta)$ of period 2π, we mention the case $f(\theta) = \{g(\theta\}^{-1}$ where $g(\theta)$ is a positive trigonometric polynomial of degree m ((SzG) 11.2, (BrK4) Note S49).

For each positive real number x the Snirelmann density is defined for a subset $\boldsymbol{A}$ of the set of positive integers $\boldsymbol{N}$ by

$$0 \le \sigma(\boldsymbol{A}) \coloneqq inf_n \frac{A(n)}{n} \le 1 \quad \text{with} \quad A(x) \coloneqq \sum_{\substack{a \in \boldsymbol{A} \\ a \le x}} 1.$$

It holds $A(n) \ge kn$ for $\sigma(\boldsymbol{A}) = k$; $A(1) = 0$ (and therefore $\sigma(\boldsymbol{A}) = 0$), if 1 is not an element of $\boldsymbol{A}$; and $A(n) = n$ (and therefore $\sigma(\boldsymbol{A}) = 1$), if $\boldsymbol{A} = \boldsymbol{N}$.

The Snirelmann-Goldbach theorem states, that the set $\boldsymbol{A} \coloneqq \{0,1\} \cup \{p + q; p, q\ prim\}$ has positive Snirelmann density. In case of a Snirelmann density ½ the binary Goldbach conjecture would be proven.

For a subset $\boldsymbol{A}$ of the set of integers $\boldsymbol{N}$ integers, if

i) the integer "1" is not an element of $\boldsymbol{A}$, the Snirelmann density of $\boldsymbol{A}$ is $= 0$

ii) if the integer "2" is not an element of $\boldsymbol{A}$, the Snirelmann density of $\boldsymbol{A}$ is $\leq \frac{1}{2}$

iii) if $\boldsymbol{A} = \boldsymbol{N}$, the Snirelmann density of $\boldsymbol{A}$ is $= 1$.

For the smallest prime $p > n$, it holds $p < 2n$, i.e. $\pi(2n) - \pi(n) \geq 1$.

The set $\{2n - 1 = [2\omega_n]\}$ resp. $\{2n - 1 = [\omega_n + \omega_{n+1} - 1]\}$ of odd integers has Snirelmann density $\leq \frac{1}{2}$, while the set $\{2n = [\omega_n + \omega_{n+1}]\}$ of even integers has Snirelmann density $= 0$.

Therefore, the sequences $\{2\omega_n\}$ and $\{\omega_n + \omega_{n+1} - 1\}$ are suggested to build a problem adequate binary number theoretical function σ^* in sync with a corresponding set with Snirelmann density $\sigma^*(\boldsymbol{A}^*) = \frac{1}{2}$.

The „Snirelmann-Stieltjes integral" density concept is related to another method to analyze binary additive problems, which is about the „dispersion method", which is about a „correlation theory of binary problems" (LiJ). Unfortunately, in its current form this method cannot be applied to the binary Goldbach problem ((LiJ), chapter X, §2).

We claim, that the proposed circle method on the unit circle with its underlying related distributional Hilbert space framework $H^{\#}_{\frac{1}{2}}(0,1)$ (going along with the Stieltjes/Plemelj „differential" potential density concept) provides the appropriate framework for a correspondingly adapted „truly" „dispersion / variance method", to solve the binary Goldbach conjecture. This is about the independence of the (number theoretical) „events" related to the (positive integer resp. real number) sets

$$\boldsymbol{A} \coloneqq \{0,1\} \cup \{p + q; p, q\ prim\} \quad , \quad \boldsymbol{A}^* \coloneqq \{0,1\} \cup \{2\omega_n\} \cup \{(\omega_n + \omega_{n+1} - 1)\}\ .$$

8. The $\mathrm{H}_{-1/2}$ Hilbert space and corresponding arithmetic functions

In (ViJ) the Prime Number Theorem (PNT) is proven on a distributional way, applying the Dirac function to derive the first derivative of the Chebyshev function. In line with the proposal of section 4.d) below we propose to replace the Dirac distributions space by the $\mathrm{H}_{-1/2}$ Hilbert space, resulting in corresponding representation of concerned arithmetic functions.

This section is about the building of a distributional density function $\theta(x)$ with

$$\theta'(x) = S[\theta](x) = \sum_{n<x} a(n) \in H_{-\frac{1}{2}} \text{ in a weak } H_0 \text{ –sense,}$$

alternatively to the Mangoldt resp. Landau distribution functions

$$\psi(x) = \sum_{n<x} \Lambda(n) \quad \text{resp.} \quad \vartheta(x) = \sum_{n=1}^{\infty} \Lambda(n)\log\left(\frac{x}{n}\right)$$

with

$$\psi'(x) = \sum_{n<x} \Lambda(n)\delta(x-n) \in H_{-\frac{1}{2}-\varepsilon} \text{ resp. } \vartheta'(x) = \frac{1}{x}\sum_{n<x} \Lambda(n).$$

The Delta function representation of $\psi'(x$ is applied in (ViJ) for „a quick distributional way to *(prove)* the prime number theorem" (see also (BrK4) Note S19).

For

$$\theta(x) := \sum_{n<x} \Lambda(n) \cdot \log\left(\sin\left(\pi \frac{x}{2\omega_n}\right)\right) \in H_{1/2}$$

one gets

$$\theta'(x) = \frac{\pi}{2}\sum_{n<x} \frac{\Lambda(n)}{\omega_n} \cot\left(\frac{\pi}{2\omega_n}x\right) \in H_{-1/2} \ .$$

For the convergence $\lim_{n\to\to\infty} \theta'(x)$ we refer to the previous sections.

The baseline formulas for the following are the representations

i) $p(x) := \left(\frac{\pi}{2}x\right)cot\left(\frac{\pi}{2}x\right) = 1 + \sum_{n=1}^{\infty}\left[\frac{x}{x+n} + \frac{x}{x-n}\right]$

ii) $w(x) := \frac{p(x)}{x} = \left(\frac{\pi}{2}\right)cot\left(\frac{\pi}{2}x\right) = \frac{1}{x} + \sum_{n=1}^{\infty}\left[\frac{1}{x+n} + \frac{1}{x-n}\right]$

iii) $\psi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[-\frac{\zeta'}{\zeta}(s)\right]x^s\frac{ds}{s} = \sum_{n<x}\Lambda(n)$ and $-\frac{\zeta'}{\zeta}(s) = \int_0^{\infty} x^{-s}d\psi = \sum_{n=2}^{\infty}\Lambda(n)n^{-s}$, $(Re(s) > 1)$, (EdH) 3.2

iv) $\vartheta(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[-\frac{\zeta'}{s\cdot\zeta}(s)\right]x^s\frac{ds}{s} = \int_0^x \psi(t)dlog(t) = \sum_{n=1}^{\infty}\Lambda(n)\log\left(\frac{x}{n}\right)$ and $-\frac{\zeta'}{s\cdot\zeta}(s) = \int_0^{\infty} x^{-s}d\vartheta$, $(Re(s) > 0)$

The Euler conjecture, i.e. the convergence of the series

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \cdots = 0$$

can be derived from the PNT ((LaE) §159). This series is linked to the zeta function by the identities

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \quad , \quad -\frac{\zeta'(s)}{s} = \sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{n^s}$$

i.e.

$$-\frac{\zeta'(s)}{s\zeta(s)} = \left[\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right]\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{n^s}\right] \quad , \quad \frac{\zeta'(s)}{\zeta(s)} = \left[\sum_{n=1}^{\infty}\mu(n)\frac{1}{n^s}\right]\cdot\left[\sum_{n=1}^{\infty}\log\left(\frac{1}{n}\right)\frac{1}{n^s}\right].$$

What cannot derived from the PNT is the convergence of the series

$$(*) \quad \sum_{n=1}^{\infty}\frac{\mu(n)}{n}log\left(\frac{1}{n}\right) = 1 \ .$$

„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived" ((LaE) §160).

The Landau statement above corresponds to the proposed replacement of the Dirac distribution by appropriate distributions of the Hilbert space $H_{-\frac{1}{2}}$, going along with the following identities ((ApT), (BrK4), (LaE) §227), while at the same time enabling the Bagchi RH criterion (BaB),

$$1 = \sum_{n=1}^{\infty} \frac{1}{n}\mu(n)\, log(\frac{1}{n}) = \sum_{1}^{\infty} \frac{1}{n} a_n b_n =: \big((u,v)\big)_{-\frac{1}{2}} := \lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} u\left(\frac{1}{2}+it\right) v\left(\frac{1}{2}-it\right) dt$$

with ($s = \frac{1}{2} + it$) and

$$u(t) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \in H_{-\frac{1}{2}} \quad , \quad v(t) := \sum_{n=1}^{\infty} \frac{log(1/n)}{n^s} \in H_{-\frac{1}{2}}.$$

Let

$$\Xi(t) := \zeta\left(\tfrac{1}{2}+it\right) := \sum_{n=1}^{\infty} \tfrac{1}{n^s} \in H_{-\frac{1}{2}-\varepsilon} \quad , \quad w(t) := \sum_{n=1}^{\infty} \log\left(\tfrac{1}{n}\right) \frac{\mu(n)}{n^s}$$

$$r(t) := \sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \frac{1}{n^s} \quad , \quad s(t) := \sum_{n=1}^{\infty} \frac{\Lambda(n)}{nlogn} \frac{1}{n^s} \ .$$

It holds $\|\Xi\|_{-1/2}^2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\Xi(t)|^2 dt = \sum_{n=1}^{\infty} \frac{1}{n} = \zeta(1) = \infty$ and $((\Xi, w))_{-\frac{1}{2}} = 1$ and therefore $w \in H_{-\frac{1}{2}-\varepsilon}$. Because of $\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} < \infty$ and $\Xi \in H_{-\frac{1}{2}-\varepsilon}$ it follows that $w \in H_{-\frac{1}{2}+\varepsilon}$. Because of

$$\big((\Xi, r)\big)_{-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} < \infty \quad , \quad \big((w,s)\big)_{-\frac{1}{2}} \sim \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \frac{\mu(n)}{n} < \infty$$

It follows $r \in H_{-\frac{1}{2}+\varepsilon}$, $w \in H_0$, $s \in H_{-\frac{1}{2}-\varepsilon}$. For

$$\sigma(x) := \sum_{n\le x} \frac{\mu(n)}{n} \log\left(\frac{x}{n}\right)$$

it holds

$$\sigma(xy) + 1 = \sigma(x) + \sigma(y) \quad , \quad \sigma'(x) = \frac{1}{x}\sum_{n\le x} \frac{\mu(n)}{n}$$

and the inverse mapping is given by

$$\sigma^{-1}(x) = \sum_{n\le x} \frac{1}{n} log\left(\frac{x}{n}\right) \ , \ x \ge 1 \ .$$

The asymtotics of a related arithmetical function is given by ((ApT) 3.12)

$$\sum_{n\le x} \frac{1}{n}\frac{\varphi(n)}{n} + \sum_{n=1}^{\infty} \frac{1}{n}\frac{\mu(n)\, log\, n}{n} = \frac{6}{\pi^2}(log\, x + \gamma) + +O(\frac{log\, x}{x}).$$

For the relationship to the alternative Zeta function theory below we note that the function $\pi\, cot(\pi x)$ is holomorphic except the pole $z = 1$ and it holds

$$\frac{1}{2} cot(\frac{x}{2}) = \frac{d}{dx} ln(2\, sin(\frac{x}{2}) = -\frac{d}{dx}\sum_{n=}^{\infty} \frac{cos(nx)}{n} = \sum_{n=}^{\infty} sin(nx) \in H_{-1}^{\#}(0,2\pi)$$

$$\sum_{n\ge x} \frac{1}{n^s} = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} z^{1-s}(-\pi\, cot(\pi z)\frac{dz}{z} =: g(s) \ .$$

The arithmetical functions $\mu(n)$, $\varphi(n)$ are multiplicative, but not completely multiplicative ($\mu(4) = 0 \neq 1 = \mu(2)\mu(2)$, $\varphi(4) = 2 \neq 1 = \varphi(2)\varphi(2)$), ((ApT) pp. 36, 88). We note that the product $\varphi(n) = n\prod_{p|n}(1-\frac{1}{p})$ is empty for $n = 1$ since there are no primes which divide 1; Some values of arithmetical functions:

n :	1	2	3	4	5	6	7	8	9	10	....
$\Lambda(n)$:	0	$log2$	$log3$	$log2$	$log5$	0	$log7$	$log2$	$log3$	0	....
$\mu(n)$:	1	-1	-1	0	-1	1	-1	0	0	1	....
$\varphi(n)$:	1	1	2	2	4	2					....

$\varphi(12) = 4,\ \varphi(60) = 16,\ \varphi(81) = 54$

$\varphi(n)$ is even for $n \ge 3$ and $\varphi(p) = p -$

9. Appendix: Formulas and properties

The proposed alternative "baseline" function is the Hilbert transform of the Gaussian function, which is the Dawson function ((OlF) p. 44)

$$F(x) \coloneqq e^{-x^2} \int_0^x e^{-t^2} dt$$

Its relationship to the *„analysis of zeros of certain trigonometric integrals and entire higher genus-1 functions*" (PoG2), is given by the identities ((GrI) 3.896)

$$F(x) \coloneqq e^{-x^2} \int_0^x e^{-t^2} dt = \int_0^\infty e^{-t^2} \sin(2xt) dt = x\,{}_1K_1\left(1, \tfrac{3}{2}; -x^2\right) = e^{-x^2} H(x)$$

with $H(x) \coloneqq x\,{}_1K_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right)$. We note that the Hilbert transform of the $sin(ax)$ –function is given by $-cos(ax)$ –function and $\int_0^\infty e^{-t^2} \cos(2xt) dt = \frac{\sqrt{\pi}}{2} e^{-x^2}$ ((GrI) 3.896). From (GrI) 7.612) we mention the reciprocal formula

$$\int_0^\infty e^{-\frac{x^2}{2}}\,{}_1F_1\left(\tfrac{1}{2}, 1, \tfrac{x^2}{2}\right) sin(yx) dx = \frac{\sqrt{\pi}}{2} e^{-\frac{x^2}{2}}\,{}_1F_1\left(\tfrac{1}{2}, 1, \tfrac{x^2}{2}\right).$$

The Kummer function related Mellin transforms can be derived from the following formulas ((GrI) 7.612):

i) $\int_0^\infty x^s\,{}_1F_1(a, c, -x) \frac{dx}{x} = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(c-s)} \frac{\Gamma(c)}{\Gamma(a)}$ $\quad 0 < Re(s) < Re(a)$

ii) $\int_0^\infty x^s\,{}_1F_1(a, a+1, -x) \frac{dx}{x} = \frac{a}{(a-s)} \Gamma(s)$ $\quad 0 < Re(s) < Re(a)$

iii) $\int_0^\infty x^{s-1/2}\,{}_1F_1\left(a, a+\frac{1}{2}, -x\right) \frac{dx}{x} = \frac{\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)} \Gamma(s-\frac{1}{2})$ $\quad \frac{1}{2} < Re(s) < Re(a+\frac{1}{2})$

iv) $\int_0^\infty x^{\frac{s}{2}}\,{}_1F_1(\tfrac{1}{2}, \tfrac{3}{2}, -x) \frac{dx}{x} = \frac{\Gamma(\frac{s}{2})}{1-s}$ $\quad 0 < Re(s) < Re(1)$

v) $h(x) + 2xh'(x) = e^{-x}$ with $h(x) \coloneqq {}_1F_1(\tfrac{1}{2}, \tfrac{3}{2}, -x)$

vi) $M\left[-xh'\right]\left(\frac{s}{2}\right) = \frac{s}{2} M[h]\left(\frac{s}{2}\right) = \frac{\Gamma(1+\frac{s}{2})}{1-s}$ $\quad 0 < Re(s) < Re(1)$

vii) $M_{\frac{1}{4}\frac{1}{4}}(z) = z^{\frac{3}{4}} e^{-z/2}\,{}_1F_1(\tfrac{1}{2}, \tfrac{3}{2}, z)$, $e^{-\frac{z}{2}}\,{}_1F_1\left(\frac{1}{2}, \frac{3}{2}, z\right) = z^{-\frac{3}{4}} M_{\frac{1}{4}\frac{1}{4}}(z) = (-z)^{-\frac{3}{4}} M_{-\frac{1}{4}\frac{1}{4}}(-z)$

for the Whittaker functions $M_{\frac{1}{4}\frac{1}{4}}(z)$ ((GrI) 9.220, 9.231)

viii) For $-1/4 < Re(\vartheta) < 1/4$, $0 < 1/2 - 2\vartheta < 1$, $1/2 < 1 + 2\vartheta < 3/2$ it holds ((GrI) 7.612):

$$\int_0^\infty G_{2\vartheta}(x) sin(yx) dx = \frac{\sqrt{\pi}}{2} G_{2\vartheta}(x) \quad \text{with } G_{2\vartheta}(x) \coloneqq x^{4\vartheta} e^{-\frac{x^2}{2}}\,{}_1F_1\left(\tfrac{1}{2} - 2\vartheta, 1 + 2\vartheta, \tfrac{x^2}{2}\right)$$

resp.

$$\int_0^\infty e^{-\frac{x^2}{2}}\,{}_1F_1\left(\tfrac{1}{2}, 1, \tfrac{x^2}{2}\right) sin(yx) dx = \frac{\sqrt{\pi}}{2} e^{-\frac{x^2}{2}}\,{}_1F_1\left(\tfrac{1}{2}, 1, \tfrac{x^2}{2}\right) \quad (\vartheta = 0)$$

ix) The asymptotics of the Kummer functions are given by ((OlF), 7 §10.1, (AbM) 13.5.1.)

$${}_1F_1(a, c; x) \sim \frac{1}{\Gamma(a)} \frac{e^x}{x^{c-a}} \qquad \text{resp.} \qquad {}_1F_1(a, a+1; x) \sim \frac{1}{\Gamma(a)} \frac{e^x}{x} \quad , \quad x \to \infty$$

x) The simple zeros of ${}_1F_1\left(1, \frac{3}{2}; z\right)$ lie in the half-plane $Re(z) < -1/2$. The simple zeros of ${}_1F_1\left(\frac{3}{2}, \frac{3}{2}; z\right)$ lie in the half-plane $Re(z) > 1/2$. All zeros z_n of the Kummer function $K_a(z) \coloneqq {}_1K_1(a, a+1; z)$ $(0 < a \leq 1)$ are simple and satisfy the asymptotic formula (SeA)

$$z_n = 2\pi i n + \left[(1-a) + \tfrac{(1-a)^2}{2\pi i n}\right] \cdot \left[\log(2\pi|n|) + \tfrac{log\Gamma(a)}{1-a} \pm \left(-\tfrac{\pi}{2}\right)\right] - \tfrac{3a+1}{2\pi i n} + O(\tfrac{log|n|}{n^2}), \quad n \to \pm\infty,$$

The zeros all lie in the horizontal stripe

$$(2n-1)\pi < |Im(z_n)| \coloneqq |2\pi\omega_n| < 2n\pi \quad .$$

((SeA), see also ((BrK4) lemma A3, Notes O5-17, O22, O23, (BrK7) Note 11)).

For the Fourier inverse of the Zeta function on the critical line (in a distributional sense) we refer to (BrK4) Notes S21, S24. The related (classical) Zeta approximation series reprensentation is provided in (TiE) 4.14 resp. (BrK4) Notes S51, O9, O27. In (OlF1) 25.6.6, an integral value representation for $\zeta(2n+1)$ is provided with $\cot(\pi x)$ "density" function.

We note the corresponding Gamma function equivalent in the form

$$log'\Gamma(1-x) - log'\Gamma(x) = \pi\cot(\pi x) = \frac{1}{x}[1 - 2\sum_{k=0}^{\infty} s_{2k}\, x^{2k}] \quad \text{(NiN) §14 (5), §19 (16).}$$

The alternative „Gamma" function $\Gamma^*\left(\frac{s}{2}\right)$ fulfills the following properities

i) $\Gamma^*\left(\frac{s}{2}\right)\Gamma^*\left(\frac{1-s}{2}\right) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)$, $\Gamma'(1) = -\gamma$, $\Gamma^{*\prime}(1) = \pi$ ($\tan(\pi) = 0$ and $tan'(\pi) = \pi$)

ii) $\Gamma\left(1+\frac{s}{2}\right) \quad \rightarrow \quad \Gamma^*\left(\frac{s}{2}\right) \coloneqq \Gamma\left(\frac{s}{2}\right) tan\left(\frac{\pi}{2}s\right) = \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{\Gamma\left(1+\frac{s-1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}$

iii) $log'\Gamma^*(s) = log'\Gamma(s) + log'\tan\left(\frac{\pi}{2}s\right) = log'\Gamma(s) + \frac{\pi}{\sin(\pi s)}$

iv) $log'\Gamma^*\left(\frac{s}{2}\right) = log'\Gamma\left(\frac{s}{2}\right) + \frac{1}{2}\frac{1}{\sin(\pi s)} = log'\Gamma\left(\frac{s}{2}\right) + \frac{1}{2}\Gamma(s)\Gamma(1-s)$, $log'\Gamma\left(\frac{3}{4}\right) - log'\Gamma\left(\frac{1}{4}\right) = \pi$

v) $\frac{\pi}{2}\Gamma^*\left(\frac{s}{2}\right) = \sum_{k=1}^{\infty}\frac{\Gamma\left(1+\frac{s}{2}\right)}{(k-\frac{1}{2})^2-(\frac{s}{2})^2}$ (because of $tan\frac{\pi}{2}x = \frac{4x}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-x^2}$).

The alternative „Gamma" function properties and some related lemmata in section 2 below in combination with the „value" property of Kummer resp. hypergeometric functions ((BeI), (WoJ)), might also enable a new tool to prove the irrationality or even the transcendence of e.g. the Euler constant γ. The approaches in ((BeI), (WoJ)) are both based on classical E-function theory, whereby the considered Kummer functions in (BeI) are explicitly excluding the considered Kummer function of this paper. The concept in (PoG4) (in combination with the conjecture in (PoG3) about power series with rational or integer coefficients in the context of a convergence radius one) might provide an alternative E-or G-function approach.

With respect to the alternative $\zeta^*(s)$ function

$$\zeta^*(1-s) = \frac{\tan(\frac{\pi}{2}s)}{s}\cdot\zeta(1-s) \quad \text{resp.} \quad \zeta^*(s) = \frac{\cot(\frac{\pi}{2}s)}{1-s}\cdot\zeta(s) = \frac{\tan(\frac{\pi}{2}(1-s))}{1-s}\cdot\zeta(s)$$

resp.

$$log\zeta^*(s) = \log(\tan\frac{\pi}{2}(1-s)) + log\left(\frac{1}{1-s}\right) + log\zeta(s)$$

We onte the following properties

i) $\frac{1}{1-s} = \int_1^{\infty} x^{-s}dx = 1 - s\int_1^{\infty} x^{-s-1}xdx$

ii) $\frac{\pi}{2}\tan\left(\frac{\pi}{2}s\right) = \frac{1}{1-x} + O(|1-x|)$ in the neightborhood of $x = 1$

iii) $\log(\tan\left(\frac{\pi}{2}s\right)) = -4\sum h_n\frac{\sin(2\pi ns)}{\pi n}$

iv) $log'\Gamma\left((1-\theta) = \frac{3}{4}\right) - log'\Gamma\left(\theta = \frac{1}{4}\right) = \pi$

v) $log'(\tan\left(\frac{\pi}{2}s\right) = -log'(\cot\left(\frac{\pi}{2}s\right) = \frac{\pi}{\sin(\pi s)} = \frac{1}{x}[+2\sum_{k=0}^{\infty}\sigma_{2k}\, x^{2k}] = \beta(x) + \beta(1-x)$

with $\beta(x) \coloneqq \sum_{k=0}^{\infty}(-1)^k\frac{1}{x+k} = \sum_{k=0}^{\infty}(-1^k)\sigma_{k+1}\, x^k$ (NiN) §14 (6), §19 (17).

Applying Riemann's building concept for the auxiliary function, defining a self-adjoint operator with Mellin tranform $\xi(s)$ ((EdH) 10.3), results into a replacement of

$$log\,(sin\,x) = logx - \frac{1}{6}x^2 - \frac{1}{180}x^4 - \frac{1}{2835}x^6 - \cdots \quad \rightarrow \quad \log\left(tan\left(\frac{\pi}{2}x\right)\right) = logx + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \cdots \ldots$$

The density of prime numbers appears to be the Gaussian density $dg = \log\left(\frac{1}{t}\right)dt$ defining the corresponding prime number counting integral function ((EdH) 1.1 (3)). The Clausen density dw, based on the Clausen integral ((AbM) 27.8)

$$w(t) = \int_0^t \log\left(2sin\frac{t}{2}\right)dt \;, \quad 0 \le t \le \pi$$

is related to the Hilbert transform of the fractional part function ((BrK4) Note O28).

For $T(x) \coloneqq \log\left(tan(\frac{\pi}{2}x)\right)$, we summaries a few properties

i) $\frac{\pi}{2}T(x) = -\sum \frac{2h_n}{n}\sin(2\pi nx) \in L_2^{\#}(0,1)$ (ElL)

with $2h_n = \sum_{k=1}^{n}\frac{2}{2k-1} = 2H_{2n} - H_n$ and $H_n = \sum_{k=1}^{n}\frac{1}{k}$ (harmonic numbers) and

$$\int_0^1 T(x)\,cos(k\pi x)\,dx = \begin{cases} -1/k & k\ odd \\ 0 & k\ even \end{cases}.$$

ii) the $log\,(tan\,x)$ −integral evaluated by series involving $\zeta(2n+1)$ is provided (ElL1)

iii) for the Hilbert transform evaluation of $T(x)$ we refer to (MaJ)

iv) from (GrI), 1.421,1.518, we recall the series representations

$$\mathrm{T(x)} = logx + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2^{2k-1}-1)2^{2k}B_{2k}x^{2k}}{k(2k)!} \quad , x^2 < (\tfrac{\pi}{2})^2$$

$$T(x) = logx + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \cdots \ldots.$$

$$log\,(sin\,x) = logx - \frac{1}{6}x^2 - \frac{1}{180}x^4 - \frac{1}{2835}x^6 - \cdots ..$$

v) For the related Fourier expansion of the $log\Gamma(x)$ function we refer to (EsO) with coefficients $a_n = \frac{1}{2n}$, $b_n = \frac{A+logn}{2cn}$ and $a_0 = log\sqrt{2\pi}$.

vi) The counterpart of the asymptotics $log\,(tan\frac{\pi}{2}x) \sim \log(\sin(x) \sim logx$ with respect to the cot −function is given by the estimate

$$\left|\frac{\pi}{2}\cot\left(\frac{\pi}{2}x\right) - \frac{1}{x}\right| \le \frac{\pi^2}{6} = \zeta(2)\ (|x| \le 1),$$

which is a result of the following inequalities

$$\left|\frac{1}{2}cot\left(\frac{x}{2}\right) - \frac{1}{x}\right| = \left|\sum_{k=1}^{\infty}\frac{1}{x+2\pi k} + \frac{1}{x-2\pi k}\right| \le \left|\sum_{-\infty}^{\infty}\frac{x}{\pi(x-2\pi k)}\right| \le \frac{\pi}{2\pi^2}\sum_{-\infty}^{\infty}\left|\frac{1}{1-\frac{x}{2k\pi}}\right|\frac{1}{k^2} \le \frac{\pi}{6} \quad , \ |x| \le \pi$$

With respect to the distributional Fourier series representation of the $\cot(\pi x)$ function we note the product representation ((GrI), 1.392)

$$sin\,(nx) = \begin{cases} n\cdot sin(x)\cdot\prod_{k=1}^{(n-1)/2}\left(1 - \dfrac{sin^2(x)}{sin^2(\pi\frac{k}{n})}\right) & n\ odd \\ \dfrac{n}{2}\cdot sin(2x)\cdot\prod_{k=1}^{(n-2)/2}\left(1 - \dfrac{sin^2(x)}{sin^2(\pi\frac{k}{n})}\right) & n\ even \end{cases}$$

$$-log\left|2\cos\left(\frac{x}{2}\right)\right| = -\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n} \qquad -log\left|2\sin\left(\frac{x}{2}\right)\right| = -\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$$

$$-log\left|2\sin\left(\frac{x}{2}\right)\right| - log\left|2\cos\left(\frac{x}{2}\right)\right| = -log|2\sin(x)| = -\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}.$$

In (ChK) VI, §2, two expansions of $\cot(z)$ are compared to prove that all coefficients of one of this expansion $\frac{\zeta(2n)}{\pi}e^{-2n}$ are rational. Corresponding formulas for odd inters are unknown.

In (EsR), 3.8 (example 78), resp. (BrK4) Notes S51, a "finite part"-"principle value" integral representation of the $\frac{\pi}{2}\cot\left(\frac{\pi}{2}x\right)$ – is given (which is zero also for positive or negative integers)

$$F.p.(P.v.\int_0^\infty t^x \frac{t}{t^2-1}dt) = \begin{cases} 0 & for \quad x\in 2Z \\ \left(\frac{\pi}{2}\right) cot\left(\frac{\pi}{2}x\right) & otherwise \end{cases}.$$

It is used as enabler to obtain the asymptotic expansion of the p.v. integral, defined by the "restricted" Hilbert transform integral of a function $u(x)$ over the positive x-axis, only. In case $u(x)$ has a structure $u(x) = v(x)\sqrt{x}$ the representation enjoys a remarkable form, where the numbers $n + 1/2$ play a key role.

In the context of Landau's „generalized number theoretical function theorem" we note the following properties: for

$$g(x) := \frac{1}{sin^2(\pi x)} \quad , \quad p(x) := (\tfrac{\pi}{2}x)cot(\tfrac{\pi}{2}x) = 1 + \sum_{n=1}^{\infty}\left[\frac{x}{x+n} + \frac{x}{x-n}\right] \quad , \quad h(x) := x\cdot p(x) \ ,$$

it holds

i) $\cot(x) = \frac{1}{2}cot\left(\frac{x}{2}\right) - \frac{1}{2}cot(\frac{\pi-x}{2})$, $-\frac{1}{2}x\frac{g'(x)}{g(x)} = P(x) = -\pi\, cot(\pi x) \underset{x\to 0}{\to} 1$

ii) $\sin\left(\pi n + \frac{1}{n}\right) = O(\frac{1}{n})$, $\sin\left(O(\frac{1}{x})\right) = O(\frac{1}{x})$ $\cos\left(O(\frac{1}{x})\right) = O(1)$ (OlF) 3.1

iii) $\int_0^1 h(x)dx = \frac{1}{2} - \sum_{k=1}^{\infty}\frac{\zeta(2k)}{2^{2k}(2k+1)}$, $\int_0^1 p(x)\,dx = \int_0^1 h(x)\frac{dx}{x} = \int_1^\infty h\left(\frac{1}{x}\right)\frac{dx}{x} = log2$ (GrI) 3.747

iv) $\int_0^1 h(x)\frac{dx}{x} + \int_1^\infty h\left(\frac{1}{x}\right)\frac{dx}{x} = 2log2$ and $h(x) \in L_1(0,1)$ (GrI) 3.748

resp. $\frac{1}{log2}\int_0^1 x(\frac{\pi}{2}x)cot(\frac{\pi}{2}x)\frac{dx}{x} = \frac{1}{log2}\int_1^\infty \frac{1}{x}(\frac{\pi}{2}\frac{1}{x})cot(\frac{\pi}{2}\frac{1}{x})\frac{dx}{x} = 1$, $\int_0^1 x\left[(\frac{\pi}{2}x)cot(\frac{\pi}{2}x)\right]dx = \frac{1}{2} - \sum_{k=1}^{\infty}\frac{\zeta(2k)}{2^{2k}(2k+1)}$.

From (ZyA) V.2, we recall for $0 < \beta < 1$ and $0 < x \le \pi$ the estimates

i) $\left|\sum_1^N \frac{\cos(nx)}{n^\beta}\right| \le c\frac{x^\beta}{x}$

ii) $\left|\sum_1^N \frac{\sin(nx)}{n^\beta}\right| \le c\frac{x^\beta}{x}$

iii) $\left|\sum_1^N \frac{\cos(nx)}{n}\right| \le \log\left(\frac{1}{x}\right) + C.$

The series $\sum_2^\infty \frac{\cos(nx)}{logn}$ is divergent, ist conjugate $\sum_2^\infty \frac{\sin(nx)}{logn}$ is not a Fourier series, and ((ZyA) V.1)

i) $\sum_2^\infty \frac{\cos(nx)}{logn} \sim \left[\frac{\pi}{2}log^{-1}(\frac{1}{x})\right]\frac{1}{x}log^{-1}(\frac{1}{x})$ $x \to +0$

ii) $\sum_2^\infty \frac{\sin(nx)}{logn} \sim \frac{1}{x}log^{-1}(\frac{1}{x})$ $x \to +0.$

Because of $\int_0^1 f(x)sin(\pi nx)dx \to 0$, $n \to \infty$ $\forall f \in L_2(0,1)$, the sequence $\{sin(\pi nx)\}$ converges weakly to zero, but not strongly, as $\|sin(\pi nx)\| = \frac{1}{\sqrt{2}}$. The same is true for

$$f(x) = \begin{cases} 1, & n < x < n+1 \\ 0, & otherwise \end{cases}$$

The specifically considered Kummer functions fulfill the following property

$$c_1 \cdot \left[{}_1F_1(a,c;x)\right]^2 \le {}_1F_1(a-\mu,c;x)\cdot {}_1F_1(a+\mu,c;x) \le \left[{}_1F_1(a,c;x)\right]^2 \ , \ a>0, \ c>a\ge\mu-1 \ , \ x\in R$$

From (BaR) we recall the inequality

$$ {}_1F_1(a-\mu,c;x)\cdot {}_1F_1(a+\mu,c;x) \le \left[{}_1F_1(a,c;x)\right]^2.$$

for $a>0$, $c>a\ge\mu-1$ and $x\in R$ or $a\ge\mu-1$, $c>-1$ $(c\neq 0)$, $x>0$, and positiv integer μ.

In (KaD) this result is extended to non-integer positive μ and complemented with a reverse inequality given asymptotically precise lower bound for the quantity

$$\frac{{}_1F_1(a-\mu,c;x)\cdot {}_1F_1(a+\mu,c;x)}{\left[{}_1F_1(a,c;x)\right]^2}.$$

From (PeO) pp. 312, 352, we recall the following Continued fractions (CF) representions:

$$log\,\frac{1+x}{1-x}\sim 2xF\left(\frac{1}{2},1,\frac{3}{2};x^2\right)=\frac{2x|}{|1}-\frac{1x^2|}{|3}-\frac{4x^2|}{|5}-\frac{9x^2|}{|7}-\frac{16x^2|}{|9}-\ldots.$$

$$actanx\sim\frac{x|}{|1}+\frac{1x^2|}{|3}+\frac{4x^2|}{|5}+\frac{9x^2|}{|7}+\frac{16x^2|}{|9}+\cdots$$

$${}_1F_1(1,c;x)\sim 1+\frac{x|}{|c}-\frac{cx|}{|c+1}+\frac{1x|}{|c+2}-\frac{(c+1)x|}{|c+3}+\frac{2x|}{|c+4}-\frac{(c+2)x|}{|c+5}+\frac{3x|}{|c+6}\ldots$$

$$\frac{1}{{}_1F_1(1,c;x)}\sim 1-\frac{\frac{1}{c}x|}{|1}+\frac{\frac{1x}{c(c+1)}\Big|}{|1}-\frac{\frac{cx}{(c+1)(c+2)}\Big|}{|1}+\frac{\frac{2x}{(c+2)(c+3)}\Big|}{|1}-\frac{\frac{(c+1)x}{(c+3)(c+4)}\Big|}{|1}+\frac{\frac{3x}{(c+4)(c+5)}\Big|}{|1}\ldots$$

$$\frac{{}_1F_1(a,c;x)}{{}_1F_1(a+1,c+1;x)}\sim 1-\frac{\frac{c-a}{c(c+1)}x|}{|1}+\frac{\frac{a+1}{(c+1)(c+2)}x|}{|1}-\frac{\frac{c-a+1}{(c+2)(c+3)}x|}{|1}+\frac{\frac{a+2}{(c+3)(c+4)}x|}{|1}\ldots.$$

with

$${}_1F_1(a+1,c+1;x)=\frac{c}{a}\frac{d}{dx}\left[{}_1F_1(a,c;x)\right].$$

and therefore

$$\frac{1}{{}_1F_1\left(1,\frac{3}{2};x\right)}\sim 1-\frac{\frac{2}{3}x|}{|1}+\frac{\frac{1x}{\frac{3}{2}\cdot\frac{5}{2}}\Big|}{|1}-\frac{\frac{\frac{3}{2}x}{\frac{5}{2}\cdot\frac{7}{2}}\Big|}{|1}+\frac{\frac{2x}{\frac{7}{2}\cdot\frac{9}{2}}\Big|}{|1}-\frac{\frac{\frac{5}{2}x}{\frac{9}{2}\cdot\frac{11}{2}}\Big|}{|1}+\frac{\frac{3x}{\frac{11}{2}\cdot\frac{13}{2}}\Big|}{|1}\ldots.$$

$$=1-\frac{\frac{2\cdot 2}{1\cdot 3}\cdot\frac{1}{2}x|}{|1}+\frac{\frac{2\cdot 2}{3\cdot 5}\cdot 1x|}{|1}-\frac{\frac{2\cdot 2}{5\cdot 7}\cdot\frac{3}{2}x|}{|1}+\frac{\frac{2\cdot 2}{7\cdot 9}\cdot 2x|}{|1}-\frac{\frac{2\cdot 2}{9\cdot 11}\cdot\frac{5}{2}x|}{|1}+\frac{\frac{2\cdot 2}{11\cdot 13}\cdot 3x|}{|1}\ldots.$$

$$\frac{2}{3}\frac{{}_1F_1\left(1,\frac{3}{2};x\right)}{\frac{d}{dx}\left[{}_1F_1\left(1,\frac{3}{2};x\right)\right]}\sim 1-\frac{\frac{\frac{1}{2}}{\frac{35}{22}}x|}{|1}+\frac{\frac{\frac{2}{57}}{22}x|}{|1}-\frac{\frac{\frac{3}{2}}{\frac{79}{22}}x|}{|1}+\frac{\frac{3}{79}}{22}x|}{|1}=1-\frac{\frac{2\cdot 2}{3\cdot 5}\cdot\frac{1}{2}x|}{|1}+\frac{\frac{2\cdot 2}{5\cdot 7}\cdot 2x|}{|1}-\frac{\frac{2\cdot 2}{5\cdot 7}\cdot\frac{3}{2}x|}{|1}+\frac{\frac{2\cdot 2}{7\cdot 9}\cdot 3x|}{|1}\ldots \quad .$$

From (ArE) we recall a few properties of log-convex functions:

Theorem 1.1:
The sum of convex functions is again convex. The limit function of a convex sequence of convex functions is convex. A convergent infinite series whose terms are all convex has a convex sum.

Theorem 1.3:
A function is convex, if, and only if, it is continuous and weakly convex.
Theorem 1.4: $f(x)$ is an convex function if, and only if, $f(x)$ has monotonically increasing one-side derivatives.

Theorem 1.5:
A function is convex, if, and only if, it is weakly convex.

Theorem 1.6:
A product of log-convex (weakly log-convex) functions is again log-convex (weakly log-convex). A convergent sequence of log-convex (weakly log-convex) functions has a log-convex (weakly log-convex) limit function, provided the limit is positive.

Theorem 1.7:
Suppose $f(x)$ is a twice differentialbe function. If the inequalities

$$f(x)f''(x) - |f'(x)|^2 \geq 0$$

hold, then $f(x)$ is log-convex.

Theorem 1.8:
Suppose $f(x)$ and $g(x)$ are functions, defined on a common interval. If both are weakly log-convex, then their sum $f(x) + g(x)$ is also weakly log-convex.

Theorem 1.9:
If $\varphi(x)$ is a positive continuous function defined on the interior of the integration interval, then

$$\int_a^b \varphi(t)t^{x-1}dt$$

is a log-convex function of x for every interval on which the proper or improper integral exists.

Theorem 1.10:
If $f(x)$ is log-convex on a a certain interval, and if c is any real number $\neq 0$, then both functions $f(x + c)$ and $f(cx)$ are log-convex on the corresponding intervals.

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