damped modes in the absence of growing ones, when the time-reversibility of the formal equations of motion might have led us naively to expect that eigenfrequencies must come in complex-conjugate pairs.

In its purest form, Landau damping represents a phase-space behavior peculiar to collisionless systems. Analogs to Landau damping exist, for example, in the interactions of stars in a galaxy at the Lindblad resonances of a spiral density wave. Such resonances in an inhomogeneous medium can produce wave absorption (in space rather than in time), which does not usually happen in fluid systems in the absence of dissipative forces (an exception is the behavior of corotation resonances for density waves in a gaseous medium). For our current phase-space problem, electrons in the distribution function moving at velocities v slightly faster than the wave "push" on it and thereby give energy to the wave, whereas electrons slightly slower the wave "drag"on it and thereby remove energy from the wave. Since longitudinal plasma oscillations (unlike spiral density waves) have only positive energies, the former tend to amplify the wave; the latter, to damp it. Because a Maxwellian has more slow particles than fast ones, $\Psi'(v_p)\,\mathrm{sgn}(v_p) < 0$; and longitudinal oscillations in a thermal plasma will always suffer a net amount of damping. Landau amplification can take place only for charged-particle distribution functions that display some sort of anomalous behavior in phase space. Strange distribution functions can result, for example, in the vicinity of collisionless shocks, and phase-space instabilities can be important in these and other plasma contexts.

30

Orbit Theory and Drift Currents

Reference: Chandrasekhar, *Plasma Physics*, pp. 14–36, 71–74.

Electric currents constitute the material sources for astronomical magnetic fields. In the collisionally dominated regime (MHD), we obtain the current density via Ohm's law,

$$\mathbf{j}_e = \sigma\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right), \tag{30.1}$$

where σ is the electrical conductivity,

$$\sigma = \frac{\omega_{pe}^2}{4\pi\nu_c}, \tag{30.2}$$

with ω_{pe} being the electron plasma frequency and ν_c being the collision frequency of electrons with ions (review Chapter 21). In the MHD approximation we assume that the mean difference in velocity between electrons and ions reaches a terminal value in which frictional forces due to collisions balance the macroscopic electromagnetic forces. Clearly, this approximation must break down in the collisionless regime—the usual assumption of plasma physics. In a collisionless plasma, the electromagnetic forces can be balanced only by inertial effects. This realization motivates us to study how free electric charges move in the presence of large-scale electromagnetic (and gravitational) fields.

SOME BASIC CONSIDERATIONS

A charge q of mass m spiraling nonrelativistically in a static uniform magnetic field $\mathbf{B}$ possesses a gyrofrequency given by Larmor's formula,

$$\omega_L = \frac{q\mathbf{B}}{[illegible]}, \tag{30.3}$$

like particles) is repulsive, but gravity is attractive. Thus, mutual Coulomb repulsion among electrons crowded together makes the compressed regions bounce back faster than velocity dispersion or pressure can accomplish alone.

Even more interesting are the implications of the imaginary part of the dispersion relationship, equation (29.24). For $k^2 \ll \omega_{\rm pe}^2/\langle v^2\rangle$, so that $\omega_{\rm R}^2 \approx \omega_{\rm pe}^2$ in equation (29.23), we may approximate equation (29.24) by

$$\omega_{\rm I} \approx -\frac{\pi}{2}\frac{\omega_{\rm pe}^3}{k^2}\Psi_0'(\omega_{\rm R}/k)\ \mathrm{sgn}(k). \tag{29.26}$$

Notice that we follow J. D. Jackson rather than L. D. Landau in not replacing $\omega_{\rm R}$ by $\omega_{\rm pe}$ in the argument of Ψ_0'; this is because we are working out in the tail of the distribution function, where small changes in the velocity can make substantial changes in the value of Ψ_0 and its derivatives.

For a one-dimensional Maxwellian distribution,

$$\Psi_0(v) = \left(2\pi\langle v^2\rangle\right)^{-1/2}\exp\left(-\frac{v^2}{2\langle v^2\rangle}\right), \tag{29.27}$$

and equation (29.26) becomes

$$\omega_{\rm I} \approx \left(\frac{\pi}{8}\right)^{1/2}\langle v^2\rangle^{-3/2}\frac{\omega_{\rm pe}^3\omega_{\rm R}}{|k|^3}\exp\left(-\frac{\omega_{\rm R}^2}{2k^2\langle v^2\rangle}\right). \tag{29.28}$$

Notice that $\omega_{\rm I} > 0$ independent of the sign of k; all waves are damped whether they travel to the right or to the left.

If we now substitute the expression (29.25) for $\omega_{\rm R}^2$, we obtain

$$\frac{\omega_{\rm I}}{\omega_{\rm R}} = \left(\frac{\pi}{8}\right)^{1/2}\left(\frac{k_{\rm D}}{|k|}\right)^3\exp\left[-\frac{3}{2}-\frac{1}{2}\left(\frac{k_{\rm D}}{|k|}\right)^2\right], \tag{29.29}$$

where $k_{\rm D}$ is the Debye wavenumber (see Chapter 1),

$$k_{\rm D} \equiv \frac{\omega_{\rm pe}}{\langle v^2\rangle^{1/2}}. \tag{29.30}$$

Equation (29.29) represents Jackson's correction (the term 3/2 in the exponent) to Landau's damping decrement for longitudinal plasma oscillations. Notice that the imaginary part of ω is (exponentially) small in comparison to the real part to the extent that $|k| \ll k_{\rm D}$. The damping decrement formally becomes of order unity for $|k| \sim k_{\rm D}$; thus only disturbances much longer than the Debye length do not suffer heavy Landau damping. At scales much larger than the Debye length, the plasma has a collective behavior; below the Debye length, collective oscillations are much less important than the graininess manifest in individual charges (see Chapter 1).

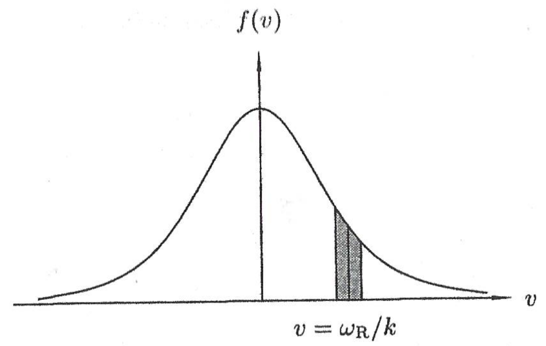

FIGURE 29.6
For a longitudinal plasma wave of infinitesimal amplitude, resonant interactions of electrons with the wave are possible only for electrons with velocities v (component along $\mathbf{k}$) within an infinitesimal range centered on $v = \omega_{\rm R}/k$. Electrons going slightly faster than the wave have an opposite effect to electrons going slightly slower than the wave; thus the Landau damping decrement depends only on the derivative of the electron distribution function evaluated at the exactly resonant value $v = \omega_{\rm R}/k$.

PHYSICAL INTERPRETATION FOR LANDAU DAMPING

Resonant wave-particle interaction underlies the physical mechanism behind Landau damping (see Figure 29.6 and the discussion cited in Stix's book at the beginning of the chapter). Electrons with random velocities v substantially different from the phase speed of the wave, $\omega_{\rm R}/k$, drift in and out of the crests and troughs of the wave. Sometimes they are accelerated by the collective electric field E_1, sometimes decelerated; but integrated over time, no net interaction results (in the linear approximation) because the time spent in crests and troughs averages out for a sinusoidal disturbance.

Consider, however, the behavior of particles with velocities v very nearly equal to the phase speed $v_{\rm p} \equiv \omega_{\rm R}/k$. Such particles maintain the same phase relationship to the wave for practically all time; they can "ride" the electrostatic field (as seen in their frame) as effectively as a surfer can ride a water wave. Such resonant particles have secular interactions with the wave that do not average out with time. This description lends insight into the necessity of referring to an *initial-value* treatment to sort out the ambiguities in the integration path. When we adopt an initial-value treatment (instead of a normal-mode or steady-state treatment), we are forced to specify an arrow of time—namely, that the secular buildup of resonant interactions began in the *past*. This specification makes possible assumed case. All longitudinal plasma oscillations must damp in time if $\Psi_0(v)$ is a monotonically decreasing function of $|v|$. The novel method by which oscillations in a collisionless plasma damp forms the subject of the next section.

LANDAU DAMPING IN THE LONG-WAVELENGTH LIMIT

We specialize our discussion of *Landau damping* to disturbances of long wavelength. For small k, we may expand

$$\frac{1}{(v-\omega/k)} = -\frac{k}{\omega}\left[1+\left(\frac{v}{\omega/k}\right)+\left(\frac{v}{\omega/k}\right)^2+\left(\frac{v}{\omega/k}\right)^3+\cdots\right]. \quad (29.19)$$

We assume again that $\Psi_0(v)$ is a monotonically decreasing function of $|v|$, disappearing at infinity faster than any power of v (e.g., a Maxwellian). We may then safely carry out the expansion (29.19) for the integrand and perform the integrals term-by-term:

$$\int_{-\infty}^{+\infty}\frac{\Psi_0'(v)\,dv}{(v-\omega/k)} = -\frac{k}{\omega}\int_{-\infty}^{+\infty}\left[1+\left(\frac{v}{\omega/k}\right)+\left(\frac{v}{\omega/k}\right)^2+\left(\frac{v}{\omega/k}\right)^3+\cdots\right]\Psi_0'(v)\,dv,$$

and we need not be careful whether we write a $\wp$ in front of the integral on the right-hand side. All the even powers in v vanish if $\Psi_0'(v)$ is odd in v; hence, if we factor out a factor of kv/ω from the remaining terms, we have

$$\int_{-\infty}^{+\infty}\frac{\Psi_0'(v)\,dv}{(v-\omega/k)} = -\frac{k^2}{\omega^2}\int_{-\infty}^{+\infty} v\left[1+\left(\frac{v}{\omega/k}\right)^2+\cdots\right]\Psi_0'(v)\,dv.$$

Integration by parts yields

$$\int_{-\infty}^{+\infty}\frac{\Psi_0'(v)\,dv}{(v-\omega/k)} = \frac{k^2}{\omega^2}\left[1+3\frac{\langle v^2\rangle}{\omega^2/k^2}+\cdots\right].$$

Collecting results, we may now express the dispersion relationship as

$$\epsilon = 1-\frac{\omega_{\rm pe}^2}{\omega^2}\left[1+3\frac{k^2\langle v\rangle^2}{\omega^2}+\cdots\right]+i\frac{\omega_{\rm pe}^2}{k^2}p = 0, \quad (29.20)$$

where p represents the pole contribution:

$$p = \pi\Psi_0'(\omega/k)\,\mathrm{sgn}(k) \qquad \text{for} \qquad \omega_{\rm I} = 0, \quad (29.21)$$

$$p = 2\pi\Psi_0'(\omega/k)\,\mathrm{sgn}(k) \qquad \text{for} \qquad \omega_{\rm I} > 0. \quad (29.22)$$

For small k, $\Psi_0'(\omega/k)$ is to be evaluated out in the tail of the distribution function; hence the self-consistent value for $\omega_{\rm I}$ will turn out to be exponentially small in comparison with $\omega_{\rm R}$. We therefore choose to solve for $\omega_{\rm R}$ and $\omega_{\rm I}$ by successive approximation. If we assume that $\omega_{\rm I} \ll \omega_{\rm R}$, the real and imaginary parts of equation (29.20) have the expressions

$$1-\frac{\omega_{\rm pe}^2}{\omega_{\rm R}^2}\left[1+3\frac{k^2\langle v^2\rangle}{\omega_{\rm R}^2}+\cdots\right] = 0, \quad (29.23)$$

$$2\frac{\omega_{\rm I}\omega_{\rm pe}^2}{\omega_{\rm R}^3} = -\pi\frac{\omega_{\rm pe}^2}{k^2}\Psi_0'(\omega_{\rm R}/k)\ \mathrm{sgn}(k), \quad (29.24)$$

where we have used the value (29.21) for p when we approximate ω/k by $\omega_{\rm R}/k$ in the pole contribution.

For small k, equation (29.23) can be solved by iteration,

$$\omega_{\rm R}^2 \approx \omega_{\rm pe}^2 + 3k^2\langle v^2\rangle, \quad (29.25)$$

which demonstrates that only waves with frequency higher than the plasma frequency can propagate (see Chapter 20 of Volume I for an analogous phenomenon involving *transverse* waves, i.e., electromagnetic radiation). The additional term $3k^2\langle v^2\rangle$ in equation (29.25) represents the effect of finite velocity dispersion, with the factor 3 first obtained correctly for a collisionless plasma by Bohm and Gross (see discussion below). In a fluid analogy the effect arises from finite "pressure," and we would substitute $\langle v^2\rangle = \mathcal{R}T$ for a Maxwellian distribution, where $\mathcal{R}$ equals Boltzmann's constant divided by the mean molecular mass and T equals the electron temperature.

Equation (29.25) therefore provides the answer to the question that motivated our introductory discussion. Notice, in particular, the similarities and differences with the dispersion relation for Jeans' problem of gravitational stability in an infinite medium (review Problem Set 3):

$$\omega^2 = -4\pi G\rho_0 + k^2 a_s^2.$$

The term $3\langle v^2\rangle$ replaces $a_s^2 = \gamma\mathcal{R}T$ because $c_v = \mathcal{R}/2$ and $c_P = c_v + \mathcal{R} = 3\mathcal{R}/2$ for the specific heats associated with the one-dimensional variation of a plasma in which the effects of compressional heating do not get shared by the other two directions through randomizing collisions. Thus, the square of the adiabatic sound speed for longitudinal plasma oscillations is larger than the square of the velocity dispersion in one dimension, $\langle v^2\rangle$, by the ratio $\gamma_{\rm eff} = c_P/c_v = 3$. The term $\omega_{\rm pe}^2 = +4\pi e^2 n_{\rm e0}/m_{\rm e}$ replaces $-4\pi G\rho_0$ because we deal here with electric forces rather than gravitational ones. The plus sign instead of a minus sign reminds us that electricity (between

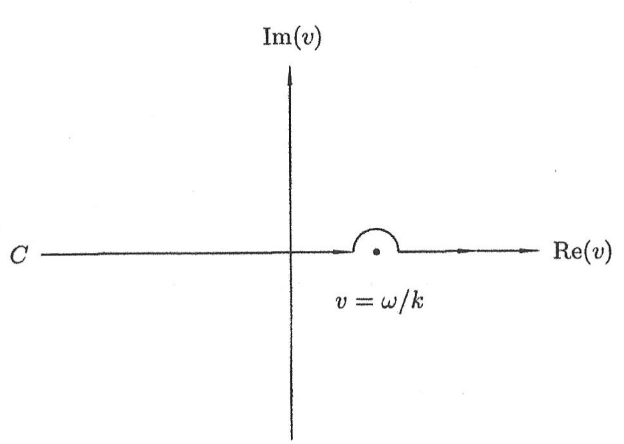

FIGURE 29.4
Analytic deformation of the path of integration for W in Figure 29.3 as the imaginary part of ω goes to zero.

As we allow ω_{I} to go to zero from the negative side, and the pole at $v = \omega/k$ approaches the real line (from below for k real and positive), however, we need to deform the contour to keep $W(\omega/k)$ analytic. In addition to the principal-value integral, this picks up half a residue contribution [negative or positive depending on whether $\mathrm{sgn}(k) = \pm 1$; see Figure 29.4].

$$W = \wp \int_{-\infty}^{+\infty} \frac{\Psi_0'(v)}{v - \omega/k}\, dv - \pi i \Psi_0'(\omega/k)\ \mathrm{sgn}(k) \qquad \text{for} \qquad \omega_{\mathrm{I}} = 0. \quad (29.17)$$

In equation (29.17), the symbol $\wp$ stands for "principal value." If ω_{I} were to continue and become positive (damped disturbance), then analytical continuation yields, in addition to the integral along the real line (which again presents no difficulty of interpretation), a full residue contribution (see Figure 29.5).

$$W = \int_{-\infty}^{+\infty} \frac{\Psi_0'(v)}{v - \omega/k}\, dv - 2\pi i \Psi_0'(\omega/k)\ \mathrm{sgn}(k) \qquad \text{for} \qquad \omega_{\mathrm{I}} > 0. \quad (29.18)$$

ABSENCE OF GROWING DISTURBANCES FOR MONOTONICALLY DECREASING DISTRIBUTION FUNCTIONS

We now wish to prove that there are no growing disturbances if $\Psi_0(v)$ is an even, positive definite, and monotonically decreasing function of v for

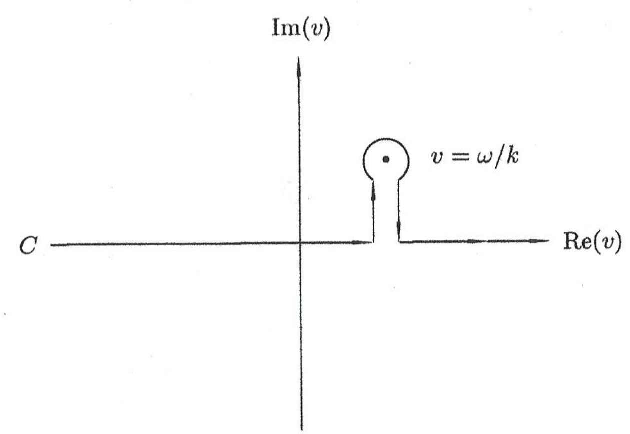

FIGURE 29.5
Continuation of the process began in Figure 29.4 as the imaginary part of ω takes on positive values (damped disturbances).

$v > 0$, as it is, for example, for a Maxwellian. For growing modes, we can multiply top and bottom by the complex conjugate of the denominator in the integrand and write equation (29.16) as

$$W = \int_{-\infty}^{+\infty} \frac{(v - \omega_{\mathrm{R}}/k)\Psi_0'(v)}{(v - \omega_{\mathrm{R}}/k)^2 + \omega_{\mathrm{I}}^2/k^2}\, dv + i\frac{\omega_{\mathrm{I}}}{k} \int_{-\infty}^{+\infty} \frac{\Psi_0'(v)\, dv}{(v - \omega_{\mathrm{R}}/k)^2 + \omega_{\mathrm{I}}^2/k^2}.$$

Since the dispersion relationship requires that a complex function $\epsilon = 1 - \omega_{\mathrm{pe}}^2 W/k^2$ vanishes, the imaginary part of W must equal 0. With $\Psi_0'(v)$ an odd function of v, we obtain

$$\frac{\omega_{\mathrm{I}}}{k}\left[\int_0^{\infty} \frac{\Psi_0'(v)\, dv}{(v - \omega_{\mathrm{R}}/k)^2 + \omega_{\mathrm{I}}^2/k^2} - \int_0^{\infty} \frac{\Psi_0'(v)\, dv}{(v + \omega_{\mathrm{R}}/k)^2 + \omega_{\mathrm{I}}^2/k^2}\right] = 0.$$

Collecting terms, we get

$$4\frac{\omega_{\mathrm{R}}\omega_{\mathrm{I}}}{k^2} \int_0^{\infty} \frac{v\Psi_0'(v)\, dv}{[(v - \omega_{\mathrm{R}}/k)^2 + \omega_{\mathrm{I}}^2/k^2][(v + \omega_{\mathrm{R}}/k)^2 + \omega_{\mathrm{I}}^2/k^2]} = 0.$$

The integral above is strictly negative if $\Psi'(v) \leq 0$ for $v \geq 0$. As a consequence, we require either ω_{I} or ω_{R} to equal 0. But $\omega_{\mathrm{I}} < 0$ by assumption that we have a growing disturbance, and $\omega_{\mathrm{R}} = 0$ would generally contradict the requirement of the real part of the dispersion relationship, $\epsilon = 0$. Therefore, we conclude that no growing modes exist in the plasma for the

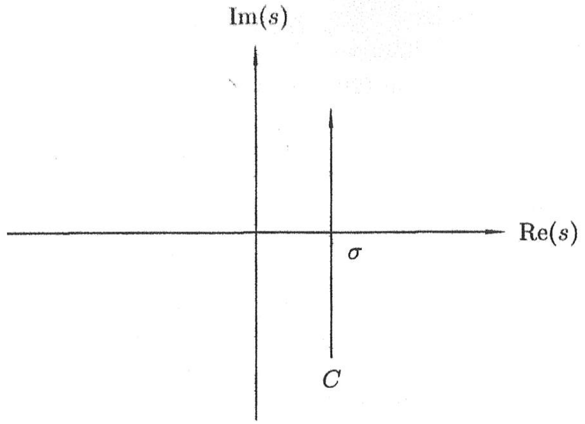

FIGURE 29.1
The starting contour in taking inverse Laplace transforms.

and the inverse transform recovers the original function,

$$q(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} Q(s)e^{st}\,ds.$$

The real value of σ in the contour integration over s must be chosen to be large and positive enough to lie to the right of all singularities of the function $Q(s)$ (see Figure 29.1). We can deform the path of integration by pushing it to the left, but analytical continuation requires that we keep all singularities to the left of the resulting contour (see Figure 29.2).

Now, let us apply the above lesson to the current problem. The combination $Q(s)e^{st}$ is analogous to the combination $W(\omega/k)e^{i\omega t}$ if we identify $s = i\omega$. The singularities of $Q(s)$ in s-space have an analogy in the pole of the integrand of W, which occurs in ω-space at $\omega = kv$, or in v-space at $v = \omega/k$. The requirement that $\sigma = \mathrm{Re}(s)$ start off positive and to the right of all singularities of $Q(s)$ then has its counterpart in the requirement that $\mathrm{Im}(\omega) \equiv \omega_{\mathrm{I}}$ start off negative, so that the pole $v = \omega/k$ lies *below* the real line (for k real and positive) in complex-v space. The integral (29.15) is easy to interpret in this case, since the complex pole does not lie along the path of integration, and we get analytic functions by simply performing the integration straightforwardly along the real-v axis (see Figure 29.3):

$$W = \int_{-\infty}^{+\infty} \frac{\Psi_0'(v)}{v-\omega/k}\,dv \qquad \text{for} \qquad \omega_{\mathrm{I}} < 0. \tag{29.16}$$

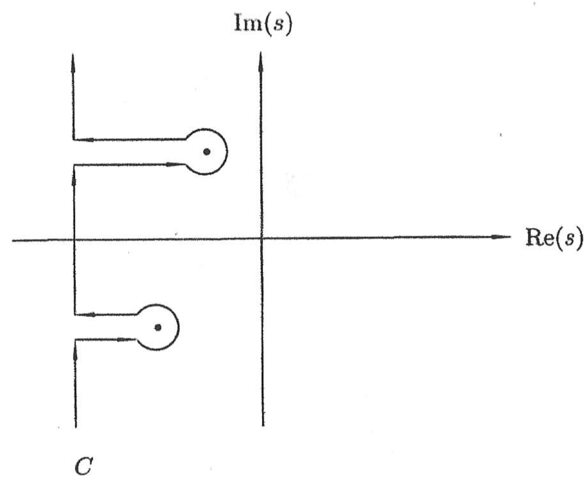

FIGURE 29.2
Analytic continuation of the inverse Laplace-transform integral if we choose to deform the contour in Figure 29.1 and the integrand contains isolated singularities (poles) denoted schematically by the dots.

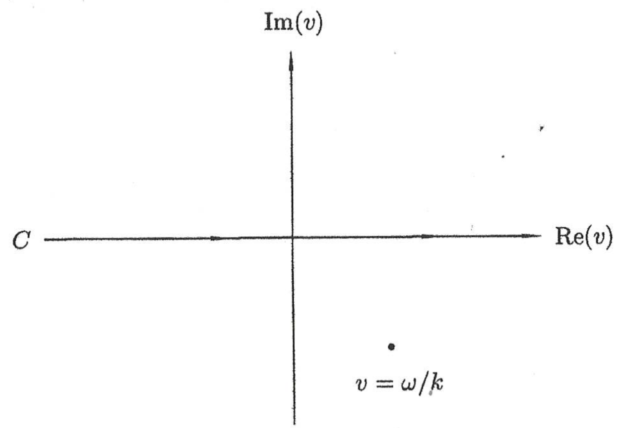

FIGURE 29.3
Interpretation of the path of integration for W in complex-v space when ω contains a negative imaginary part (growing disturbances).

$\mathbf{E}_1 = E_1\mathbf{e}_x$. Without loss of generality, we suppose that ω_R, the real part of ω, is positive, but we allow k (restricted to real values) to be either positive or negative. We now define the one-dimensional distribution functions:

$$F_0(v_x) \equiv \iint_{-\infty}^{+\infty} f_0(v_x, v_y, v_z)\, dv_y\, dv_z,$$

$$F_1(x, v_x, t) \equiv \iint_{-\infty}^{+\infty} f_1(x, v_x, v_y, v_z, t)\, dv_y\, dv_z.$$

Upon integration over v_y and v_z, equations (29.7) and (29.8) become (if we let v_x be denoted by v)

$$F_1(x, v, t) = -i\frac{e}{m_e}E_1(x,t)\frac{F_0'(v)}{(\omega - kv)}, \tag{29.9}$$

$$-ikE_1 + 4\pi e \int_{-\infty}^{+\infty} F_1\, dv = 0. \tag{29.10}$$

Substituting equation (29.9) into equation (29.10), we now obtain

$$-ik\epsilon E_1 = 0, \tag{29.11}$$

where ϵ is the plasma dielectric for longitudinal oscillations,

$$\epsilon \equiv 1 + \frac{4\pi e^2}{m_e k}\int_{-\infty}^{+\infty} \frac{F_0'(v)}{\omega - kv}\, dv. \tag{29.12}$$

We can interpret equation (29.11) by introducing the idea of a displacement vector $\mathbf{D}_1 \equiv \epsilon\mathbf{E}_1$ that accounts for the polarization of the plasma by the electric field $\mathbf{E}_1$, so that the macroscopic divergence equation reads

$$\nabla \cdot \mathbf{D}_1 = 0. \tag{29.13}$$

If we are to have nontrivial solutions for E_1 in equation (29.11), we require $\epsilon = 0$. Equation (29.12) with $\epsilon = 0$ represents the *dispersion relationship* between the wave frequency ω and the wavenumber k of longitudinal plasma oscillations. Notice that the functional form of this dispersion relationship depends on the derivative of the equilibrium velocity distribution $F_0'(v)$.

LANDAU'S ANALYSIS

Before we embark on a detailed analysis of the dispersion relation $\epsilon = 0$, we introduce some simplifying notation. We first define a normalized ...ilibrium distribution function $\Psi_0(v)$ by

so that

$$\int_{-\infty}^{+\infty} \Psi_0(v)\, dv = 1.$$

We can then rewrite equation (29.12) as

$$\epsilon = 1 - \frac{\omega_{\text{pe}}^2}{k^2}W(\omega/k) = 0, \tag{29.14}$$

where we have defined $W(\omega/k)$ as the integral function,

$$W(\omega/k) \equiv \int_{-\infty}^{+\infty} \frac{\Psi_0'(v)\, dv}{v - \omega/k}. \tag{29.15}$$

In Vlasov's analysis of equation (29.14), he naturally but erroneously assumed that longitudinal oscillations set up initially in a plasma with a nonpathological electron distribution function should be able to persist forever in the absence of dissipative collisions. In other words, it should be possible to consider real values for both ω and k. For ω/k real and finite (equal to the phase velocity of the longitudinal wave), however, the mathematical meaning of equation (29.15) presents a substantive difficulty. How should we interpret the integral for W if the pole at $v = \omega/k$ lies along the path of integration (along the real v axis)? In other words, what should we do about electrons that travel at a material speed exactly equal to the wave speed?

Vlasov proposed that one should interpret the integral as a principal-value integral, i.e., integrate from $-\infty$ to a little less than ω/k; jump over the singularity at $v = \omega/k$; and continue to integrate from a little greater than ω/k to $+\infty$. The procedure yields a finite value for the integral W since the change in signs of the two sides of the singularity cancel their individual, logarithmically divergent, contributions as $v \to \omega/k$. Nevertheless, in a classic paper, Landau showed that Vlasov's proposal overlooks an important physical phenomenon.

By directly analyzing the initial-value problem, Landau demonstrated that if modes with real ω exist, they should be obtained by first considering slightly growing modes, $\omega = \omega_{\text{R}} + i\omega_{\text{I}}$ with $\omega_{\text{I}} < 0$ (so that $e^{-\omega_{\text{I}}t}$ yields a growing exponential), and then taking the limit of vanishing growth rate,

$$\omega_{\text{I}} \to 0^-.$$

We now briefly describe why this procedure is the correct one mathematically.

When we formulate the problem as an initial-value problem rather than as a normal-mode problem, we take Laplace transforms (in time) rather than Fourier transforms. The Laplace coefficient $Q(s)$ associated with any quantity $q(t)$ is given by

$$Q(s) \equiv \int^{\infty} q(t)e^{-st}\, dt,$$

29

Longitudinal Plasma Oscillations and Landau Damping

Reference: Stix, *The Theory of Plasma Waves*, pp. 118–124, 131–148

In Chapter 28 we showed that the formal solution to Vlasov's kinetic equation reduces to finding the relevant integrals of particle motion if the force fields are regarded as known. Much of the complication of plasma physics stems, however, from the dominance of *internal* electromagnetic fields in affecting the motions of charged particles. These fields have a dynamics self-consistent with the dynamics of the particles. Finding the integrals of motion under these conditions may not be the most convenient way to solve the problem; a direct attack on the governing partial differential equations may offer a better approach.

In this chapter we discuss an example of such a fully dynamic phenomenon, the high-frequency oscillations that can take place if charge separation occurs with a periodic structure. In particular, we wish to reconsider the problem of longitudinal plasma oscillations when the electrons possess nonzero random motions. For a cold plasma in which we regard only the electrons to be mobile, spatial displacement of the electrons from the ions introduces restoring electric forces that set up oscillations at a frequency (see Chapter 1 and Volume I)

$$\omega_{pe} = (4\pi e^2 n_{e0}/m_e)^{1/2}. \tag{29.1}$$

How does a finite temperature for the electrons modify the above result?

PERTURBATIONAL VLASOV TREATMENT

To simplify the considerations, we continue to adopt the model of a two-component plasma in which the ions provide only a positively charged

background of uniform density. The electrons have a phase space description given by a distribution function $f(\mathbf{x}, \mathbf{v}, t)$. In the absence of magnetic fields or collisions, f satisfies Vlasov's kinetic equation,

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m_e}\mathbf{E}\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{29.2}$$

where the self-consistent electric field arises from the charge distribution of electrons and ions:

$$\nabla\cdot\mathbf{E} = 4\pi e(Zn_i - n_e) \qquad \text{with} \qquad n_e = \int f\, d^3v. \tag{29.3}$$

We assume the equilibrium state to be uniform and electrically neutral:

$$f_0 = f_0(\mathbf{v}), \qquad Zn_i = n_{e0} = \int f_0\, d^3v. \tag{29.4}$$

With the ions immobile, the linearized equations governing small-amplitude perturbations read

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}} = \frac{e}{m_e}\mathbf{E}_1\cdot\frac{\partial f_0}{\partial \mathbf{v}}, \tag{29.5}$$

$$\nabla\cdot\mathbf{E}_1 = -4\pi e\int f_1\, d^3v. \tag{29.6}$$

We look for solutions with temporal and spatial variations given by

$$\exp[i(\omega t - \mathbf{k}\cdot\mathbf{x})].$$

Equations (29.5) and (29.6) then become

$$f_1 = -i\frac{e}{m_e}\frac{\mathbf{E}_1\cdot\partial f_0/\partial\mathbf{v}}{(\omega - \mathbf{k}\cdot\mathbf{v})}, \tag{29.7}$$

$$-i\mathbf{k}\cdot\mathbf{E}_1 = -4\pi e\int f_1\, d^3v. \tag{29.8}$$

Faraday's law of induction,

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},$$

with $\mathbf{B} = 0$ if we consider disturbances that generate no magnetic fields, requires

$$\mathbf{k}\times\mathbf{E}_1 = 0.$$

This result implies that the oscillations are purely *longitudinal*; i.e., $\mathbf{E}_1$ is parallel to $\mathbf{k}$. Set up a rectangular coordinate system such that $\mathbf{k} = k\mathbf{e}_x$,

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