

The three Millennium problem solutions (RH, NSE, YME) and a Hilbert scale based quantum geometrodynamics

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This homepage addresses the problem & solution areas

- A. The Riemann Hypothesis (RH)
- B. The 3D-Navier-Stokes equations (NSE) navier-stokes-equations.com
- C. The Yang-Mills equations (YME) quantum-gravitation.de
- D. Plasma dynamics
- E. Geometrodynamics

building on a common Hilbert scale framework.

The proposed framework provides an answer to Derbyshire's question, ("Prime Obsession")

... "The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?"

The story line of this page is structured as follows:

1. A Kummer function based alternative Zeta function theory
2. The $H_{-1/2}$ Hilbert space and corresponding arithmetic functions
3. An alternative approach to prove the binary Goldbach conjecture
4. The $H_{-\frac{1}{2}} = H_0 \otimes H_0^\perp$ decomposition for a quantum space-time model
 - a. The Berry-Keating conjecture and the $H_{-1/2}$ quantum state space
 - b. The $H_{-1/2}$ Hilbert space and a new ground state energy model
 - c. The $H_{-1/2}$ space and a consistent plasma & geometrodynamics enabling space-time background independency
 - d. The $H_{-1/2}$ Hilbert space replacing the Dirac distributions space
5. NSE, YME and plasma/geometrodynamics problem/solution areas
 - a. The related NSE problem/solution area (Serrin gap)
 - b. The related YME problem/solution area (mass gap)
 - c. The related plasma/geometrodynamics problem/solution areas (space-time stage background dependency)
6. References

1. A Kummer function based alternative Zeta function theory

In order to prove the Riemann Hypothesis (RH) the Polya criterion can not be applied in combination with the Müntz formula ((TiE) 2.11). This is due to the divergence of the Müntz formula in the critical stripe due to the asymptotics behavior of the baseline function, which is the Gaussian function. The conceptual challenge (not only in this specific case) is about the not vanishing constant Fourier term of the Gaussian function and its related impact with respect to the Poisson summation formula. The latter formula applied to the Gaussian function leads to the Riemann duality equation ((EdH) 1.7). A proposed alternative "baseline" function than the Gaussian function, which is its related Hilbert transform, the Dawson function, addresses this issue in an alternative way as Riemann did. As the Hilbert transform is a convolution integral in a correspondingly defined distributional Hilbert space frame it enables the Hilbert-Polya conjecture (e.g. (CaD)). The corresponding distributional ("periodical") Hilbert space framework, where the Gaussian / Dawson functions are replaced by the fractional part / $\log(2\sin x)$ – functions enables the Bagchi reformulation of the Nyman-Beurling RH criterion.

The corresponding formulas, when replacing the Gaussian function by its Hilbert transform, are well known: the Hilbert transform of the Gaussian is given by the Dawson integral (GaW). Its properties are e.g. provided in ((AbM) chapter 7, (BrK4) lemma D1). The Dawson function is related to a special Kummer function in a similar form than the (error function) $\operatorname{erf}(x)$ –function resp. the $\operatorname{li}(x)$ –function ((AbM) (9.13.1), (9.13.3), (9.13.7), (LeN) 9.8, 9.13). A characterization of the Dawson function as an sin-integral (over the positive x-axis) of the Gaussian function is given in ((GrI) 3.896 3.). Its Mellin transform is provided in ((GrI) 7.612, (BrK4) lemma S2). The asymptotics of the zeros of those degenerated hypergeometric functions are given in (SeA) resp. ((BrK4) lemma A4). The fractional part function related Zeta function theory is provided in ((TiE) II).

The Dawson function $F(x)$ (i.e. the Hilbert transform of the Gaussian function $f(x) := e^{-x^2}$) is related to the two Kummer functions ${}_1K_1\left(\frac{1}{2}, \frac{3}{2}; x\right)$ and ${}_1K_1\left(1, \frac{3}{2}; x\right)$ by the identities ((GrI), 9.212, (LeN) (9.13.3))

$$F(x) := x \cdot {}_1K_1\left(1, \frac{3}{2}; -x^2\right) = x \cdot e^{-x^2} \cdot {}_1K_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right)$$

whereby ([AbM] p.507, [OIF] p. 44/67)

$${}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1 n!} \quad \text{and} \quad \frac{d}{dx} F(x) + 2xF(x) = 1$$

A kind of „reverse“ identity is related to the non-normalized (exponential) error function ([AbM] 13.6)

$$\operatorname{erf}(x) := \int_0^x e^{-t^2} dt = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k! 2k+1}$$

in the form ([AbM] 7.15)

$$\operatorname{erf}(x) = x {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right) = x e^{-x^2} {}_1F_1\left(1, \frac{3}{2}; x^2\right)$$

The relationship to the Whittaker functions is given by

$$M_{\frac{1}{4}, \frac{1}{4}}(x) = x^{\frac{3}{4}} e^{-\frac{x}{2}} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; x\right), \quad M_{-\frac{1}{4}, -\frac{1}{4}}(x) = x^{\frac{3}{4}} e^{-\frac{x}{2}} {}_1F_1\left(1; \frac{3}{2}; x\right)$$

From ((BrK4), lemma K2) we recall the fomula

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right) + 2x \frac{d}{dx} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right) = e^x$$

whereby

$$\frac{d}{dx} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right) = \frac{1}{3} \cdot {}_1F_1\left(\frac{3}{2}, \frac{5}{2}; x\right).$$

The Kummer function related Mellin transforms can be derived from the following formula ((GrI) 7.612):

$$\int_0^\infty x^s {}_1F_1(a; c, -x) \frac{dx}{x} = \frac{\Gamma(s)\Gamma(a-s)\Gamma(c)}{\Gamma(c-s)\Gamma(a)} \quad , \quad 0 < \operatorname{Re}(b) < \operatorname{Re}(a)$$

from which one also gets in the critical stripe

$$\int_0^\infty x^{s/2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -x\right) \frac{dx}{x} = \frac{\Gamma(s/2)}{1-s} .$$

Both Kummer functions also enjoy appreciated non-asymptotics of its zeros (SeA): the imaginary part of the zeros of both functions fulfill the inequality $(2n-1) \cdot \pi < |\operatorname{Im}(z_n)| < 2n\pi$, while the real parts fulfill $\operatorname{Re}(z) < -1/2$ resp. $\operatorname{Re}(z) > 1/2$ for ${}_1F_1\left(1, \frac{3}{2}; z\right)$ resp. ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; z\right)$. In other words, there are no zeros of ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; z\right)$ on the critical line $s=1/2+it$, resp. there are no zeros of ${}_1F_1\left(1, \frac{3}{2}; z\right)$ on the "dual" line $(1-s)=1/2-it$ (see also (BrK4) Notes O5, O22, O23, (BrK7) Note 11).

In (BuH) p.184) the product representation of the Kummer functions are provided:

$${}_1F_1(a, c, z) = \frac{1}{\Gamma(c)} e^{\frac{z}{c}} \prod \left(1 - \frac{z}{\alpha_n}\right) e^{z/\alpha_n}$$

where α_n denotes the infinite set of zeros of ${}_1F_1(a, c, z)$.

The asymptotics of the special Kummer functions are given by ((OIF), 7 §10.1, (AbM) 13.5.1.)

$${}_1F_1(a, c; x) \sim \frac{1}{\Gamma(a)} \frac{e^x}{x^{c-a}} \quad \text{resp.} \quad {}_1F_1(a, a+1; x) \sim \frac{1}{\Gamma(a)} \frac{e^x}{x} \quad , \quad x \rightarrow \infty .$$

Those asymptotics provide an option to replace the auxiliary functions ((EdH) 1.14, 1.16)

$$G(\beta) := \int_{c^+} \frac{t^{\beta-1}}{\log t} dt \quad \text{resp.} \quad E(\beta) := - \int_x^\infty \frac{t^{\beta-1}}{\log t} dt$$

to derive Riemann's formula for $J(x)$ resp. $\pi(x)$ by those special Kummer functions.

Let H and M denote the Hilbert and the Mellin transform operators. For the Gaussian function $f(x)$ it holds

$$M[f](s) = \frac{1}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) , \quad M[-xf'(x)](s) = \frac{s}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) = \frac{1}{2} \pi^{-s/2} \Pi\left(\frac{s}{2}\right) .$$

The corresponding entire Zeta function is given by ([EdH] 1.8)

$$\xi(s) := \frac{s}{2} \Gamma\left(\frac{s}{2}\right) (s-1) \pi^{-s/2} \zeta(s) = (1-s) \cdot \zeta(s) M[-xf'(x)](s) = \xi(1-s).$$

The central idea is to replace

$$M[-xf'(x)](s) \rightarrow M[f_H(x)](s)$$

with $f_H(x) := H[f](x)$, $\hat{f}_H(0) = 0$ and

$$M[f_H(x)](s) = 2\pi \cdot M\left[x \cdot {}_1F_1\left(1, \frac{3}{2}, -\pi x^2\right)\right](s) = \frac{1}{2} \pi^{\frac{1-s}{2}} \Gamma\left(\frac{s}{2}\right) \tan\left(\frac{\pi}{2}s\right) .$$

This enables the definition of an alternative *entire* Zeta function in the form

$$\xi^*(s) := \frac{1}{2}(1-s)\pi^{\frac{1-s}{2}}\Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right) \cdot \zeta(s)$$

with same zeros as $\xi(s)$, fulfilling the identity

$$\xi^*(s)\xi^*(1-s) = \pi \frac{\xi(s)\xi(1-s)}{s(1-s)}.$$

The product representation of the alternatively proposed modified „Gamma“ term

$$\Gamma\left(1 + \frac{s}{2}\right) \rightarrow \Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right) = \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1 - \frac{s}{2}\right)}$$

can be built on the product representations of the Gamma function in the form ((EdH) 1.3)

$$\Gamma(1+s) = \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^s}{(1+\frac{s}{n})} \quad \Gamma(1-s) = \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^{-s}}{(1-\frac{s}{n})}.$$

resulting into an alternative Riemann approximation error function enjoying improved convergence behavior.

One proof of the Riemann functional equation is based on the fractional part function $\rho(x)$, whereby the zeta function $zeta(s) := \zeta(s)$ in the critical stripe is given by the Mellin transform $zeta(1-s) = M\left[-x \cdot \frac{d}{dx}\rho(x)\right](s-1)$, ((TiE) (2.1.5)). The functional equation is given by $\zeta(s) = \chi(s)\zeta(1-s)$, whereby $\chi(s)$ is defined according to ((TiE) (2.1.12)). The Hilbert transform of the fractional part function is given by the $\log(\sin x)$ –function. After some calculations (see also (BrK4) lemma 1.4, lemma 3.1 (GrI) 1.441, 3.761 4./9., 8.334, 8.335) the corresponding alternative $zeta^*(s) := \zeta^*(s)$ function is given by

$$zeta^*(1-s) = \zeta^*(1-s) = \frac{\tan(\frac{\pi}{2}s)}{s} \cdot \zeta(1-s) \quad \text{resp.} \quad zeta^*(s) = \zeta^*(s) = \frac{\cot(\frac{\pi}{2}s)}{1-s} \cdot \zeta(s)$$

We further note the identity ($s = \frac{1}{2} + it$)

$$\frac{\pi}{2} \log'(\tan(\frac{\pi}{2}s)) = \frac{\pi}{\sin(\pi s)} = \Gamma(\frac{1}{2} + it)\Gamma(\frac{1}{2} - it)$$

With respect to the considered distributional Hilbert spaces $H_{-1/2}$ and H_{-1} we note that the Zeta function is an integral function of order 1 and an element of the distributional Hilbert space H_{-1} . This property is an outcome of the relationship between the Hilbert spaces above, the Dirichlet series theory (HaG) and the Hardy space isometry as provided in e.g. ((LaE), §227, Satz 40). With respect to the physical aspects below we refer to (NaS), where the $H_{-1/2}$ dual space of $H_{-1/2}$ on the circle (with its inner product defined by a Stieltjes integral) is considered in the context of Teichmüller theory and the universal period mapping via quantum calculus. For the corresponding Fourier series analysis we refer to ((ZyA) XIII, 11). The approximation by polynomials in a complex domain leads to several notions and theorems of convergence related to Newton-Gaussian and cardinal series. The latter one are closely connected with certain aspects of the theory of Fourier series and integrals. Under sufficiently strong conditions the cardinal function can be resolved by Fourier's integral. Those conditions can be considerably relaxed by introducing Stieltjes integrals resulting in (C,1) summable series ((WhJ1) theorems 16 & 17, (BrK4) remarks 3.6/3.7).

The RH is connected to the quantum theory via the Hilbert-Polya conjecture resp. the Berry-Keating conjecture. It is about the hypothesis, that the imaginary parts t of the zeros $1/2 + it$ of the Zeta function $Z(t)$ corresponds to eigenvalues of an unbounded self-adjoint operator, which is an appropriate Hermitian operator basically defined by $QP +$

PQ , whereby Q denotes the location, and P denotes the (Schrödinger) momentum operator. In (BrK3) the corresponding model (convolution integral) operator $S(1)$ (of order 1 with "density" $dcot$ for the one-dimensional harmonic quantum oscillator model is provided. Riemann's "workaround" function $h(x) := -x \frac{d}{dx} f(x)$ do have an obvious linkage to the "commutator" concept in quantum theory.

In this context the Gaussian function $f(x)$ can be characterized as "minimal function" for the Heissenberg uncertainty inequality. Applying the same solution concept as above then leads to an alternative Hilbert operator based representation in $H_{-1/2}$, resp. to a H_{-1} based definition of the commutator operator with extended domain. The common denominator of the alternatively proposed Hilbert space framework $H_{-1/2}$ goes along with the definition of a correspondingly defined "momentum" operator (of order 1) $P: H_{1/2} \rightarrow H_{-1/2}$ defined in a variational form. In the one-dimensional case the Hilbert transform H (in the $n > 1$ case the Riesz operators R) is linked to such an operator given by $(Pu, v)_{-1/2} = (Hu, v)$. With respect to quantum theory this indicates an alternative Schrödinger momentum operator (where the gradient operator "grad" is basically replaced by the Hilbert transformed gradient, i.e. $\mathbf{P} := -i * \mathbf{R}(\text{grad})$) and a corresponding alternative commutator representation $QP - PQ$ in a weak $H_{-1/2}$ form.

We note that the Riesz operators \mathbf{R} commute with translations and homothesis and enjoy nice properties relative to rotations.

The theory of spectral expansions of non-bounded self-adjoint operator is connected with the notions "Lebesgue-Stieltjes integral" and "functional Hilbert equation for resolvents ((LuL) (7.8)). The corresponding Hilbert scale framework plays also a key role on the inverse problem for the double layer potential. The corresponding model problem (w/o any compact disturbance operator) with the Newton kernel enjoys a double layer potential integral operator with the eigenvalue $1/2$ (EbP).

The RH is equivalent to the Li criterion governing a sequence of real constants, that are certain logarithmic derivatives of $Z(s)$ evaluated at unity (LiX). This equivalence results from a necessary and sufficient condition that the logarithmic of the function $Z\left(\frac{1}{1-s}\right)$ be analytic in the unit disk. The proof of the Li criterion is built on the two representations of the Zeta function, its (product) representation over all its nontrivial zeros ((HdE) 1.10) and Riemann's integral representation derived from the Riemann duality equation, based on the Jacobi theta function ((EdH) 1.8). Based on Riemann's integral representation involving Jacobi's theta function and its derivatives in (BiP) some particular probability laws governing sums of independent exponential variables are considered. In (KeJ) corresponding Li/Keiper constants are considered. The proposed alternative entire Zeta function $Z^*(s)$ is suggested to derive an analogue Li criterion.

The density function $J(x)$ of the Fourier inverse integral representation of $\log \zeta(s)$ can be reformulated into a representation of the function $\pi(x)$ (that is, for the "number of primes counting function" less than any given magnitude x ((EdH) 1.17)). Riemann's proof of the formula for $J(x)$ results into the famous Riemann approximation error function ((HdE) 1.17 (3)) based on the product formula representation of the Gamma function $\Gamma(1 + s/2)$ ((HdE) 1.3 (4), (GrI) 8.322). The challenge to prove the corresponding $li(x)$ function approximation criterion (BrK4) p.10

$$|li(x) - \pi(x)| = O(\sqrt{x} \log x) = O\left(x^{\frac{1}{2} + \varepsilon}\right); \varepsilon > 0$$

is about the (exponential) asymptotics of the Gaussian function ((EdH) 1.16, (BrK4) note S25). In this context we note that the Dawson function enjoys an only polynomial asymptotics in the form $O(x^{-1})$.

Alternatively to the Gaussian density $dg = \log\left(\frac{1}{t}\right)dt$ the above indicates to consider the Clausen density dw , where $w(t) = \int_0^t \log\left(2\sin\frac{t}{2}\right)dt$ ($0 \leq t \leq \pi$) denotes the periodical continuation of the Clausen integral ((AbM) 27.8). Obviously the Clausen integral is related to the Hilbert transform of the fractional part function.

The density of prime numbers appears to be the Gaussian density $dg = \log\left(\frac{1}{t}\right)dt$ defining the corresponding prime number counting integral function ((EdH) 1.1 (3)).

We mention the Kummer function based representation of the li-function in the form $li(x) = x {}_1F_1(1,1; -\log x)$ ((LeN) (9.13.7)) with the asymptotics

$$li(x) = -x {}_1F_1(1; 1, -\log x) = Ei(\log x) \approx \frac{1}{\log x}.$$

By substitution of the integration variable in the li-function integral ($t \rightarrow e^y$) this results into an alternative prime number approximation function in the form

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; \log x\right) = x - \log x \frac{2}{3} {}_1F_1\left(\frac{3}{2}, \frac{5}{2}; \log x\right).$$

For the related equations with respect to the incomplete Gamma function we refer to (OIF1) 7.2.2, 8.4.15, (AbM) 6.5.12, 13.6.10). The incomplete Gamma function play a key role to compute the action of the Leray projection operator on the Gaussian functions (LeN1). Those action formulas can be applied to derive in the context of the well-posedness topic of the NSE and related (based on tempered distribution and a Carleson measure characterization of the BMO space) estimates ((LeN1), (KoH), theorems 1 and 2, see also (BrK4) pp. 26, 58, 64, 99, 121).

We further note, that the generalized asymptotic (Poincaré) expansion admits expansions that have no conceivable value, in an analytical or numerical sense, concerning the functions they represent. In (OIF) §10, the expansion of $\frac{\sin x}{x}$ is provided with first summand term $\frac{e^{-x}}{\log x}$.

Corresponding formulas for the $\tan x$ resp. the $\log(\tan x)$ –functions are provided in ((GrI) 1.421,1.518).

In (EIL) the Fourier expansion of the $\log(\tan)$ function is provided, giving a note to its related Hilbert space H_α regularity. In (EIL1) $\log(\tan x)$ –integrals are evaluated by series involving $\zeta(2n+1) := \zeta(2n+1)$. Its graph looks like a beautiful white noise diagram. In (EsO), formulas (6.3), (6.4), the Fourier expansion of the $\log\Gamma(x)$ function is provided with coefficients $a_n = \frac{1}{2n}$, $b_n = \frac{A+\log n}{2cn}$ and $a_0 = \log\sqrt{2\pi}$. For a corresponding Hilbert transform evaluation we refer to (MaJ).

In (TiE) theorem 4.11, an approximation to the ζ function series in the critical stripe by a partial sum of its Dirichlet series is given ((BrK4) remark 3.8). One proof of this theorem is built on a simple application of the theorem of residues, where the ζ series is expressed as a (Mellin transform type) contour integral of the $\cot\left(\frac{\pi}{2}x\right)$ – function ((TiE) 4.14). As the \cot and the ζ function are both elements of the distributional Hilbert space H_{-1} the contour integral above with a properly chosen contour provides a contour integral representation for the ζ in a weak H_{-1} sense. In (ChK) VI, §2, two expansions of $\cot(z)$ are compared to prove that all coefficients of one of this expansion $\frac{\zeta(2n)}{\pi}e^{-2n}$ are rational. Corresponding formulas for odd inters are unknown. In (EsR), example 78, a "finite part"- "principle value" integral representation of the $\frac{\pi}{2}\cot\left(\frac{\pi}{2}x\right)$ – is given (which is zero also for positive or negative integers). It is used as enabler to obtain the asymptotic

expansion of the p.v. integral, defined by the "restricted" Hilbert transform integral of a function $u(x)$ over the positive x -axis, only. In case $u(x)$ has a structure $u(x) = v(x)\sqrt{x}$ the representation enjoys a remarkable form, where the numbers $n+1/2$ play a key role. In (OIF1) 25.6.6, an integral value representation for $\zeta(2n+1)$ is provided with $\cot(\pi x)$ "density" function.

In ((BrK4) lemma 3.4, lemma A12/19) the function $P(x)$ is considered in the context of (appreciated) quasi-asymptotics of (corresponding) distributions ((ViV) p. 56/57) and the Riemann mapping theorem resp. the Schwarz lemma. The considered "function" $g(x) := -\frac{d}{dx} \cot(\frac{\pi}{2}x)$ (whereby the \cot -function is an element of H_{-1}) is auto-model (or regular varying) of order -1 . This condition and its corresponding asymptotics property ((BrK) lemma 3.4) provide the prerequisites of the RH Polya criterion ((PoG), (BrK5) theorem 6). The above quasi-asymptotics indicates a replacement of the differential $d(\log x)$ by $d(\log(\sin x))$. The $\cot(z)$ function expansions (ChK) VI, §2) in combination with Ramanujan's formula ((EdH) 10.10) resp. its generalization theorem ((EdH) p.220) is proposed to be applied to define an appropriate analytical (Mellin transform) function in the stripe $1/2 < \text{Re}(s) < 1$.

In (GrI) 8.334, the relationship between the \cot - and the Gamma function is provided. From (BeB) 8. Entry17(iii)) we quote: "*the indefinite Fourier series of the $\cot(\frac{\pi}{2}x)$ -function may be formally established by differentiating the corresponding Fourier series equation for (the $L_2 = H_0$ -function) $-\log(2\sin(\frac{\pi}{2}x))$ " ((BrK4) remark 3.8). The proposed distributional Hilbert scales provide the proper framework to justify Ramanujan's related parenthetical remark "*for the same limit*" (in a H_{-1} -sense).*

2. The $H_{-1/2}$ Hilbert space and corresponding arithmetic functions

In (ViJ) the Prime Number Theorem (PNT) is proven on a distributional way, applying the Dirac function to derive the first derivative of the Chebyshev function. In line with the proposal of section 4.d) below we propose to replace the Dirac distributions space by the $H_{-1/2}$ Hilbert space, resulting in corresponding representation of concerned arithmetic functions:

From the PNT the Euler conjecture, i.e. the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \dots = 0$$

can be derived ((LaE) §159). This series is linked to the zeta function by the identities

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad , \quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{n^s}$$

i.e.

$$-\frac{\zeta'(s)}{s\zeta(s)} = \left[\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right] \cdot \left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{n^s} \right] \quad , \quad \frac{\zeta'(s)}{\zeta(s)} = \left[\sum_{n=1}^{\infty} \mu(n) \frac{1}{n^s} \right] \cdot \left[\sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right) \frac{1}{n^s} \right].$$

What cannot be derived from the PNT is the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\left(\frac{1}{n}\right) = 1 \quad .$$

„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived“ ((LaE) §160). This statement corresponds to the proposed replacement of the Dirac distribution by appropriate distributions of the Hilbert space $H_{-\frac{1}{2}}$, going along with the following identities ((ApT), (BrK4), (LaE) §227), while at the same time enabling the Bagchi RH criterion,

$$1 = \sum_{n=1}^{\infty} \frac{1}{n} \mu(n) \log\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} a_n b_n =: ((u, v))_{-\frac{1}{2}} := \lim_{\omega \rightarrow \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} u\left(\frac{1}{2} + it\right) v\left(\frac{1}{2} - it\right) dt$$

with $(s = \frac{1}{2} + it)$

$$u\left(\frac{1}{2} + it\right) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \in H_{-\frac{1}{2}} \quad , \quad v\left(\frac{1}{2} - it\right) := \sum_{n=1}^{\infty} \frac{\log(1/n)}{n^s} \in H_{-\frac{1}{2}}.$$

Let

$$z\left(\frac{1}{2} + it\right) := \sum_{n=1}^{\infty} \frac{1}{n^s} \in H_{-1} \quad , \quad w\left(\frac{1}{2} - it\right) := \sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right) \frac{\mu(n)}{n^s}$$

$$r\left(\frac{1}{2} - it\right) := \sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \frac{1}{n^s} \quad , \quad s\left(\frac{1}{2} - it\right) := \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n \log n} \frac{1}{n^s}.$$

Then it holds $((z, w))_{-\frac{1}{2}} = 1$. Because of $\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} < \infty$ and $z \in H_{-1}$, it follows that

$$((z, r))_{-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} < \infty \quad , \quad ((w, s))_{-\frac{1}{2}} \sim \sum_{n=1}^{\infty} \frac{\Lambda(n) \mu(n)}{n} < \infty$$

and therefore $r \in H_0$, $w \in H_0$, $s \in H_{-1}$.

For

$$\sigma(x) := \sum_{n \leq x} \frac{\mu(n)}{n} \log\left(\frac{x}{n}\right)$$

it holds

$$\sigma(xy) + 1 = \sigma(x) + \sigma(y) \quad , \quad \sigma'(x) = \frac{1}{x} \sum_{n \leq x} \frac{\mu(n)}{n}$$

and the inverse mapping is given by

$$\sigma^{-1}(x) = \sum_{n \leq x} \frac{1}{n} \log\left(\frac{x}{n}\right) \quad , \quad x \geq 1 \quad .$$

The asymptotics of a related arithmetical function is given by ((ApT) 3.12)

$$\sum_{n \leq x} \frac{1}{n} \frac{\varphi(n)}{n} + \sum_{n=1}^{\infty} \frac{1}{n} \frac{\mu(n) \log n}{n} = \frac{6}{\pi^2} (\log x + \gamma) + o\left(\frac{\log x}{x}\right).$$

For the relationship to the alternative Zeta function theory below we note that the function $\pi \cot(\pi x)$ is holomorphic except the pole $z = 1$ and it holds

$$\frac{1}{2} \cot\left(\frac{x}{2}\right) = \frac{d}{dx} \ln\left(2 \sin\left(\frac{x}{2}\right)\right) = -\frac{d}{dx} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = \sum_{n=1}^{\infty} \sin(nx) \in H_{-1}^{\#}(0, 2\pi)$$

From ((TiE) 4.14) we recall the identity ($s = \frac{1}{2} + it$)

$$\sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} z^{1-s} (-\pi \cot(\pi z)) \frac{dz}{z} =: g(s)$$

leading to $((z, g))_{-1} < \infty$, i.e. $z, g \in H_{-1}$ resp. $((z, \omega))_{-1}, ((g, \omega))_{-1}, \forall \omega \in H_{-1}$. The latter representation enables an orthogonal projections $z^{(*)}, g^{(*)} \in H_{-1/2}$ in the form $((z^{(*)}, \omega))_{-1}, ((g^{(*)}, \omega))_{-1}, \forall \omega \in H_{-1/2}$.

Putting

$$g(x) := \frac{1}{4 \sin^2(\pi x)} > 0$$

then it holds

$$\text{i) } -2 \log(2 \sin(\pi x)) = -\log(4 \sin^2(\pi x)) = \log(g(x))$$

$$\text{ii) } \frac{1}{2} x \frac{g'(x)}{g(x)} = -\pi x \cot(\pi x) \xrightarrow{x \rightarrow \infty} -1$$

Therefore, $g(x)$ is auto-model (or regular varying) of order -1 ([VIV] p. 56/57), i.e.

$$\lim_{x \rightarrow \infty} \frac{g(ax)}{g(x)} = \frac{1}{a}.$$

3. An alternative approach to prove the binary Goldbach conjecture

The asymptotics of the special Kummer functions ${}_1K_1(a, a+1; x) \sim e^x/x$ is proposed as alternative tool for the additive number theory. Landau predicted the proof of the binary Goldbach conjecture (with high probability) based on the Stäckel approximation formula in combination with his own corresponding additions (LaE1). With the notation of (LaE1) the prime pair (p, q) counting function $H(x)$ with the condition $p+q \leq x$ is given by

$$H(x) = \sum_{p \leq x} \pi(x-p)$$

The Stäckel formula shows the corresponding asymptotics with respect to the (number theoretical) Euler $\varphi(n)$ –function in the form

$$\sim \frac{n}{\log n} \cdot \frac{n}{\log n} \cdot \frac{1}{\varphi(n)},$$

We suggest to apply a modified "density" function $\frac{x}{\log x} \cdot \frac{x}{\log x}$ in the form

$$c(a, b) \cdot {}_1K_1(a, a+1; \log x) \cdot {}_1K_1(b, b+1; \log x)$$

to anticipate the fact, that the two factors $\frac{n}{\log n}$ relates to two different prime numbers. From

$${}_1K_1(a, a+1; x) \cdot {}_1K_1(a+1, a+2; x) = \frac{a+1}{a} {}_1K_1(a, a+1; x) \cdot \frac{d}{dx} {}_1K_1(a, a+1; x)$$

it follows for $a = \frac{1}{2}$

$$\frac{x}{\log x} \cdot \frac{x}{\log x} \sim \frac{3}{2} \cdot \frac{d}{dx} \left[{}_1K_1\left(\frac{1}{2}, \frac{3}{2}; \log x\right) \right]^2 =: \frac{3}{2} \cdot \frac{d}{dx} K(\log x).$$

From ((ApT) 3.12) we recall the asymptotics

$$\sum_{n \leq x} \frac{1}{n} \frac{\phi(n)}{n} + \sum_{n=1}^{\infty} \frac{1}{n} \frac{\mu(n) \log n}{n} = \frac{6}{\pi^2} (\log x + \gamma) + o\left(\frac{\log x}{x}\right),$$

i.e. $\frac{\phi(n)}{n} \sim \frac{\mu(n) \log n}{n}$ are in balance. With respect to the Stäckel formula above this leads to

$$\sim \frac{d}{dn} \left[{}_1K_1\left(\frac{1}{2}, \frac{3}{2}; \log n\right) \right]^2 \cdot \frac{1}{\varphi(n)} \sim \frac{d}{dn} \left[{}_1K_1\left(\frac{1}{2}, \frac{3}{2}; \log n\right) \right]^2 \cdot \frac{1}{\mu(n) \log n}.$$

With respect to the Hilbert space $H_{-1/2}$ we note the identities $((dK, v))_{\alpha-1/2} = (K, v)_{\alpha}$ for $K, v \in H_{\alpha}$.

Riemann inverted the formula ((EdH), 1.13)

$$J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s} = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{1}{t(t^2-1)} \frac{dt}{\log t}$$

by means of the Möbius inversion formula, getting ((EdH), 1.17)

$$\pi(x) = Li(x) - \sum_{n=2}^{\infty} Li(x^{1/n}), \quad Li(x) := \int_2^x \frac{dt}{\log t} - Li(2), \quad Li(2) \sim 1.04 \dots$$

Riemann suggested a more natural and better approximation in the form

$$\pi(x) \sim R(x) := Li(x) - \sum_{n=2}^{\infty} \frac{\mu(n)}{n} Li(x^{1/n}).$$

However, he was aware of the defects of this approximation and his analysis of it, which is basically due to fact, that he has no estimate at all of the size of the „periodic“ terms $\sum_{n=1}^{\infty} \sum_{\rho} Li(x^{\rho/n})$ ((EdH), 1.17).

From the above we recall the alternatively proposed $\text{li}(x)$ function

$$\text{li}^*(x) := \text{Ei}^*(\log x) = {}_1F_1\left(\frac{1}{2}; \frac{3}{2}, \log x\right) \approx \frac{x}{\log x} ,$$

as well the Landau density function $\sigma(x) := \sum_{n \leq x} \frac{\mu(n)}{n} \log\left(\frac{x}{n}\right)$ with $\sigma(1) = 1$. The natural corresponding alternative approximation function to $R(x)$ is then given by

$$\pi(x) \sim B(x) := {}_1F_1\left(\frac{1}{2}; \frac{3}{2}, \log x\right) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}, \log\left(\frac{x}{n}\right)\right) ,$$

enjoying appropriate estimates for the terms ${}_1F_1\left(\frac{1}{2}; \frac{3}{2}, \log\left(\frac{x}{n}\right)\right)$ (as the term $\frac{\log x}{n}$ is replaced by the term $\log\left(\frac{x}{n}\right)$).

The prime pair (p, q) counting function $H(x)$ with the condition $p + q \leq x$ is then given by (LaE1)

$$H(x) = \sum_{p \leq x} \pi(x - p) \sim \sum_{p \leq x} B(x - p).$$

The formula is proposed to extend the Schnirelmann theorem, that the set of integers, which are the sums of two primes, has the appropriate positive density α by applying and corresponding „Chebychev“ estimate $\pi(x) \gg \frac{x}{\log x}$.

4. The $H_{-\frac{1}{2}} = H_0 \otimes H_0^\perp$ decomposition for a quantum space-time model

The proposed Hilbert space framework (with its relationship to the Zeta function theory and the Hilbert-Polya/ Berry-Keating conjecture) enables a combined usage of spectral theory, variational methods for non-linear operators (VaM), Galerkin-Ritz approximation theory (VeW), and tools like Pseudo-Differential operators ((EsG), (LoA), (PeB)), degenerated hypergeometric functions (GrI), Hilbert (resp. Riesz) transform(s) and wavelets (HoM). The link between PDO and the Galerkin-Ritz approximation theory is given by the Garding inequality and the concept of hypoellipticity ((AzA), (GaL), (PeB)). The norms of the Hilbert scale H_α can be enriched with an additional norm enjoying an "exponential decay" behavior. Each Hilbert space norm with $\alpha < 0$ is governed by the norm of the Hilbert space H_0 and this "exp-decay" norm. This property is proposed to be applied in the context of the decomposition of the Hilbert space $H_{-1/2} = H_0 \otimes H_0^\perp$ ((BrK), (BrK1), (BrK3), (BrK7)).

We briefly sketch the key elements of the Hilbert space/scale theory with relationship to corresponding approximation theory ((NiJ), (NiJ1)), and wavelet theory, (FaM), (HoM).

If there exists an operator A with $D(A) = H_1$, $R(A) = H_0$ and $\|x\|_1 = \|Ax\|_0$, whereby the operator A is positive definite, self-adjoint and A^{-1} is compact.

Then the corresponding eigenvalue problem $A\varphi_i = \sigma_i\varphi_i$ has infinite solutions $\{\sigma_i, \varphi_i\}$ with $\sigma_i \rightarrow \infty$ and $(\varphi_i, \varphi_k) = \delta_{i,k}$, and for each element $x \in H_1 = A^{-1}H_0$ it holds the representation

$$x = \sum_{i=1}^{\infty} (x, \varphi_i) \varphi_i$$

Inner products with corresponding norms of a distributional Hilbert scale can be defined based on the eigen-pairs of an appropriately defined operator in the form

$$(x, y)_\alpha := \sum_{i=1}^{\infty} \lambda_i^\alpha (x, \varphi_i) (y, \varphi_i) = \sum_{i=1}^{\infty} \lambda_i^\alpha x_i y_i$$

Additionally, for $t > 0$ there can be an inner product resp. norm defined for an additional governing Hilbert space with an "exponential decay" behavior in the form $e^{-\sqrt{\lambda_i}t}$ given by

$$(x, y)_{(t)}^2 := \sum_{i=1}^{\infty} e^{-\sqrt{\lambda_i}t} (x, \varphi_i) (y, \varphi_i) \quad , \quad \|x\|_{(t)}^2 := (x, x)_{(t)}^2$$

The distributional $H_{-\frac{1}{2}}$ - Hilbert space is proposed to model quantum states, alternatively to the Hilbert space H_0 . A mathematical (wavelet microscopic) analysis of those states is then about an analysis of the "objects"

$$x = x_0 + x_0^\perp \in H_0 \otimes H_0^\perp$$

with

$$\|x_0\|_0 = 1 \quad \sigma := \|x_0^\perp\|_{-1/2}^2$$

As it holds for any $t, \delta, \alpha > 0$ and $\lambda \geq 1$ the inequality

$$\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{t(\delta^{-1} - \sqrt{\lambda})}$$

the following inequality is valid for any $x \in H_0$, governing the approximation "quality" of a quantum state with respect to the norm of H_0 :

$$\|x\|_{-1/2}^2 \leq \delta \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2 = \sigma \|x\|_0^2 + e \|x\|_{(\sigma)}^2 = \sigma \|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}\sigma} x_i^2$$

It balances the "continuous" view of the overall state with its "discrete" components.

For a related approximation theory we refer to (BrK8), (NiJ), (NiJ1).

The decomposition of the quantum state space $H_{-1/2} = H_0 \otimes H_0^\perp$ resp. the quantum energy space $H_{1/2} = H_1 \otimes H_1^\perp$ goes along with the Fourier wave resp. the Calderón wavelet tool. The (wavelet) admissibility condition is strongly related to the Hilbert space $H_{\frac{1}{2}}$. The admissibility condition ensures the validity of the inverse wavelet transform which then is valid for all Hilbert scale values.

A L_2 – based Fourier wave analysis is the baseline for statistical analysis, as well as for PDE and PDO theory. The Fourier transform of a wavelet transformed function f is given by:

$$\widehat{W_\vartheta[f]}(a, \omega) := (2\pi|a|)^{\frac{1}{2}} c_\vartheta^{-\frac{1}{2}} \hat{\vartheta}(-a\omega) \hat{f}(\omega) \quad .$$

For $\varphi, \vartheta \in L_2(\mathbb{R})$, $f_1, f_2 \in L_2(\mathbb{R})$,

$$0 < |c_{\vartheta\varphi}| := 2\pi \left| \int_{\mathbb{R}} \frac{\hat{\vartheta}(\omega) \overline{\hat{\varphi}(\omega)}}{|\omega|} d\omega \right| < \infty$$

and $|c_{\vartheta\varphi}| \leq c_\vartheta c_\varphi$ one gets the duality relationship $(W_\vartheta f_1, W_\varphi^* f_2)_{L_2(\mathbb{R}^2, \frac{da db}{a^2})} = c_{\vartheta\varphi} (f_1, f_2)_{L_2}$ i.e.

$$W_\varphi^* W_\vartheta [f] = c_{\vartheta\varphi} f \quad \text{in a } L_2 \text{ –sense.}$$

For $\varphi, \vartheta \in L_2(\mathbb{R})$, $f_1, f_2 \in L_2(\mathbb{R})$,

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$$W_\varphi^* W_\vartheta [f] = c_{\vartheta\varphi} f \quad \text{in a } L_2 \text{ –sense.}$$

This identity provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "microscope observations" of two wavelet (optics) functions ϑ, φ can be compared with each other by the above "reproducing" ("duality") formula. The prize to be paid is about additional efforts, when re-building the reconstruction wavelet. We further note that for a convenient choice of the two wavelet functions the Gibbs phenomenon disappears.

We note the Gaussian function related "Mexican hat" (wavelet) function

$$g(x) := -\frac{d^2}{dx^2} (e^{-\frac{x^2}{2}}) = (1 - x^2) e^{-\frac{x^2}{2}}.$$

has being successfully applied e.g. in wavelet theory. We further mention that the Hilbert transform of a wavelet is again a wavelet.

The decomposition of the quantum state space $H_0 \otimes H_0^\perp$ resp. the quantum energy space $H_1 \otimes H_1^\perp$ is very much related to the „hidden variables in quantum theory“ concept of D. Bohm (BoD) with the notions of implicate and explicate order:

((BoD), A2): „It is important to emphasize, however, that mathematics and physics are not being regarded here as separate but mutually related structures (so that, for example, one could be said to apply mathematics to physics as paint is applied to wood). Rather, it is being suggested that mathematics and physics are to be considered as aspects of a single undivided whole“.

The implicate or enfolded order is about „a new notion of order, that may be appropriate to a universe of unbroken wholeness. In the enfolded order, space and time are no longer the dominant factors determining the relationships of dependence or independence of different elements.“

((BoD), A3): „Implicate order is generally to be described not in terms of simple geometric transformations, such as translations, rotations, and dilations, but rather in terms of a different kind of operations. ... What happens in the broader context of implicate order we shall call a metamorphosis. ... An example of such a metamorphosis metamorphosis M is determined by the Green's function relating amplitudes at the illuminated structure to those at the photographic plate“.

In our case this related to the closed sub-spaces H_0^\perp and H_1^\perp .

Rather, an entirely different sort of basic connection of elements is possible, from which our ordinary notions of space and time, along with those of separately existent material particles, are abstracted as forms derived from the deeper order. These ordinary notions in fact appear in what is called the explicate or unfolded order, which is a special and distinguished form contained within the general totality of all the implicate orders... Explicate order arises primarily as a certain aspect of sense perception and of experience with the content of such sense perception. It may be added that, in physics, explicate order generally reveals itself in the sensibly observable results of functioning of an instrument. ... „What is common to the functioning of instruments generally used in physical research is that the sensibly perceptible content is ultimately describable in terms of a Euclidean system of order and measure, i.e., one that can adequately be understood in terms of ordinary Euclidean geometry. ... The general transformations are considered to be the essential determining features of a geometry in a Euclidean space of three dimensions; those are displacement operators, rotation operators and the dilatation operator ((BoD) A.2).

In our case this is related to the Hilbert spaces H_0 and H_1 . The Euclidean systems of order and measure are strongly related to the Archimedian principle.

„Of course, in the quantum theory, the algebraic terms are interpreted as standing for 'physical observables' to which they correspond. However, in the approach that is being suggested here, such terms are not to be regarded as standing for anything in particular. ... This means, of course, that we do not regard terms like 'particle', 'charge', 'mass', 'position', 'momentum', etc., as having primary relevance in the algebraic language. Rather, at best, they will have to come out as high-level abstractions.“ ((BoD) A4).

4a. The Berry-Keating conjecture and the $H_{-1/2}$ quantum state space

The Berry-Keating conjecture puts the zeros of the Zeta function (on the critical line, if the RH is true) in relationship to the (energy level) eigenvalues associated with the classical Hermitian operator $H(x, p = x \cdot p \sim x \cdot \frac{d}{dx}$, where x denotes the position coordinate and p the conjugate momentum. The Friedrichs extension of the variational representation of the Zeta function (on the critical) with L_2 -test space indicated a $H_{-1/2}$ quantum state space with related $H_{1/2}$ energy space. The today's standard quantum state resp. energy spaces are $H_0 = L_2$ resp. H_1 , i.e. those Hilbert spaces are compactly embedded subspaces of the proposed new ones.

Applying the physical quantum (fluid) Hilbert (state) space $H_{-1/2}$ to the 3-D non-linear, non-stationary NSE enables a well posed variational representation of the NSE with appropriate valid energy inequality, closing the Serrin gap problem. The correspondingly variational representation of the Maxwell equations enables a quantum field model (modified YME), enabling a differentiation of "elementary particles" with and w/o mass (modelled by the orthogonal decomposition of the Hilbert spaces $H_{-1/2} = H_0 \otimes H_0^\perp$ resp. $H_{1/2} = H_1 \otimes H_1^\perp$). It enables the concept of orthogonal projection, which can be interpreted as "mass generation process during observation". Purely energy interaction of the "EP" are "acting" in the orthogonal space (which might be interpreted as zero point energy pool, "wave package resp. eigen-differential space). The macroscopic and microscopic state of quanta relate to corresponding frequencies of its vibrations. The corresponding action variables of the system ((HeW) II.1.c) define the related kinematical (physical) and thermodynamical concept of "time", as described in ((RoC), (SmL)), (RoC1), section 13): *"Our interaction with the world is partial, which is why we see it in blurred way. To this blurring is added quantum indeterminacy. The ignorance that follows from this determines the existence of a particular variable - thermal time - and of an entropy that quantifies our uncertainty. Perhaps we belong to a particular subset of the world that interacts with the rest of it in such a way that this entropy is lower in one direction of our thermal time."*

4b. The $H_{-1/2}$ Hilbert space and a new ground quantum state H_0^\perp & ground quantum energy H_1^\perp model

The Schrödinger (differentiation) operator is not bounded with respect to the norm of L_2 , i.e. only on a dense subspace of L_2 a corresponding spectral representation of this operator can be defined. The not vanishing constant Fourier term of the baseline Hermite polynomial (which is the Gaussian function) leads to mathematical challenges with respect to the creation and annihilation operators of the related Hamiltonian operator of the quantum oscillator model. The Hilbert transform of a function f has always vanishing constant Fourier terms. As a consequence, the Hilbert-transformed Schrödinger operator form with extended domain $H_{-1/2}$ is bounded (with respect to the norm of L_2) leading to a bounded Hermitian operator with corresponding spectral form representation.

Based on the newly defined common Hilbert space domain spectral theory can be applied, while

- the (physical) test space keeps the same, i.e. $L_2 = H_0$
- the current domains of the considered operators are extended to enable a (convergent) energy norm $\|x\|_{1/2}$ and a corresponding weak variation representation of the considered operator equations with respect to the inner product $(x, y)_{-1/2}$.

The corresponding notions from variation theory are "energy norm" and "operator norm" with correspondingly defined minimization problems ("energy" resp. "action" minimization problems). The corresponding eigenvalue problem of an operator T is then related to the inner product $(Tx, x)_{-1/2}$.

With respect to the newly proposed Pseudo-differential and Fourier multiplier operators with extended fractional Hilbert scale domain we note the following:

- The Maxwell equations are represented by differential equations or integral equations. Both representations are considered as equivalent.
- The Lagrange ("force") and the Hamiltonian ("energy") formalisms are considered as equivalent. The mathematical proof is based on the Legendre transform, i.e. the equivalence is only valid if the assumptions of the Legendre transform are fulfilled.

In both cases, corresponding (mathematical) regularity assumptions are required to enable those propositions. A restriction of the domain regularity of the considered operators leads to no longer well-defined classical differential equations resp. to no longer valid Lagrange formalism. In other words, the provided consistent model in the distributional framework represents the mathematical/transcendental view of the considered physical world, while the corresponding classical solutions of the several differential equations are mathematical approximations to those physical models. This concept also overcomes the "physical interpretation" challenge of the "Neumann PDE" representation of the pressure p in the NSE model.

As a consequence there is only a "one-energy" (field) concept and corresponding (PDE specific) manifestations/ forms of considered "Nature forces".

The proposed variation Hilbert space frame is built on the space-time frame with dimension $n = m + 1 = 4$. Therefore the Huygens' Principle (which is also valid for the initial value problem of the wave equation) is valid for all considered "wave" PDE, overcoming e.g. the $n > 10$ requirement of the string theory. At the same time, the characteristics roles of a space-time dimension = 4 is also underlined by the specific role of undistorted spherical travelling waves (Courant-Hilbert, "methods of mathematical physics", II, VI, §10.3).

Schrödinger's "purely quantum wave" vision is about half-odd integers, rather than integers to be applied to wave-mechanical vibrations which correspond to the motion of

particles of a gas resp. the eigenvalues and eigen-functions of the harmonic quantum oscillator still governed by the Heisenberg uncertainty inequality. The alternatively proposed $H_{1/2}$ energy space enables Schrödinger's vision ((ScE) (7.23) ff):

let ω denotes the angular frequency, h the (\hbar) Planck constant and

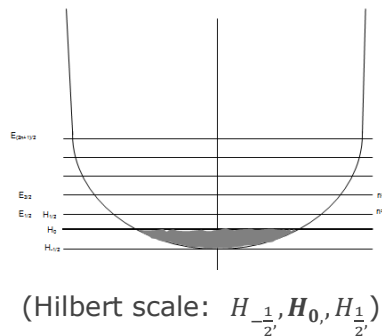
$$e := \frac{\omega h}{2}$$

Then Schrödinger's "half-odd integer vision" is about the following replacement:

| | | | |
|---------|----------------------|---------------|--|
| $n = 0$ | $E_0 = e$ | \rightarrow | $E_{1/2} = 1 * e$ |
| $n > 0$ | $E_1 = 1 * \omega h$ | \rightarrow | $E_{3/2} = 2 * e$ |
| | $E_2 = 2 * \omega h$ | \rightarrow | $E_{5/2} = 3 * e$ |
| ... | | | |
| | $E_n = n * \omega h$ | \rightarrow | $E_{(2n+1)/2} = (n + 1) * e, n = 0,1,2,..$ |

As a consequence the corresponding eigenvalue and eigenfunction solutions of the number operator (i.e. the product of generation and annihilation operators) start with index $n = 1$, not already with $n = 0$.

With respect to the ladder operators of the harmonic quantum oscillator the proposed alternative quantum state and related energy Hilbert scales can be visualized by



A successfully applied least action principle (being interpreted as a maxime of Kant's reflective judgment) results into appropriate consistent mathematical-physical models, those models can be declared as law of natures. The above is related to the three "forces of nature" as modelled by the SMEP. The nature of those elementary particles and the way they move, is described by quantum mechanics, but quantum mechanics cannot deal with the curvature of space-time. Space-time are manifestations of a physical field, the gravitational field. At the same time, physical fields have quantum character: granular, probabilistic, manifesting through interactions. The to be defined common mathematical solution framework needs to provide a quantum state of a gravitational field, i.e. a quantum state of space. The crucial difference between the photons characterized by the Maxwell equations (the quanta of the electromagnetic field) and the to be defined quanta of gravity is, that photons exists in space, whereas the quanta of gravity constitute space themselves ((RoC2) p. 148). The proposed mathematical framework provides a common baseline to integrate quantum mechanics & thermodynamics with gravity & thermodynamics. From a physical model problem perspective this is about a common mathematical framework for black body radiation ((BrK4) remark 2.6, Note O55, O71, O72) and black hole radiation ((RoC3) p. 56, 60 ff)). The thermodynamics is the common physical theory denominator with the Planck concept of zero point energy of the harmonic quantum oscillator (BrK), (BrK1), and the Boltzmann entropy concept. An integrated model needs to combine the underlying Bose-Einstein and the Dirac-Fermi statistic. In this context already Schrödinger suggested half-odd quantum numbers rather than integers. *"From the point of analogy one would very much prefer to do so. For, the "zero point energy" of a Planck oscillator is not only borne*

out by direct observation in the case of crystal lattices, it is also so intimately linked up with the Heisenberg uncertainty relation that one hates to dispense with it. On the other hand, if we adopt it straightaway, we get into serious trouble, especially on contemplating changes of the volume (e.g. adiabatic compression of a given volume of black-body radiation" ((ScE) p. 50).

In the context of the newly proposed "energy-space" $H_{1/2} = H_1 + H_1^\dagger$ we also refer to the Bose-Einstein condensation, where below the critical temperature T_c BEC "normal gas" particles coexist in equilibrium with "condensed" particles. Unlike a liquid droplet in a gas, here the "condensed" particles are not separated in space ($H_{-1/2} = H_0 + H_0^\dagger$) from normal particles. Instead they are separated in momentum space. The condensed particles all occupy a single quantum state of zero momentum, while normal particles all have finite momentum.

4c. The $H_{-1/2}$ quantum state space and a consistent plasma & geometrodynamics enabling space-time background independency

The Hilbert space $H_{-1/2}$ is suggested as physical quantum (Hilbert) state space model accompanied by correspondingly defined variational (Differential, Pseudo Differential or singular integral operator) equations. Beside the NSE and the YME problem areas the following other related areas are considered: (1) plasma dynamics and (2) space-time quantum geometrodynamics.

Concerning (1) we consider the model problem of the Landau damping phenomenon (which is about "*wave damping w/o energy dissipation by collision in plasma*") proposing a proof of this phenomenon, based on a $H_{-1/2}$ (weak) problem-adequate (variational) Landau equations representation.

Concerning (2) we propose a $H_{-1/2}$ (weak) quantum gravity model, which overcomes the current „quantum state stage background dependency“ problem (as a consequence of key principles of GRT and all related „philosophical“ aspects of „space-time and the transcendental external world“ ((AnE), (BoD), (RoC0-3), (ScE1-2), (SmL), (WeH3), (WhJ0-2)). The proposed quantum space-time gravity model ensures „quantum state stage background independency“, going along with non-linear elasticity PDO equations (note: the origin of the notion „tensor“ came for elasticity theory), in our case, embedded in a weak $H_{-1/2}$ variational PDO framework, governed by the least action principle.

Concerning the problem area (1) we note that based on the classical Vlasov (partial differential) equation (describing the time evolution of the distribution function of plasma consisting of charged particles with long-range interaction) the non-linear Landau damping phenomenon has been proven in (MoC). The central element of the proof is about analytical (!) norm estimates in sync with the underlying Gaussian distribution function regularity. At the same point in time, the Vlasov equation in its classical PDE representation overlooks the important physical phenomenon of "*electrons travelling with exactly the material speed and the wave speed*" ((ShF) p. 392). The not physical problem adequate (analytical) norm estimates in combination with the physical modelling gap on plasma collision level show that the Vlasov equation is a not appropriate mathematical model for the non-linear Landau damping phenomenon. In (BrK6) an alternative model (based on the original Boltzmann-Landau (collision) equations; (LiP) (LiP1)) is proposed. The non-linear Boltzmann-Landau collision operator is given by

$$Q(f, f) = \frac{\partial}{\partial v_i} \left\{ \int_{R^N} a_{ij}(v-w) \left[f(w) \frac{\partial f(v)}{\partial v_j} - f(v) \frac{\partial f(w)}{\partial w_j} \right] dw \right\}$$

with

$$a_{ij}(z) = \frac{a(z)}{|z|} \left\{ \delta_{ij} - \frac{z_i z_j}{|z|^2} \right\} = \frac{a(z)}{|z|} P(z) := \frac{1-[1-a(z)]}{|z|} [Id - Q](z) \quad Q(z) := (R_i R_j)_{1 \leq i, j \leq N}$$

and $a(z)$ symmetric, non-negative and even in z and R_i denote the Riesz operators, and with an unknown function f corresponding at each time t to the density of particle at the point x with velocity v . It can be approximated by a linear Pseudo Differential Operator (PDO) of order zero with symbol

$$b_{ij}(z) = z \cdot a_{ij}(z) = \frac{z}{|z|} \left\{ \delta_{ij} - \frac{z_i z_j}{|z|^2} \right\} = \frac{z}{|z|} P(z) := \frac{z}{|z|} [Id - Q](z)$$

whereby $a_{ij}(z)$ denotes the symbol of the Oseen kernel (LeN). Corresponding Hilbert space norm estimates are provided to build a problem adequate proof of the Landau damping phenomenon.

An appropriate plasma collisions (dynamics) model is a central building block for the related geometrodynamics problem/solution area (2). The proposed framework is also suggested to be applied to build a unified quantum field and gravity field theory based on the conceptual thoughts of Wheeler/deWitt (CIi), and and the related Loop Quantum Theory (LQT), which is a modern version of the theory of Wheeler and deWitt, where "*the*

variables of the theory describe the fields that form matter, photons, electrons, other components of atoms and the gravitational field - all on the same level" ((RoC1) section 8, "dynamics as relation").

(CiI) 2.8: *Einstein's "general relativity" or "geometric geometry of gravitation" or "geometrodynamics", has two central ideas: (1) Space-time geometry "tells" mass-energy how to move, (2) mass-energy "tells" space-time geometry how to curve. The concept (1) is automatically obtained by the Einstein field equations, (CiI) (2.3.14), basically as the covariant divergence of the Einstein tensor is zero. At the same point in time there are multiple tests of the geometrical structure and of the geodesic equation of motion, e.g. gravitational deflection and delay of electromagnetic waves, de Sitter and Lense-Thirring effect, perihelion advance of Mercury, Lunar Laser Ranging with its relativistic parameters: time dilation or gravitational redshift, periastron advance, time delay in propagation of pulse, and rate of change of orbital period, (CiI) 3.4.*

(CiI) 3.5: *"Hilbert used a variational principle and Einstein the requirement that the conservation laws for momentum and energy for both, gravitational field and mass-energy, be satisfied as a direct consequence of the field equations. ... Einstein geometrodynamics, ..., has the important and beautiful property the the equations of motion are a direct mathematical consequence of the Bianchi identities."*

With respect to the overall conceptual idea of this homepage a Hilbert space based geometrodynamics is proposed to be built on "space-time states", which are represented by elements of $H_{-1/2}$, while their corresponding "space-time energy" elements are represented by the corresponding "dual" (wavelets) elements in $H_{1/2}$. The Einstein field equations are proposed to be re-formulated as a weak (!) least action minimization problem by correspondingly defined variational equations representation. With respect to the Bianchi identities we emphasize that if $(u, v)_{-1/2}$ denotes the inner product of $H_{-1/2}$ the following relationships hold true: $(div(u), v)_{-1/2} \sim (u, \nabla v)_{-1/2} \sim (u, \nabla v)_0$. The methods of functional analysis are basically the same as those in the elasticity theory (MaJ1). The building principles for an appropriately defined variational representation is about that the way, (1) how "Space-time geometry "tells" mass-energy how to move", can be obtained by those representation and that the multiple tests (observed phenomena) of the geometrical structure and of the geodesic equation of motion ((2) "where mass-energy "tells" space-time geometry how to curve") is modelled (as a kind of symmetry break down) as approximation solution in the compactly embedded sub-spaces H_0 resp. H_1 of $H_{-1/2}$ resp. $H_{1/2}$.

4d. The $H_{-1/2}$ quantum state space replacing the Dirac distributions space

With respect to the below we note that the Dirac theory with its underlying concept of a Dirac "function" is proposed to be replaced by (fluid/quantum/... state) "elements" of the distributional Hilbert space $H_{-1/2}$. We note that the regularity of the Dirac distribution "function" depends from the space dimension, i.e. it is an element of $H_{-1/2-\varepsilon}$ ($\varepsilon > 0$, $n =$ space dimension). Therefore, the alternative $H_{-1/2}$ quantum state concept avoids space dimension depending regularity assumptions for quantum mechanics "wave packages" / "eigen-functions" / "momentum functions" with corresponding continuous spectrum. We note that for signals on \mathbb{R} the spectrum of the Hilbert transform is (up to a constant) given by the distribution $\text{v. p.} \left(\frac{1}{x}\right)$, whereby the symbol "v.p." denotes the Cauchy principal value of the integral over \mathbb{R} . Its corresponding Fourier series is given by $-i \cdot \text{sgn}(k)$ with its relationship to "positive" and "negative" Dirac "functions" and the unit step function $Y(x)$. The $H_{-1/2}$ framework, replacing the Dirac "function" concept, enables a generalization to dimensions $n > 1$ without any corresponding additional regularity requirements.

5. NSE, YME and plasma/geometroynamics problem/solution areas

The common Hilbert scale is about the Hilbert spaces $H(a)$ with $a=1, 1/2, 0, -1/2, -1$ with its corresponding inner products $((u,v)), (u,v), (u,v), ((u,v)), (((u,v)))$. The proposed mathematical concepts and tools are especially correlated to the names of Plemelj, Stieltjes and Calderón.

The newly proposed "fluid/quantum state" Hilbert space $H_{-1/2}$ with its closed orthogonal subspace of H_0 goes also along with a combined usage of L_2 waves governing the H_0 Hilbert space and "orthogonal" wavelets governing the $H_{-1/2} - H_0$ space. The wavelet "reproducing" ("duality") formula provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets, where the "*microscope observations*" of two wavelet (optics) functions can be compared with each other (LoA). The prize to be paid is about additional efforts, when re-building the reconstruction wavelet.

We propose modified Maxwell equations with correspondingly extended domains according to the above. This model is proposed as alternative to SMEP, i.e. the modified Maxwell equation are proposed to be a "Non-standard Model of Elementary Particles (NMEP)", i.e. an alternative to the Yang-Mills (field) equations. The conceptual approach is also applicable for the Einstein field equations. Mathematical speaking this is about potential functions built on corresponding "density" functions. The source density is the most prominent one. Physical speaking the source is the root cause of the corresponding source field. Another example is the invertebrate density (=rotation) with its corresponding rotation field. The Poincare lemma in a 3-D framework states that source fields are rotation-free and rotation fields are source-free. The physical interpretation of the rotation field in the modified Maxwell equations is about rotating "mass elements w/o mass" (in the sense of Plemelj) with corresponding potential function. In a certain sense this concept can be seen as a generalization of the Helmholtz decomposition (which is about a representation of a vector field as a sum of an irrotational (curl-free) and a solenoidal (divergence-free) vector field): it is derived applying the delta "function" concept. In the context of the proposed distributional Hilbert space framework, the Dirac function concept (where the regularity of those "function" depends from the space dimension) is replaced by the quantum state Hilbert space $H_{-1/2}$. The solution $u \in H_{-1/2}$ of the Helmholtz equation in terms of the double layer potential is provided in ((LiI), 7.3.4). From the Sobolev embedding theorem it follows, that for any space dimension $n > 0$ the modified Helmholtz equation is valid for not continuous vector fields.

5a. The related NSE problem/solution area

This section is about a straightforward solution of the NSE Millenium problem by closing the Serrin gap provided that the $H_{-1/2}$ Hilbert space is the accepted underlying fluid state model.

The Navier-Stokes Equations (NSE) describes a flow of incompressible, viscous fluid. The three key foundational questions of every PDE is existence, and uniqueness of solutions, as well as whether solutions corresponding to smooth initial data can develop singularities in finite time, and what these might mean. For the NSE satisfactory answers to those questions are available in two dimensions, i.e. 2D-NSE with smooth initial data possesses unique solutions which stay smooth forever. In three dimensions, those questions are still open. Only local existence and uniqueness results are known. Global existence of strong solutions has been proven only, when initial and external forces data are sufficiently smooth. Uniqueness and regularity of non-local Leray-Hopf solutions are still open problems.

Basically the existence of 3D solutions is proven only for "large" Banach spaces. The uniqueness is proven only in "small" Banach spaces. The question of global existence of smooth solutions vs. finite time blow up is one of the Clay Institute millennium problems.

The existence of weak solutions can be provided, essentially by the energy inequality. If solutions would be classical ones, it is possible to prove their uniqueness. On the other side for existing weak solutions it is not clear that the derivatives appearing in the inequalities have any meaning. Basically all existence proofs of weak solutions of the Navier-Stokes equations are given as limit (in the corresponding weak topology) of existing approximation solutions built on finite dimensional approximation spaces. The approximations are basically built by the Galerkin-Ritz method, whereby the approximation spaces are e.g. built on eigenfunctions of the Stokes operator or generalized Fourier series approximations. It has been questioned whether the NSE really describes general flows: The difficulty with ideal fluids, and the source of the d'Alembert paradox, is that in such fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by a streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the NSE blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost. Given a smooth datum at time zero, will the solution of the NSE continue to be smooth and unique for all time?

The NSE are derived from the (Cauchy) stress tensor (resp. the shear viscosity tensor) leading to liquid pressure force. In electrodynamics & kinetic plasma physics the linear resp. the angular momentum laws are linked to the electrostatic (mass "particles", collision, static, quantum mechanics, displacement related; "fermions") Coulomb potential resp. to the magnetic (mass-less "particles", collision-less, dynamic, quantum dynamics, rotation related; "bosons") Lorentz potential.

We note that the solution of the Navier-Stokes equation are related to the considered degenerated hypergeometric functions by its corresponding integral function representation (PeR1).

With respect to the open Millenium 3D non-stationary, non-linear NSE problem we note that the alternatively proposed "fluid state" Hilbert space $H(-1/2)$ with corresponding alternative energy ("velocity") space $H(1/2)$ enables a (currently missing) energy

inequality based on existing contribution of the non-linear term. In the standard weak NSE representation this term is zero, which is a great thing from a mathematical perspective, avoiding sophisticated estimating techniques, but a doubtful thing from a physical modelling perspective, as this term is the critical one, which jeopardized all attempts to extend the 3D problem based on existing results from the 2D case into the 3D case. The corresponding estimates are based on Sobolev embedding theorems; the Sobolevskii estimate provides the appropriate estimate given that the "fluid state" space is $H(-1/2)$ in a corresponding weak variational representation.

There is no uniqueness proof for weak solutions except for over small time intervals. The simplest possible model example how a singularity can appear, is the ODE

$$y'(t) = y^2(t), \quad y(0) = y_0$$

with the solution

$$y(t) = \frac{y_0}{1 - t \cdot y_0}$$

which becomes infinite in finite time.

Based on the Hilbert scale $H(\alpha)$, $\alpha \in \mathbb{R}$, defined by the orthogonal set of eigenpairs of the Stokes operator we present a global unique weak $H(-1/2)$ – solution of the generalized 3D Navier-Stokes initial value problem

$$\begin{aligned} (\dot{u}, v)_{-1/2} + (Au, v)_{-1/2} + (Bu, v)_{-1/2} &= 0 \\ &\text{for all } v \in H_{-1/2} \\ (u(0), v)_{-1/2} &= (u_0, v)_{-1/2} \end{aligned}$$

resp.

$$-(\Delta u, v)_{-1/2} + (\nabla p, v)_{-1/2} \cong (\nabla u, \nabla v)_{-1/2} + (\nabla p, v)_{-1/2} \cong (u, v)_{1/2} + (p, v)_0.$$

The pressure p can be expressed in terms of the velocity by the formula

$$p = - \sum_{j,k=1}^3 R_j R_k (u_j u_k)$$

where (R_1, R_2, R_3) is the Riesz transform. The Leray-Hopf projector is the matrix valued Fourier multiplier given by

$$P(\xi) = Id - \frac{\xi \otimes \xi}{|\xi|^2} = (\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2})_{1 \leq j, k \leq n}, \quad P = Id - R \otimes R =: Id - Q$$

whereby Q is an orthogonal projector, i.e. it holds $Q := R \otimes R = (R_j R_k)_{1 \leq j, k \leq 1} = Q^2$. As a result the Leray-Hopf operator

$$P = Id - R \otimes R =: Id - Q = Id - \frac{D \otimes D}{D^2} Id - \Delta^{-1} (\nabla \times \nabla)$$

is also an orthogonal projection.

The global boundedness is a consequence of the Sobolevskii-estimate of the non-linear term ([SoP]) enabling the generalized energy inequality

$$\frac{1}{2} \frac{d}{dt} \|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq |(Bu, u)_{-1/2}| \leq c \cdot \|u\|_{-1/2} \|u\|_1^2.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets

$$y'(t) \leq c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$$

resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \leq \|u_0\|_{-1/2} + \int_0^t \|u\|_1^2(s) ds \leq c \left\{ \|u_0\|_{-1/2} + \|u_0\|_0^2 \right\}.$$

A "3D challenge" like the NSE above is also valid, when solving the monochromatic scattering problem on surfaces of arbitrary shape applying electric field integral equations. From (IvV) we recall that the (integral) operators A and $A(t): H(-1/2) \rightarrow H(1/2)$ are bounded Fredholm operators with index zero. The underlying framework is still the standard one, as the domains are surfaces, only. An analog approach as above with correspondingly defined surface domain regularity is proposed.

The initial boundary value problem determines the initial pressure $p_0(x)$ by the Neumann problem

$$\begin{aligned} \Delta p_0 &= (f_0 - u_0 \cdot \nabla u_0) && \text{in } \Omega \\ \frac{\partial p_0}{\partial n} &= [\Delta u_0 - u_0 \cdot \nabla u_0 + f_0] \cdot n && \text{at } \partial\Omega \end{aligned}$$

with $f_0 := \lim_{t \rightarrow 0} f(\cdot, t)$. Applying formally the div-operator to the classical NSE the pressure field must satisfy the following Neumann problem ((GaG))

$$\begin{aligned} \Delta p &= (u \cdot \nabla) u - f && \text{in } \Omega \\ \frac{\partial p}{\partial n} &= [\Delta u - (u \cdot \nabla) u + f] \cdot n && \text{at } \partial\Omega \end{aligned}$$

where n denotes the outward unit normal to $\partial\Omega$. As it holds that

$$[\Delta u - (u \cdot \nabla) u + f] \cdot n|_{\partial\Omega} \rightarrow [\Delta u_0 - (u_0 \cdot \nabla) u_0 + f_0] \cdot n|_{\partial\Omega} \text{ in } H_{-1/2}(\partial\Omega)$$

and

$$\nabla \cdot [f - u \cdot \nabla u]|_{\partial\Omega} \rightarrow \nabla \cdot [f_0 - u_0 \cdot \nabla u_0]|_{\partial\Omega} \text{ in } H_{-1/2}(\partial\Omega)$$

the pressure p tends to p_0 in the sense that $\|\nabla(p(\cdot, t) - p_0)\| \rightarrow 0$ as $t \rightarrow 0$.

As a consequence the prescription of the pressure at the boundary walls or at the initial time independently of u , could be incompatible with and, therefore, could render the NSE problem ill-posed.

With respect to the relationship to the considered Hilbert space $H_{-1/2}$ we emphasize that the Prandtl operator with domain $H_{1/2}$ and range $H_{-1/2}$ is bounded and coercive and the corresponding exterior Neumann problem admit one and only one generalized solution (BrK2).

A $H_{-1/2}$ (fluid state) Hilbert space framework is also applied to derive optimal finite element approximation estimates for non-linear parabolic problems with not regular initial value data (BrK2).

Kolmogorov's turbulence theory is a purely statistical model (based on the H_0 (observation/test) Hilbert space), which describes (only!) the qualitative behavior of turbulent flows. There is no linkage to the quantitative fluid behavior as it is described by the Euler or the Navier-Stokes equations. The physical counterpart of his low- and high-pass filtering Fourier coefficients analysis is a "local Fourier spectrum", which is a contradiction in itself, as, either it is non-Fourier, or it is nonlocal ((FaM)). WE propose to combine the wavelet based solution concept of (FaM) with a revisited CLM equation model in a physical $H_{-1/2}$ Hilbert space framework to enable a turbulent $H_{-1/2}$ signal which can be split into two components: coherent bursts and incoherent noise. The model enables a localized Heisenberg uncertainty inequality in the closed (noise) subspace $L(2;ortho) = H(0;ortho) = H(-1/2) - H_0$, while the momentum-location commutator vanishes in the (coherent bursts) test space H_0 .

5b. The related YME problem/solution area

This section is about a straightforward solution of the YME mass gap problem provided that the $H_{-1/2}$ Hilbert space is the accepted underlying quantum state model.

We propose an alternative mathematical framework for the Standard Model of Elementary Particles (SMEP), which replaces gauge theory and variational principles: The underlying concepts of exterior derivatives and tensor algebra are replaced by (distributional) Hilbert scales and (purely Hamiltonian) variational principles. As a consequence, the vacuum energy becomes an intrinsic part of the variational principles, i.e. it is identical for all considered Lagrange resp. Hamiltonian mechanisms of all related differential equations, while the corresponding "force" becomes an observable of the considered (Hamiltonian) minimization problem.

In some problem statements of the YME there are basically two assumptions made:

1. the energy of the vacuum energy is zero
2. all energy states can be thought of as particles in plane-waves.

As a consequence the mass gap is the mass of the lightest particle.

Our challenge of proposition 1 is about the measure of the vacuum energy, which gives the value "zero". While the energy norm in the standard $H(1)$ Hilbert space might be zero, the value of the quantum state with respect to the energy norm of the sub-space $H(1/2)$ still can be >0 .

Our challenge of proposition 2 is going the same way: a particle with mass can be measured (condensed energy), i.e. it is an element of the test space H_0 , while there still can be "wavelets" in the closed complementary space $H(-1/2) - H_0$, where the test space is "just" compactly embedded. Those "wavelets" might be interpreted as all kinds of today's massless "particles" (neutrinos and photons) with related "dark energy". As a consequence there is no YME mass gap anymore, but there is a new concept of vacuum energy (wave packages, eigen-differentials, rotation differential) governed by the Heisenberg uncertainty principle. This is about an alternative harmonic quantum energy model enabling a finite "quantum fluctuation = total energy", while replacing Dirac's Delta function by $H(-1/2)$ distributions enabling and an alternative Schrödinger's momentum operator (BrK7).

A physical interpretation could be about "rotating differentials" ("quantum fluctuations"), which corresponds mathematically to Leibniz's concept of monads. The mathematical counterparts are the ideal points (or hyper-real numbers). This leads to non-standard analysis, whereby the number field has same cardinality than the real numbers. It is "just" the Archimedean principle which is no longer valid.

The proposed mathematical concepts and tools are especially correlated to the names of Schrödinger and Weyl (e.g. in the context of "half-odd integers quantum numbers for the Bose statistics" and resp. Weyl's contributions on the concepts of matter, the structure of the world and the principle of action (WeH), (WeH1), (WeH2)). It enables an alternative (quantum) ground state energy model embedded in the proposed distributional Hilbert scale frame of this homepage covering all variational physical-mathematical PDE and Pseudo Differential Operator (PDO) equations (e.g. also the Maxwell equations).

The electromagnetic interaction has gauge invariance for the probability density and for the Dirac equation. The wave equation for the gauge bosons, i.e. the generalization of the Maxwell equations, can be derived by forming a gauge-invariant field tensor using generalized derivative. There is a parallel to the definition of the covariant derivative in

general relativity. With respect to the above there is an alternative approach indicated, where the fermions are modelled as elements of the Hilbert space H_0 , while the complementary closed subspace H_0^\perp is a model for the "interaction particles, bosons". For gauge symmetries the fundamental equations are symmetric, but e.g. the ground state wave function breaks the symmetry. When a gauge symmetry is broken the gauge bosons are able to acquire an effective mass, even though gauge symmetry does not allow a boson mass in the fundamental equations. Following the above alternative concept the "symmetry state space" is modelled by H_0 , while the the ground state wave function is an element of the closed subspace H_0^\perp of $H_{-1/2}$ (BrK).

When one wants to treat the time-harmonic Maxwell equations with variational methods, one has to face the problem that the natural bilinear form is not coercive on the whole Sobolev space. One can, however, make it coercive by adding a certain bilinear form on the boundary of the domain (vanishing on a subspace of H_{-1}), which causes a change in the natural boundary conditions.

In SMEP (Standard Model of Elementary Particles) symmetry plays a key role. Conceptually, the SMEP starts with a set of fermions (e.g. the electron in quantum electrodynamics). If a theory is invariant under transformations by a symmetry group one obtains a conservation law and quantum numbers. Gauge symmetries are local symmetries that act differently at each space-time point. They automatically determine the interaction between particles by introducing bosons that mediate the interaction. $U(1)$ (where probability of the wave function (i.e. the complex unit circle numbers) is conserved) describes the electromagnetic interaction with 1 boson (photon) and 1 quantum number (charge Q). The group $SU(2)$ of complex, unitary (2x2) matrices with determinant 1 describes the weak force interaction with 3 bosons ($W(+)$, $W(-)$, Z), while the group $SU(3)$ of complex, unitary (3x3) matrices describes the strong force interaction with 8 gluon bosons.

Reformulated Maxwell or gravitation field equations in a weak $H_{-1/2}$ -sense leads to the same effect, as dealing with an isometric mapping $g \rightarrow H(g)$ in a weak H_0 sense (H denotes the Hilbert transform) alternatively to a second order operator in the form $x^*P(g(x))$ in a weak $H_{-1/2}$ sense. This results into some opportunities as

- the solutions of the Maxwell equations in a vacuum do not need any calibration transforms to ensure wave equation character; therefore, the arbitrarily chosen Lorentz condition for the electromagnetic potential (to ensure Lorentz invariance in wave equations) and its corresponding scalar function ((FeR), 7th lecture) can be avoided
- enabling alternative concepts in GRT to e.g. current ("flexible") metrical affinity, affine connexions and local isometric 3D unit spheres dealing with rigid infinitesimal pieces, being replaced by geometrical manifolds, enabling isometrical stitching of rigid infinitesimal pieces ((CiI), (ScP)).

The gauge invariance is the main principle in current SMEP theory.

(BID) 10.3: *"It is fine that the gauge field of electromagnetism has zero mass because there the force is mediated by photons, which are massless. However, Yang-Mills type forces must arise from the exchange of massive particles because of the observed short range of these forces. The Higgs mechanism helps in two ways. First, gauge fields can acquire mass by the symmetry breaking. Second, the undesirable Goldstone bosons (which arise in the symmetry-breaking process) can be usually gauged away."*

The Higgs effect (or mechanism) builds on an extended from global to local $U(1)$ transformations symmetry group of the underlying Lagrangian. It explains the mass of the gauge W^- and Z -(weak interaction) bosons of the weak "nuclear-force".

(HiP): *"Within the framework of quantum field theory a "spontaneous" breakdown of symmetry occurs if a Lagrangian, fully invariant under the internal symmetry group, has such a structure that physical vacuum is a member of a set of (physically equivalent) states which transform according to a nontrivial representation of the group. This degeneracy of the vacuum permits non-trivial multiplets of scalar fields to have nonzero vacuum expectation values (or "vacuons"), whose appearance leads to symmetry-breaking terms in propagators and vertices. ... When the symmetry group of the Lagrangian is extended from global to local transformations by introduction of coupling with a vector gauge field the original scalar massless boson as a result of spontaneous breakdown of symmetry then becomes the longitudinal state of a massive vector (Higgs) boson whose transverse states are the quanta of the transverse gauge field. A perturbative treatment of the model is developed in which the major features of these phenomena are present in zero order."*

The Higgs boson is supposed to be a heavy elementary particle (with non-zero rest mass of about 125 GeV with spin 0). The Higgs field is supposed to fill the whole universe interacting with each particle, which "moves" through it by a kind of frictional resistance, i.e. which has kinetic energy. Therefore, the Higgs effect (i.e. generating mass particles) requires a Higgs field with not vanishing amplitudes in the ground state.

5c. The related plasma/ geometrodynamics problem/solution area

This section is about a space-time quantum model, enabling a space-time stage background independency, provided that the $H_{-1/2}$ Hilbert space is the accepted underlying quantum state model.

Replacing the affine connexion and the underlying covariant derivative concept by a geometric structure with corresponding inner product puts the spot on the

Thurston conjecture: *The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structure (ThW).*

This conjecture asserts that any compact 3-manifold can be cut in a reasonably canonical way into a union of geometric pieces. In fact, the decomposition does exist. The point of the conjecture is that the pieces should all be geometric. There are precisely eight homogeneous spaces (X, G) which are needed for geometric structures on 3-manifolds. The symmetry group $SU(2)$ of quaternions of absolute value one (the model for the weak nuclear force interaction between an electron and a neutrino) is diffeomorph to S^3 , the unit sphere in $R(4)$. The latter one is one of the eight geometric manifolds above (ScP). We mention the two other relevant geometries, the Euclidean space E^3 and the hyperbolic space H^3 . It might be that our universe is not an either... or ..., but a combined one, where then the "connection" dots would become some physical interpretation. Looking from an Einstein field equation perspective the Ricci tensor is a second order tensor, which is very much linked to the Poincare conjecture, its solution by Perelman and to S^3 (AnM). The geometrodynamics provides alternative (pseudo) tensor operators to the Weyl tensor related to H^3 (CiI). In (CaJ) the concept of a Ricci potential is provided in the context of the Ricci curvature equation with rotational symmetry. The single scalar equation for the Ricci potential is equivalent to the original Ricci system in the rotationally symmetric case when the Ricci candidate is nonsingular. For an overview of the Ricci flow regarding e.g. entropy formula, finite extinction time for solutions on certain 3-manifolds in the context of Perelman's proof of the Poincare conjecture we refer to (KIB), (MoJ).

The single scalar equation for the Ricci potential (CaJ) might be interpreted as the counterpart of the CLM vorticity equation as a simple one-dimensional turbulent flow model in the context of the NSE.

The link back to a Hilbert space based theory might be provided by the theory of spaces with an indefinite metric ((DrM), (AzT), (DrM), (VaM)). In case of the $L(2)$ Hilbert space H , this is about a decomposition of H into an orthogonal sum of two spaces H_1 and H_2 with corresponding projection operators P_1 and P_2 relates to the concepts which appear in the problem of S. L. Sobolev concerning Hermitean operators in spaces with indefinite metric ((VaM) IV). For x being an element of H this is about a defined "potential" ((VaM) (11.1))

$$\varphi(x) := ((x))^2 = \|P_1x\|^2 - \|P_2x\|^2$$

and a corresponding "grad" potential operator $W(x)$, given by

$$W(x) = \frac{1}{2} \text{grad}\varphi(x) := P_1(x) - P_2(x) \quad , \quad (\text{VaM}) (11.4).$$

The potential criterion $\varphi(x) = c > 0$ defines a manifold, which represents a hyperboloid in the Hilbert space H with corresponding hyperbolic and conical regions. The tool set for an appropriate generalization of the above "grad" definition is about the (homogeneous, not always non-linear in h) Gateaux differential (or weak differential) $\mathbf{VF}(x, h)$ of a functional F at a point x in the direction h ((VaM) §3)). The appropriate weak inner product might be the inner product of the "velocity" space $H_{1/2}$. We note the Sobolev embedding theorem, i.e. H_k is a sub-space of C^0 (continuous functions) for $k > n/2$, i.e. there is no concept of "continuous velocity/momentum" in the proposed Hilbert space framework, i.e. there is no Frechet differential existing ((VaM) 3.3). This refers to one of

the several proposals, which have been made to drop some of the common sense notions about the universe ((KaM) 1.1), which is about continuity, i.e. space-time must be granular. The size of these grains would provide a natural cutoff for the Feynman integrals, allowing to have a finite S-matrix.

A selfadjoint operator B defined on all of the Hilbert space H is bounded. Thus, the operator B induces a decomposition of H into the direct sum of the subspaces, and therefore generates related hyperboloids ((VaM) 11.2). Following the investigations of Pontrjagin and Iohvidov on linear operators in a Hilbert space with an indefinite inner product, M. G. Krein proved the Pontrjagin-Iohvidov-Krein theorem (FaK).

In an universe model with appropriately connected geometric manifolds the corresponding symmetries breakdowns at those "connection dots" would govern corresponding different conservation laws in both of the two connected manifolds. The Noether theorem provides the corresponding mathematical concept (symmetry \rightarrow conservation laws; energy conservation in GRT, symmetries in particle physics, global and gauge symmetries, exact and broken). Those symmetries are associated with "non-observables". Currently applied symmetries are described by finite- (rotation group, Lorentz group, ...) and by infinite-dimensional (gauged $U(1)$, gauged $SU(3)$, diffeomorphisms of GR, general coordinate invariance...) Lie groups.

A manifold geometry is defined as a pair (X,G) , where X is a manifold and G acts transitively on X with compact point stabilisers (ScP). Related to the key tool "Hilbert transform" resp. "conjugate functions" of this page we recall from (ScP), that Kulkarni (unpublished) has carried out a finer classification in which one considers pairs (G,H) where G is a Lie group, H is a compact subgroup and G/H is a simple connected 3-manifold and pairs (G_1,H_1) and (G_2,H_2) are equivalent if there is an isomorphism $G_1 \rightarrow G_2$ sending H_1 to a conjugate of H_2 . Thus for example, the geometry S^3 arises from three distinct such pairs, (S^3,e) , $(U(2),SO(2))$, $(SO(4),SO(3))$. Another example is given by the Bianchi classification consisting of all simply connected 3-dimensional Lie groups up to an isomorphism.

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