A geometric Hilbert space based quantum and gravity model

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Dedicated to my son Mario
on the occasion of his 29th birthday, Dec 2, 2020

Prolog

In the book „*The Mathematical Reality, Why Space and Time are an Illusion*", (UnA1), $^{(*)}$, the concept of „*Vision – Mathematization – Simplification*" is proclaimed. The overall „*Vision*" is about a simplification of the incompatible SMEP and the cosmology model, by reducing the number of current „constants of nature", especially regarding the „*constant speed of light*" (→ variable speed of light, (UnA)) and the „*Planck constant*").

This and other related papers address the „Mathematization" piece of the above „Vision" regarding the ground-breaking idea going back to Einstein (EiA3), $^{(**)}$, to explain the gravitation directly from characteristics of the universe (UnA). It is basically achieved by the mathematical concept of a coarse-grained „*kinematical*" energy Hilbert space H_1, compactly embedded into a $H_{1/2}$- (energy) Hilbert space. Physical notions, like „*space-time*", „*density*", „*action*", „*Planck's quantum of action*", „*forces*", … become mathematical & physical reality in this („coarse-grained", see also (BrK), and truly „fermions") kinematical H_1 „world", complementary to a purely (mathematical) truly „bosons" potential energy $H_1^{\perp}$ „world". The link to Einstein's SRT (resp. the Lorentz transformation) is given by the famous PCT (charge, parity, time) theorem, (StR).

The mathematical models are weak variational representations of related well defined PDE, which include well defined domains and appropriate boundary/initial value conditions. The mathematical reality beyond the physical reality is about the extended variational representation in the overall (energy) $H_{1/2}$ „world". Then, the „*Planck constant*" might be possible to be interpreted as a physical unit of measure in a corresponding H_1 based action functional. The „*speed of light*" might be possible to be interpreted as a potential barrier $\varphi(x) = \bar{c} > 0$ defining a manifold, which represents a hyperboloid in a corresponding Hilbert space H with corresponding hyperbolic and conical regions $^{(***)}$.

(UnA1) §10: „*The solution is about a qualitative justification for the fact, that (these) two phenomena, c and h, occur in nature*". …. the proposed strategy in (UnA1) leads to the 4-dimensional unit sphere S^3, with its underlying „natural" quaternions numbers. … „*The quaternionic multiplication of a spatiotemporal derivative with electromagnetic potential is given by*

$$\left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \times \left(\varphi, \vec{A}\right) = \frac{\partial \varphi}{\partial t} - \vec{\nabla} \cdot \vec{A} \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\varphi + \vec{\nabla} \times \vec{A} \ .$$

The last two terms precisely match the known expressions for the electric and magnetic fields $\vec{E}$ and $\vec{B}$."

The primarily affected PDO in the context of the proposed quantum gravity theory are the Laplace operator (the baseline operator for Newton's gravity theory), the Prandtl operator, and the (linearized) Boltzmann collision operator.

The proposed quantum gravity theory for an integrated gravity and quantum theory foresees a complexity reduction concerning

- a common Hilbert space framework, i.e. a change from a current metric space framework (equipped with an exterior product, only) to a Hilbert space framework with an inner product (defining a corresponding norm/metric) enabling Hilbert space based (weak) variational representation of Partial Differential Equations (PDE) or Pseudo Differential Equations
- a replacement of Dirac's physical „point charge" model of „ideal functions" $\delta \in \mathrm{H}_{-n/2-\varepsilon}$, ($n$ denotes the space dimension, and $\varepsilon > 0$, (DiP1)) by a „EP point charge" model $\in \mathrm{H}_{-1/2} = L_2 \otimes L_2^{\perp}$.

The only prize to be paid for this complexity reduction is the following:

the (thermo-) statistics Hilbert space L_2 is extended to $H_{-1/2} = L_2 \otimes L_2^{\perp} = H_0 \otimes H_0^{\perp}$, and standard PDE variational representations in the form $u \in D_0(\mathrm{B})\colon (Bu, v)_0, \ \forall\, v \in H_0$, are considered as approximations to extended (weak) variational representations in the form $u \in D_{-1/2}(\mathrm{B})\colon (Bu, v)_{-1/2}, \ \forall\, v \in H_{-1/2}$.

(*) … however, perception is (physical) reality, like the Michelson-Morley experiment, but with different possible interpretations, (SuL) 1.6: (1) Einstein: Maxwell equations describe a (provisionally; see $^{(+)}$, next page) physical law (2) Lorentz: „*light speed is caused by the movements of bodies through the ether*".

(**) … „*Nothing forces us to assume that … clocks have to be seen as running at the same speed*", A. Einstein

(***) A decomposition of a Hilbert space H into an orthonal sum of two spaces H^1 and H^2 with corresponding projection operators P^1 and P^2 enables a definition of a „potential" and a related „potential operator": for x being an element of H its "potential" is about an indefinite metric given by ((VaM) (11.1)) $\varphi(x) \coloneqq \left((x)\right)^2 = \|P^1x\|^2 - \|P^2x\|^2$ with a related potential operator $\boldsymbol{W}(x)$ in the form (VaM) (11.4) $\boldsymbol{W}(x) := \frac{1}{2}\mathrm{grad}\varphi(x) \coloneqq P^1(x) - P^2(x)$.

The proposed quantum gravity model reduces the zoo of elementary particles of the SMEP to one single EP (which Plemelj called „mass element") with or w/o existing (mathematically defined) classical density, while only the first one can be affected resp. allows the definition of kinematical notions in correspondingly defined PDE models. The impact on the proposed one-single EP model and a revisited Newton potential equation in a weak $\mathrm{H}_{-1/2} = L_2 \otimes L_2^{\perp}$ based framework puts the spot on *„Einstein's lost key"*, (EiA3), (UnA), which is about the concept of a *variable speed of light* based on clocks of various types at points with different gravitation potentials (UnA), (UnA1) (*).

In (DeH) it is pointed out that the Mach principle is a cosmological principle, which, as there are multiple cosmological models, it becomes also a selection principle to select the few physical relevant cosmological models. Therefore, in the sense of Kant, it it not a *„constitutive"* principle (like the general co-variance of the field equations), but a *„regulative"* principle. In this sense the Mach principle is a principle for the very large (cosmology). In (DeH) it is also pointed out that the Planck action constant is independent from from any weak or strong gravitation field. It somehow mirrors the fundamental difference of physical mraco and micro world (**). Schrödinger's formula (ScE3) told us, that the negative potential of the total mass of the universe at a given point of observation (calculated with the valid graviation constant G at this point) corresponds to half of the quadrat of the speed of light, $\frac{1}{2} \cdot c^2$. This approach of Schrödinger in (ScE3) was rediscovered by R. Dicke, (DiR), (UnA1); (we note that the momentum is given by $\frac{\mathrm{m}}{2} \cdot v^2$) (***).

One of the central notions in theoretical physics is about the „potential", which is more specifically about a „potential", „potential functions" and „potential operators", e.g. (ChJ), (SuL), (VaM). In case of the Poission equation the potential function is about the solution of the Poisson equation, the Laplace operator $-\Delta := -div(grad)$ (with appropriately defined Dirichlet or Neumann boundary conditions, as part of the underlying operator domain), is about the potential operator, and the potential $p(u)$ itself of the potential function u is defined by $p(u) := \frac{1}{2} \|u\|_{H_1}^2$.

The proposed quantum element „energy" concept is about the sum $\|x\|_{1/2}^2 = \|x_0\|_1^2 + \|x_0^{\perp}\|_{1/2}^2$, whereby x denotes a quantum element $x = x_0 + x_0^{\perp} \in H_{-1/2}$ with its related quantum element energy $x_e^2 = e_0^2 + (e_0^{\perp})^2 = \|x_0\|_1^2 + \|x_0^{\perp}\|_{1/2}^2$. A decomposition of a Hilbert space H into an orthonal sum of two spaces H^1 and H^2 with corresponding projection operators P^1 and P^2 enables a definition of a *„potential"* $\varphi(x)$ and a related *„potential operator"* $\boldsymbol{W}(x)$, (VaM) (11.1), (11.4), $\varphi(x) := \big((x)\big)^2 = \|P^1 x\|^2 - \|P^2 x\|^2$ and $\boldsymbol{W}(x) := \frac{1}{2}\mathrm{grad}(\varphi(x)) := P^1(x) - P^2(x)$. The potential criterion $\varphi(x) = c_1 > 0$ defines a manifold, which represents a hyperboloid in the Hilbert space H with corresponding hyperbolic and conical regions.

Regarding the *„important conjecture"*, $h \sim \pi/2 \cdot c \cdot m_p \cdot r_p$, (see (UnA1) chapter 6), we note that the potential criterion above can be used to model the two *„constitutive"* principles: (1) there is no gravitation potential between a proton p and an electron e : this „world" is governed by the Planck action constant; (2) the gravitation potential of an atom is calculated by the two radius' $r_1 := \overline{p,e}$ and $r_2 := R_U$: this „world" is *approximately* governed by the parameters derived from the Mach principle, including the speed of light accompanied with the concepts of time and events. In this model the further below proposed repulsive/attractive model replaces Dirac's spin concept to model the Pauli exclusion principle.

(+) (EiA) p. 52: *„Die Gesetze aber, nach denen sich Ströme und Ladungen verhalten, sind uns nicht bekannt. Wir wissen wohl, dass die Elektrizitäten in Elementarkörperchen (Elektronen, positiven Kernen) bestehen, aber wir begreifen es nicht vom theoretischen Standpunkt aus. Wir kennen die energetischen Faktoren nicht, welche die Anordnung der Elektrizität in Körperchen von bestimmter Grösse und Ladung bewirken, und alle Versuche, die Theorie nach dieser Seite hin zu vervollständigen, sind bisher gescheitert. Wir kennen daher, falls wir überhaupt die Maxwellschen Gleichungen zugrunde legen dürfen, den Energietensor für die elektromagnetischen Felder nur ausserhalb der Elementarteilchen. An diesen Stellen, den einzigen, wo wir einen vollständigen Ausdruck für den Energietensor aufgestellt zu haben glauben, gilt $\frac{\partial T_{ij}}{\partial x_j} = 0$. „... p.54: „wir wissen heute, dass die Materie aus elektrischen Elementarteilchen aufgebaut ist, sind aber nicht im Besitz der Feldgesetze, auf welchen die Konstitution jener Elementarteilchen beruht." ... p. 81. „Für ein Feldgesetz der Gravitation muss die Poissongleichung der Newtontheorie zum Muster dienen. ... Die Untersuchungen der speziellen Relativitätstheorie haben uns gezeigt, dass an die Stelle des Skalars der Massendichte der Tensor der Energiedichte zu treten hat. In diesem ist nicht nur der Tensor der Energie der ponderabeln Materie, sondern auch der der elektromagnetischen Energie enthalten. Wir haben sogar gesehen, dass unter dem Gesichtspunkte einer tieferen Analyse der Energietensor der Materie nur ein vorläufiges, wenig tiefgreifendes Darstellungsmittel für die Materie anzusehen ist. In Wahrheit besteht ja die Materie aus elektrischen Elementarteilchen und ist selbst Teil, ja als der Hauptteil des elektromagnetischen Feldes anzusehen. Nur der Umstand, dass die wahren Gesetze des elektromagnetischen Feldes für sehr intensive Felder noch nicht hinreichend bekannt sind, zwingt uns vorläufig dazu, die wahre Struktur dieses Tensors bei der Darstellung der Theorie unbestimmt zu lassen.*"

(UnA1): The expansion and the total mass of the universe are related to Newton's gravitation constant G. Einstein's equivalence principle (inert mass is equivalent to heavy mass) corresponds to the Mach principle, stating that the route cause of gravitation is the result of the total mass of the universe. Schrödinger's formula (ScE3) told us, that the negative potential of the total mass of the universe at a given point of observation (calculated with the valid graviation constant G at this point) corresponds to half of the quadrat of the speed of light, $\frac{1}{2} \cdot c^2$; we note that the momentum is given by $\frac{\mathrm{m}}{2} \cdot v^2$.

(EiA3): *„Nach dem soeben Gesagten müssen wir aber an Stellen verschiedenen Gravitationspotentials uns verschieden beschaffener Uhren zur Zeitmessung bedienen. Wir müssen zur Zeitmessung an einem Orte, der relativ zum Koordinatenursprung das Gravitationspotential φ besitzt, eine Uhr verwenden, die – an den Koordinatenursprung versetzt - $(1 + \frac{\varphi}{c^2})$ mal langsamer läuft als jene Uhr, mit welcher am Koordinatenursprung die Zeit gemessen wird. Nennen wir c_0 die Lichtgeschwindigkeit im Koordinatenursprung, so wird daher die Lichtgeschwindigkeit c in einem Orte vom Gravitationspotential φ durch die Beziehung $c = c_0(1 + \frac{\varphi}{c^2})$ gegeben sein. Das Prinzip von der Konstanz der Lichtgeschwindigkeit gilt nach dieser Theorie nicht in derjenigen Fassung, wie es der gewöhnlichen Relativitätstheorie zugrunde gelegt zu werden pflegt.*".

(*) From (UnA) chapter 8 we quote: „The article (DeH) does no less than explain all known tests of the theory with variable speed of light!".

(**) In this context we quote the very last two sentences in (DEH): „The quantum theory gets primacy regarding the classical theory with its most perfect design, the general relativity theory. Therefore, the laws of of the metric field, which are in principle independent from the laws of the quantum theory, have no absolute validity. The regularity of the metric field – indeed in a statistical way – would be tied with elementary particle interaction, like it is furthermore „located" in the sense of the Mach principle."

(***) (UnA), p. 78: „The principle of the constancy of the speed of light can be maintained only by restricting to space-time regions with a constant gravitational potential", Annalen der Physik 38 (1912) p. 355-369; (UnA) p. 121: „Einstein must also have assumed the coincidence $\frac{c^2}{G} \sim \frac{M_{universe}}{R_{universe}}$, i.e. $G \sim c^2 \frac{R_{universe}}{M_{universe}}$"; this means that gravitation constant is related to the total mass of the universe, (which puts the spot on Mach's principle), i.e. the gravitation constant is that small, because the total of the universe is that large (see also (BaJ), (ScD); the formula $c^2/G \sim M_U/R_U$ allows an alternative interpretation of the observed deviation of the forecasted and measured speed of the Pioneer sondes (KrK), (ScL), (ToV); see also (UnA2).

The cosmology field equation model in (DeH) accompanied by properly defined domains enables weak variational representations in a Hilbert space framework. The calculus of variations is analogous to the elasticity theory regarding stress and strain tensors accompanied by the two Korn's inequalities, (see e.g. (VeW)).

According to the described meaning of the „Mach principle" the corresponding physical meaning of the classical space-time continuum framework is „just" a continuum approximation of the action of elementary particle interactions, (DeH). However, the weak variational representation of the considered extended Einstein SRT-Newton model enables a common modelling Hilbert space based framework with the quantum mechanics and Feynman's related quantum electrodynamics.

(UnA2): *„Feyman's theory worked so well that particle physicists decided to use it as a blue print for all other interactions*". It results into a particle zoo, which is about 36 hadrons, 6 leptons and anti partners, W- and Z-particles and another series of colorful gluons (*).

The extended framework $\mathrm{H}_{-1/2} = L_2 \otimes L_2^{\perp}$ enables an „only" two-type „elementary particle *elements*" model, which is about Hilbert space elements with („fermions") and w/o („bosons") kinematical energy.

The in (DeH) proposed linearized (Lorentz-invariant) field equations contain the Newton/Poisson equation

$$\Delta\gamma_{44} = \left[\frac{8\pi G}{c^4}\right]\rho c^2 .$$

In order to ensure a unique solution corresponding boundary conditions are required. In the context of radiation and transport partial differential equations the Neumann boundary condition is considered as more problem adequate than the Dirichlet boundary condition.

The Neumann potential operator is related to the Prandtl operator

$$(\prod v)(x) := \frac{1}{4\pi} \oiint_S v(y) \frac{\cos\phi_{xy}}{|x-y|^2} dS_y = f(x),$$

when seeking the solution of the Neumann boundary value problem

$$(*) \qquad \begin{aligned} \Delta u &= 0 && \text{in } R^3 - S \\ \frac{\partial u}{\partial n} &= f && \text{on } S. \end{aligned}$$

With respect to the proposed energy Hilbert space $H_{1/2}$ we note that the Prandtl operator

$$\prod : \; H_{1/2} \rightarrow H_{-1/2}$$

is bounded, the solution function is represented as double layer potential

$$u(x) := \frac{1}{4\pi} \oiint_S v(y) \frac{\cos\phi_{xy}}{|x-y|^2} dS_y \in H_1(R^3 - S),$$

with unknown function $v(y)$ to be determined by the Neumnn problem, and the exterior Neumann problem admits one and only on generalized solution, (LiI), chapter 4.

(*) (UnA2): *„But unlike quantum electrodynamics, the results of its extension to nuclear physics, called quantum chromodynamics, are anything but precise. … Thought the experimental agreement is disappointing, usually the „uniformity" is praised as a flash of inspiriation. Very funny is the comment of Feynman on how his own ideas were pushed to a too general level:*

> *„So when some fool physics gives a lecture at UCLA (University of California Los Angeles) in 1983 and says, „This is the way it works, and look how wonderfully similar the theories are," it's not because Nature is really similar; it's because the physicists habe only been able to think of the same damn thing, over and over again."*

As Murray Gell-Mann frankly admitted during a talk in Munich 2008, Heisenberg considered the entire idea of fractional charges assigned to quarks to be nonsense. … It is unlikely that he felt biases against fractional quantities, as he, in his freshman years, had proposed the famous „half-integer spin" on an electron, which back then stood in sharp contrast to the established wisdom. However, half-integer spins make sense observationally, whereas no one has ever seen a fraction of a charge."

Half of the four Maxwell equations,

$$div(\vec{B}) = 0 \ , \ rot(\vec{E}) + \frac{\partial}{\partial t}\vec{B} = 0 \ ,$$

are „just" a mathematical consequence of the definition of the magnetic field $\vec{B}$. They are derived via a differentiating process, applying the div- resp. the rot-operator to the definition of the magnetic field $\vec{B} := rot\vec{A}$, whereby $\vec{A}$ denotes an arbitrary (differentiable) vector field. In other words, there are no magnetic charges foreseen telling the fields, how to vary, (SuL).

The other half of the Maxwell equations,

$$div(\vec{E}) = \rho, \ rot(\vec{B}) - \frac{\partial}{\partial t}\vec{E} = \vec{j} \ ,$$

are the consequences of a more specifically defined vector field $\vec{A}$. In this case there is an underlying scalar field of $\vec{A}$ regarding the time variable, reflecting the space-time geometry structure. It enables the definition of an electric field $\vec{E}$ given by, (SuL)

$$\vec{E} := -\frac{\partial\vec{A}}{\partial t} - grad(A_0).$$

In other words, only electric charges tell the electro-magnetic fields, how to vary. Reversely, there is only the Lorentz force

$$\vec{F} = e(\vec{v} \times \vec{B}),$$

where „the magnetic field tells the electrons, how to move". From a physical modelling perspective, this „imbalance" challenge has been overcome by the concept of „displacement current". The Maxwell equations provided the baseline concepts for Einstein's gravity theory.

The electrodynamic in the special relativity theory is described by the four-vector formalism of the space-time given by the D'Alembert operator equation,

$$(\frac{\partial^2}{\partial t^2} - \Delta)\vec{A} = \frac{4\pi}{c}\vec{j},$$

with the four-vector potential $\vec{A}$, where its curvature determines the electric and magnetic field forces, and j denotes the four-current-density.

Regarding the physical notions of „*flux*" and „*mass element*" we refer to the extended definitions from J. PLemelj (PlJ). Plemelj's (Neumann boundary condition based) notion „*flux*" is defined by $\bar{U}(\sigma) := -\oint_{\sigma_0}^{\sigma} \frac{\partial U}{\partial n} d\sigma$ $(\sigma_0, \sigma_0 \in surface)$, whereby $\bar{U}$ relates to the conjugate of $U(\sigma)$, resp. its Hilbert transform. In case $\bar{U}(\sigma)$ is differentiable, this „*flux*" definition corresponds to the standard Neumann boundary operator $\frac{d\bar{U}(\sigma)}{d\sigma} = -\frac{dU}{dn}$. However, in case $\frac{dU}{dn}$ is not defined (i.e. $\bar{U}(\sigma)$ is not differentiable), the „*flux*" $\bar{U}(\sigma)$ is a still well defined term. Plemelj's concept was developed for the logarithmic potential $(n = 2)$, which is related to the Cauchy-Riemann Differential equations. The generalization to dimensions $n > 2$ $(divA = 0, rotA = 0$ (RuC)) leads to the concept of Riesz transforms (StE1).

Mathematically speaking, quantum theory is about a Hilbert space based linear operator theory. Sobolev space based classical (non-linear) partial differential operators can be equivalently re-formulated in a weak variational form. The Sobolev baseline Hilbert space, the Lebesgue space $L_2 =: H_0$, is reflexive with respect to its underlying inner product $(u,v)_0 := (u,v)_{L_2}$. In case a considered non-linear partial differential operator B can be represented in the form $B = A + K + \tilde{K}$ with A linear, self-adjoint and positive definite, A^{-1}, K linear, and $A^{-1}, K, \tilde{K}$ compact on appropriately defined domains, then there exist discrete spectra and corresponding eigen-function based orthogonal systems, enabling the definition of corresponding isomorph Hilbert scales $H_\beta^{(A^{-1})}$, $H_\beta^{(K)}$, $\beta \in R$. For a related variational calculus, which can be applied to Hamiltonian systems, nonlinear wave equations and problems related to surface of prescribed mean curvature (i.e. going far beyond purely elliptic PDE), we refer to (ChJ).

The considered decompositions $H_{-1/2} = L_2 \otimes L_2^\perp$ resp. $H_{1/2} = H_1 \otimes H_1^\perp$ are about the „coarse-grained" (discrete spectrum/orthogonal eigenfunctions based) Hilbert space L_2 resp. H_1, and closed sub-spaces $L_2^\perp$ resp. $H_1^\perp$ of $H_{-1/2}$.

> (WeH) p. 171: „*On the basis of rather convincing general considerations, G. Mie in 1912 pointed out a way of modifying the Maxwell equations in such a manner that they might possibly solve the problem of matter, by explaining why the field possesses a granular structure and why the knots of energy remain intact in spite of the back-and-forth flux of energy and momentum*".

In the Einstein's field equations „space-time geometry tells mass-energy how to move" and „mass-energy tells space-time geometry how to curve". In the Maxwell equations „charges tell the electromagnetic fields how to vary". Usually both equations systems are considered w/o any boundary or initial value conditions; but such conditions are prerequites to ensure well defined problems (*).

The Einstein operator is given by $G = R_{ik} - R\frac{g_{ik}}{2}$ with the corresponding gravity field equations $G = -\kappa T_{ik}$ and the corresponding motion equations $\frac{d}{d\tau}\left(g_{\mu,\nu}\frac{dx^{\mu}}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\frac{\partial x^{\alpha}}{\partial\tau}\frac{\partial x^{\beta}}{\partial\tau}$ for the path $x^{\mu} = x^{\mu}(t)$ of a particle.

The change from the Newton model is about a change from the Newton potential equation $-\Delta\Phi = -4\pi k\rho$ (applying the Dirac (delta) function on the right side of the PDE) to the Einstein equation $G = -\kappa T_{ik}$, going along with a change from the motion equations from

$$\frac{d^2\vec{x}}{dt^2} = -grad\Phi \qquad \to \qquad \frac{d}{d\tau}\left(g_{\mu,\nu}\frac{dx^{\mu}}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\frac{\partial x^{\alpha}}{\partial\tau}\frac{\partial x^{\beta}}{\partial\tau}.$$

Instead of one potential equation we now have 10 equations with 10 potentials Φ_{ik}; instead of a linear operator, we now have a non-linear operator, i.e. the gravity potential is no longer the sum of single gravitation potentials. Additionally, there is a circle structure, i.e. the potentials are functions of the T_{ik} ($\Phi_{ik} = f(T_{ik})$), while the space-time structure are functions of the potentials ($f(\Phi_{ik})$). The matter, as described by the energy-momentum tensor T_{ik}, reflecting the principles of energy and momentum conservation, generates a curvature of the space-time and particles move along of geodesics. Therefore, things become more complicate, a circle principle is added (the stage enables the actors, while the actors build resp. influence the stage), and the PDE model is no longer well defined (no boundary and initial values, etc.), the vaccum energy problem occurs, … and all just to achieve an improved ($\sim 10^{-7} \to \sim 10^{-14}$) mathematical model being validated by very few observed gravitational effects. On the other side there were/are alternative gravitation models existing, (UnA1):

> (DiR): „*The great difficulty with constructing a theory of gravitation is the paucity of experimental evidence. After 40 years there are still only four famous observational checks of the theory of relativity. Of these only two have any real accuracy*".

Einstein concluded the constancy of the light velocity proclaiming that the Maxwell equations describe a physical law, later „confirmed" by the Michelson-Morley experiment. His great physical achievement in this context was the discovery of the space-time symmetry structure of this assumed physical law given by the Lorentz transform. Lorentz himself did not accepted this physical law, he only considered it as approximation to whatever. Space-time structure is the mathematical pre-requisite defining the electric field. Therefore, proclaiming the conclusion „*space-time symmetry*" out of the Maxwell equation proclaimed as a physical principle is a kind of self-fulfilling prophecy.

Lorentz's interpretation of the Michelson-Morley experiment was, (SuL) 1.6:

„*light speed is caused by the movements of bodies through the ether*".

Unzicker A., „*Vom Urknall zum Durchknall*" (english: Bankrupting Physics.), 2009: Dirac applied the special relativity theory to the Schrödinger equation leading to the attribute of „*spin*" of an elementary particle. The ratio of the masses of a proton and an electron (about $1836{,}15\ldots$) are still w/o any model explanations. The Planck (action quantum) constant h corresponds approximationally to $h \sim c \cdot m_p \cdot r_p$, wherehy $r_p \sim 1{,}3 \cdot 10^{-15}m$. It leads Dirac to an estimate of the total number of elementary particle in the universe $\frac{M_{universe}}{m_{proton}} \sim 10^{80}$, (DiP).

An essential concept of the 'standard' model of cosmology is „dark energy". Its existence is postulated to explain the cosmic acceleration, inferred from the Hubble diagram of Type Ia supernovae data. In (NiJT) it is shown that those data are still quite consistent with a constant rate of expansion (***).

(*) How the magnetic field of the earth, which enabled and still ensures all „life" on earth, has been formed, while its existence is obviously guaranteed by the dynamo effort caused by the inertly rotation of the (thermodynamical generated) hot core of the earth? In the theory of Sciama „on the origin of inertia" (ScD) (see also (UnA)), inertial effects arise from the gravitational field of a moving body, where for simplicity, gravitational effects are calculated in flat space-time by means of Maxwell-type field equations. One considered case of possible motion of this system, in which the universe and body rotate with constant angular velocity about an axis through the centre of the body perpendicular to the line joining it to the particle, is modelled by a not zero gravomagnetic field, defined by a magnetic field of a rotating charge distribution in the form $\vec{H} = curl\vec{A} = 2\vec{\omega}$.

(**) (DiP) The modern study of cosmology is dominated by Hubble's observations of a shift to the red in the spectra of the spiral nebulae—the farthest parts of the universe—indicating that they are receding from us with velocities proportional to their distances from us. These observations show us, in the first place, that all the matter in a particular part of space has the same velocity (to a certain degree of accuracy) and suggest a model of the universe in which there is a natural velocity for the matter at any point, varying continuously from one point to a neighbouring point. Referred to a four-dimensional space-time picture, this natural velocity provides us with a preferred time-axis at each point, namely, the time-axis with respect to which the matter in the neighbourhood of the point is at rest. By measuring along this preferred time-axis we get an absolute measure of time, called the epoch. Such ideas of a preferred time-axis and absolute time depart very much from the principles of both special and general relativity and lead one to expect that relativity will play only a subsidiary role in the subject of cosmology. This first point of view, which differs markedly from that of the early workers in this field, has been much emphasized recently by Milne.

(DiP1) One of the most attractive ideas in the Lorentz model of the electron, the idea that all mass is of electromagnetic origin, appears at the present time to be worng, for two separate reasons. First, the discovery of the neutron has provided us with a form of mass which it is very hard to believe could be of electromagnetic nature. Secondly, we have the theory of the positron a theory in agreement with experiment so far it is known – in which positive and negative values for the mass of an electron play symmetrical roles. This cannot be fitted in which the electromagnetic idea of mass, which insists on all mass being positive, even in abstract theory. … We are faced with the difficulty that, if we accept Maxwell's theory, the field in the immediate neighborhood of the electron has an infinite mass.

(***) The 'standard' model of cosmology is founded on the basis that the expansion rate of the universe is accelerating at present — as was inferred originally from the Hubble diagram of Type Ia supernovae. There exists now a much bigger database of supernovae so we can perform rigorous statistical tests to check whether these 'standardisable candles' indeed indicate cosmic acceleration. Taking account of the empirical procedure by which corrections are made to their absolute magnitudes to allow for the varying shape of the light curve and extinction by dust, we find, rather surprisingly, that the data are still quite consistent with a constant rate of expansion.

The general solution of the Schrödinger equation is given by

$$\phi(\vec{x},t)=\sum_n c_n e^{-i\lambda_n(\frac{h}{2\pi})t}\phi_n(\vec{x}).$$

The Schrödinger field equation for the electrons wave functions $\psi(\vec{x},t)$ reflects in the right way the experimental verified relationship between the group velocity and the wave number. The wave functions themself do have no physical meaning. But the intensities of fields, as e.g. (from Maxwell theory) the energy density and the Poynting vector or (from quantum mechanics) the Hamiltonian operator of a free string

$$H:=\frac{1}{2\rho}P_0^2+\sum_1^\infty\frac{1}{2\rho}P_n^2+\frac{\rho}{2l}\omega_n^2Q_n^2=\frac{1}{2\rho}P_0^2+\frac{1}{2}\sum_1^\infty\hbar\omega_n(A_n^*A_n+A_n^*A_n)$$

i.e.

$$H=\frac{1}{2\rho}P_0^2+\sum_1^\infty\hbar\omega_nA_n^*A_n+\frac{1}{2}\sum_1^\infty\hbar\omega_n.$$

are modeled as squares of field quantities. We note that the series

$$E_0:=\frac{1}{2}\sum_1^\infty\hbar\omega_n$$

is divergent. The current interpretation of the "square concept" above is that the quantity

$$\rho(x,t):=|\psi(\vec{x},t)|^2$$

models *the density of the matter field of electrons*. Based on this interpretation the continuity equations (which is the Schrödinger equation) is given by

$$\dot{\rho}+div(\frac{\hbar}{2mi}\psi^\bullet\vec{\nabla}\psi-\psi\vec{\nabla}\psi^\bullet)=0.$$

The ground state energy is not measurable. Its chosen value is therefore arbitrarily, motivated by the fact, to keep calculations as easily as possible, and, mainly, to ensure convergent integrals/series. Energies of freely composed systems should be additive. For photons in a box section (cavity) there are infinite numbers of frequencies ω_i. If one assigns any frequency a ground state energy value $\hbar\omega_i/2$, then the ground state energy without photons has the infinite energy

$$\frac{1}{2}\sum_i\hbar\omega_i=\infty.$$

The miss-understanding, that that the ground state energy is fixed and uniquely defined, starts already in the classical physics: The definition of the Hamiltonian

$$H=\frac{p^2}{2m}+\frac{1}{2}\omega^2x^2=:T+V$$

defines the not measurable ground state energy in that way, that the state of lowest energy, the points $(x,p)=(0,0)$ in the phase space, is *defined* as „zero". The underlying quantum mechanics model is about *Hermitian operators*, *physical observables* and *wave packages*.

The spectrum of a hermitian, positive definite operator $A: D(A)\rightarrow H$ with domain $D(A)$ in a complex-valued Hilbert space H is discrete. This property enables an axiomatic building of the quantum mechanics, whereby, roughly speaking, physical states are modeled by the elements of the Hilbert space, observables of states are modelled by the hermitian operator A and the mean value of the observable A at the state ψ with $\|\psi\|=1$ is modelled by the inner product $\langle A\psi,\psi\rangle$.

In other words, the expectation value of an operator $\hat{A}$ is given by

$$\langle A\rangle=\int\psi^*(\vec{r})\cdot\hat{A}[\psi](\vec{r})d\vec{r}$$

and all physical observables are represented by such expectation values. Obviously, the value of a physical observable such as energy or density must be real, so it's required $\langle A\rangle$ to be real. This means that it must be $\langle A\rangle=\langle A\rangle^*$, or

$$\langle A\rangle=\int\psi^*(\vec{r})\cdot\hat{A}[\psi](\vec{r})dr=\int\left[\hat{A}\psi(\vec{r})\right]^*\psi(\vec{r})d\vec{r}=\langle A\rangle^*.$$

Operators $\hat{A}$, which satisfy this condition are called *Hermitian*. One can also show that for a Hermitian operator,

$$\int\psi_1^*(\vec{r})\cdot\hat{A}[\psi_2^*](\vec{r})dr=\int\left[\hat{A}\psi_1(\vec{r})\right]^*\psi_2(\vec{r})d\vec{r}$$

for any two states ψ_1 and ψ_2 .

For the eigenvalue problem of a self-adjoint, positive operator A

$$A\phi = \lambda\phi$$

the eigenvalues $\{\lambda\}$ are the discrete spectrum λ_n with either finite or countable infinite set of values

$$A\phi_n = \lambda\phi_n \ , \ \|\phi_n\|^2 = 1.$$

In this case the mean value $\bar{A}$ of A is given by

$$\bar{A} := \langle A\psi, \psi\rangle$$

Let w_n the probability, that the eigenvalue occurs of a measurement of the observables A then it holds for the mean value $\bar{A}$ of A

$$\bar{A} := \sum_n w_n\lambda_n = \sum_n w_n\langle\phi_n, A\phi_n\rangle \quad , \ \phi = \sum_n \alpha_n\phi_n.$$

Because of

$$\bar{A} = \langle\psi, A\psi\rangle = \langle\sum_n \alpha_n\phi_n \, , A(\sum_n \alpha_n\phi_n)\rangle = \sum_n \alpha_n^*\alpha_n\langle\phi_n, A\phi_n\rangle = \sum_n \alpha_n^*\alpha_n\lambda_n\langle\phi_n, \phi_n\rangle = \sum_n \alpha_n^*\alpha_n\lambda_n$$

it follows

$$\bar{A} = \sum_n \alpha_n^*\alpha_n\lambda_n$$

i.e.

$$w_n = |\alpha_n|^2 = |\langle\phi_n, \phi\rangle|^2 \ .$$

In case the operator A is only hermitian (without being positive definite resp. A^{-1} is not compact), Hilbert, von Neumann and Dirac developed a corresponding spectral theory. It leads to a *continuous spectrum* $\lambda(v)$, indexed by a continuous v. In this case $\phi(\vec{x}; v) = \phi_v(\vec{x})$ denotes an eigen-function to the eigen-value $\lambda(v)$. The norm of this function is infinite, i.e. the function is not an element of the Hilbert space. An *approximation* to this function with finite norm is given (for sufficiently small Δv) by the *eigen-differential*

$$\phi_{\Delta v}(\vec{x}) = \frac{1}{\Delta v}\int_{v-\Delta v/2}^{v+\Delta v/2} \varphi(x; v')dv'.$$

All for the Hilbert space related properties are valid for the eigen-differentials, but not for the eigen-function itself. The scalar product of the eigen-function is „normed" to a Dirac δ-function by

$$\langle\varphi(x; v'), \varphi(x; v'')\rangle = \delta(v' - v'').$$

The norm of the related eigen-differentials is given by

$$\langle\phi_{\Delta v}(\vec{x}, v), \phi_{\Delta v}(\vec{x}, v')\rangle = \frac{1}{\Delta v}\int_{v-\Delta v/2}^{v+\Delta v/2}\int_{v'-\Delta v/2}^{v'+\Delta v/2} \varphi(x; \mu')\,\varphi(x; \mu'')d\mu' d\mu'' = \frac{1}{\Delta v}\int_{v-\Delta v/2}^{v+\Delta v/2} \delta(\mu' - \mu'').$$

The integral is 1 for $v = v'$ (with appropriate norm factor) and is 0 if $|v - v'| > \Delta v$.

In case if v is a momentum the eigen-differential gives a wave package with finite distance Δv in the momentum space and therefore with finite distance $\Delta x \sim \frac{1}{\Delta v}$ in the particle space. Such a package can be normed to the value 1 (1 particle). Δx (and correspondingly Δv) has to be larger than all other typical distances of the problem. In this sense eigen-differentials correspond to the formalism of wave package modelling.

The following eigenpair relations are valid:

$$A\phi_i = \lambda_i\phi_i \qquad A\phi_\lambda = \lambda\phi_\lambda \qquad \|\phi_\lambda\|^2 = \infty \ , (\phi_\lambda, \phi_\mu) = \delta(\phi_\lambda - \phi_\mu).$$

The ϕ_λ are not elements of the Hilbert space. The so-called eigen-differentials are built as superposition of such eigenfunctions.

The eigen-functions of the discrete and continuous spectrum build an extended Hilbert space to ensure that for every ψ it holds

$$\psi(x)=\sum_n \alpha_n\varphi_n+\int c(v')\varphi(x;v')dv'.$$

With

$$c_n:=\langle\varphi_n(x),\psi(x)\rangle\ ,\ c(v):=\langle\varphi(x;v),\psi(x)\rangle$$

it holds the Parceval identity

$$\langle\psi,\psi\rangle=\sum_n|c_n|^2+\int|c(v')|^2\,dv',$$

and the eigen-differential are orthogonal wave packages.

If for every L_2- function such a representation is possible, one call the system $\{\varphi\}$ a complete orthogonal system.

Such a complete orthogonal system $\{\varphi\}$ is not uniquely defined.

There is always the degree of freedom

- *to choose arbitrarily the phase of each eigen-function*
- the set of the non-standard eigenvalues can be orthogonalized on different ways
- to replace the index v of the continuous spectrum by an index $\mu(v)$ with $\mu(v)$ differentiable, monotone function of v. Then

$$\varphi(x;\mu)=\frac{\varphi(x;v)}{\sqrt{d\mu/dv}}.$$

For not all hermitian operators there exist a complete orthogonal system of eigen-functions. For all operators, which represent physical observables, there exist a complete orthogonal system.

The building of Hilbert scales is based on the Friedrichs extension of the domain of hermitian operators. Those domains can be extented to energetic Hilbert spaces (where the domain of the Hermitian operator is densely embedded into the energetic Hilbert space), that the symmetric operator is extented to a self-adjoint operator. The corresponding eigen-pairs of the constructed self-adjoint operator enable the definition of a Hilbert scale.

The not well defined wave package concept (only approximation solutions, divergent norms, the underlying wave functions themself do have no physical meaning, the regularity of the Dirac „function" depends from the space dimension, index v of the continuous spectrum can be replaced by an index $\mu(v)$ …) is replaced by a quantum element Hilbert space $H_{-1/2}=L_2\otimes L_2^{\perp}=H_0\otimes H_0^{\perp}$ accompanied by a corresponding quantum energy Hilbert space $H_{1/2}=H_1\otimes H_1^{\perp}$. Both Hilbert spaces are decomposed into classical Hilbert subspaces L_2 and H_1 (allowing „physical observables" modelling) and related complementary subspaces, modelling not measurable physical relevant notions like quantum elements w/o density or ground state energy.

The quantum mechanics „energy density" concept (basically the H_1-norm of the potential function) is replaced by a sum given by

$$\|x\|_{1/2}^2=\|x_0\|_1^2+\|x_0^{\perp}\|_{1/2}^2\ ,$$

whereby x denotes a quantum element $x=x_0+x_0^{\perp}\in H_{-1/2}$ with its related quantum energy

$$e=\sqrt{e_0^2+(e_0^{\perp})^2}=\sqrt{\|x_0\|_1^2+\|x_0^{\perp}\|_{1/2}^2}.$$

The prominent examples of Hermitian operators are the Laplace operator (to model the elastic energy of the string) and the single layer (singular Symm integral) potential operator S to formulate the boundary integral equations of the homogeneous Dirichlet or Neumann boundary value problem.

The transport type and the Maxwell equations are also concerned with the gradient operator ∇, which is only skew symmetric. However, the properties of the Hilbert transform operator (e.g. skew symmetric, isometric, rotation invariant), and its related Riesz transform operators for space dimensions $n\geq 2$), enable an inner product definition (coming along with a corresponding „energetical" domain extension) in case the derivative operator $u\rightarrow u'$ is replaced by $u\rightarrow(Hu)'$. In the above propose $H_{-1/2}$-based variational representation this goes along with a replacement in the form

$$(u',v)_{-1/2}=(u',Sv)_0\ \rightarrow\ ([Hu]',v)_{-1/2}=(Hu',v)_{-1/2}=(Hu',Sv)_0=(HSu',v)_0=-(HHu,v)_0=(u,v)_0,$$

i.e. the modified differentation operator $u\rightarrow(Hu)'$ with domain H_0 defines an inner product (BrK), (BrK1), (BrK3).

The replacement $u \rightarrow u'$ by $u \rightarrow (Hu)'$ is also proposed to define modified (Schrödinger) differential operators

$$\frac{ih}{2\pi}\left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}\right) \rightarrow \frac{ih}{2\pi}\left(\frac{\partial}{\partial t}, \frac{1}{i}H\left[\frac{\partial}{\partial x}\right]\right).$$

The *energy-momentum* relationship *of a classical non-relativistic particle with mass* m is given by $E^2 = \frac{p^2}{2m} + U$. Substituting the (Schrödinger) differential operators $i(\frac{h}{2\pi})\frac{\partial}{\partial t}$, $-i(\frac{h}{2\pi})\frac{\partial}{\partial x}$ into this equation leads to the wave equation. Because of $H^2 = -I$, substituting the proposed modified (Schrödinger) differential operators into the equation $E^2 = \frac{p^2}{2m} + U$ results into the wave equation in the form

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right](\psi) + U(\psi) = 0 \ .$$

The *energy-momentum* relationship *of a relativistic particle with mass* m is given by $E^2 = p^2 + m^2$. The substitution of the (Schrödinger) differential operators $i(\frac{h}{2\pi})\frac{\partial}{\partial t}$, $-i(\frac{h}{2\pi})\frac{\partial}{\partial x}$ leads to the relativistic Klein-Gordon equation describing spin-0 particles in relativistic quantum field theory. However, the *relativistic particle energy-momentum* relationship allows positive and negative energy solutions $E = \pm\sqrt{p^2 + m^2}$ resulting in negative probability densities $\rho(x,t) := |\psi(\vec{x},t)|^2$. This issue has been addressed by the Dirac equation, in which the time and space derivatives are first order. The Dirac equation can be thought of in terms of a „square root" of the Klein-Grodon equation. We emphasis, that the above *energy-momentum relationship of a relativistic particle,* given by $E^2 = p^2 + m^2$, is derived by applying the Legendre transform, which is only valid in the (corse-grained) H_1 framework. In the proposed $H_{1/2} = H_1 \otimes H_1^{\perp}$ energy Hilbert space framework only the Hamiltonian formalism can be applied. The Hamiltonian and the Lagrange formalisms are only equivalent, if the Legendre transformation can be applied requiring certain regularity assumptions to the underlying domains.

Putting $c := \frac{h}{2\pi m}$, the weak L_2-based variational representation of the Schrödinger equation with potential energy $U(\psi) = 0$ is given by

$$i(\dot{\psi},\varphi)_0 = \frac{c}{2}(-\Delta\psi,\varphi)_0 = \frac{c}{2}(-i\Delta\psi,-i\varphi)_0 = \frac{c}{2}(-i\nabla\psi,-i\nabla\varphi)_0 = -\frac{c}{2}(\nabla\psi,\nabla\varphi)_0 = -\frac{c}{2}(\psi,\varphi)_1 \ \forall\varphi \in H_1,$$

i.e. for $\psi \in H_1$ it holds $\frac{i}{2}\frac{d}{dt}\|\psi\|_0^2 = -\frac{c}{2}(\nabla\psi,\nabla\psi)_0 = -\frac{c}{2}\|\psi\|_1^2$.

Because of $H^2\psi = -\psi$, the corresponding weak $H_{-1/2}$-based variational representation of the modified Schrödinger operator (in case pf space dimension $n = 1$) is given by

$$i(\dot{\psi},\varphi)_{-\frac{1}{2}} = \frac{c}{2}(iH\psi',-i\varphi')_{-1/2} = \frac{c}{2}(iHS\psi',-i\varphi')_0 = \frac{c}{2}(HS\psi',\varphi')_0 = -\frac{c}{2}(H^2\psi,\varphi')_0 = \frac{c}{2}(\psi,\varphi)_{1/2} \qquad \forall\varphi \in H_{1/2}.$$

whereby S denotes the Symm integral operator. Putting $\varphi \coloneqq \psi \in H_{1/2}$ it follows $\frac{i}{2}\frac{d}{dt}\|\psi\|_{-1/2}^2 = \frac{c}{2}\|\psi\|_{1/2}^2$, i.e. it holds $\frac{d}{dt}\|\psi\|_{-1/2}^2 = \frac{1}{2\pi i}\frac{h}{m}\|\psi\|_{1/2}^2$ for $\psi \in H_{1/2}$. The H_0-based weak definition of the commutator $[x,P][\psi](x) = i\frac{h}{2\pi}[\psi](x)$ is given by

$$([x,P][\psi],\varphi)_0 = i\frac{h}{2\pi}(\psi,\varphi)_0,$$

i.e. it especially holds $([x,P][\psi],\psi)_0 = i\frac{h}{2\pi}\|\psi\|_0^2$.

As P^* is self-adjoint with respect to $H_{-1/2}$, the coresponding weak $H_{-1/2}$-representation of the modified commutator $[x,P^*][\psi](x) = \frac{h}{2\pi}[H,x][\psi'](x)$ is given by

$$(*) \qquad ([x,P^*][\psi],\varphi)_{-1/2} = \frac{h}{2\pi}([H,x][\psi'],\varphi)_{-1/2} = -\frac{h}{2\pi}([H,x]S[\psi'],\varphi)_0 = \frac{h}{2\pi}([H,x]H[\psi],\varphi)_0 \ .$$

For the commutator $[x,H]\theta \coloneqq [xH - Hx]\theta$ it holds

$$[x,H][\theta](x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\theta(y)dy \ .$$

A vanishing constant Fourier term of a function $\theta \in L_2$ is a sufficient criterion that $\theta \in L_2$ is a wavelet. At the same time, the Hilbert transform of every function has a vanishing constant Fourier term. In other words, $H[\psi]$ in (*) above is a wavelet function with vanishing constant Fourier term. It therefore follows that the modified commutator vanishes, i.e.

$$[x,P^*][\psi](x) = 0 \quad \text{in a weak } H_{-1/2}\text{-sense.}$$

The equation indicates related properties of the correspondingly modified creation resp. annihilation operators $\hat{a},\hat{a}^+$, accompanied by the related Hamiltionian function $\hat{H} = \hat{a}^+\hat{a} + \frac{1}{2}$ and the commutator operator property $[\hat{a}\hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$.

The concept of Hilbert scales H_α, $\alpha \in R$, is built on the appropriate hermitian operator properties. The polynomial norms $\|x\|_\alpha^2$ are governed by an exponential $\|x\|_{(t)}^2$-norm, (NiJ), (NiJ1). The approximation "quality" of the specific proposed $H_{-1/2}$ –quantum element Hilbert space with respect to the „observable space" norm of H_0 is governed by the inequality

$$\|x\|_{-1/2}^2 \leq \delta\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2 \text{ , e.g. } \|x\|_{-1/2}^2 \leq t\|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}t} x_i^2 \text{ .}$$

The proposed $H_{1/2}$ –quantum energy Hilbert space overcomes the current challenges of a mathematically not well defined ground state energy model, which are accompanied by the miss-understanding that the ground state energy is fixed and uniquely defined.

The Friedrichs extension (canonial self-adjoint extension of a non-negative densely defined symmetric operator) can be applied to extend potential operators with domains $D(A)$ (a subspace of a Hilbert space H) to self-adjoint operators with an extended domain $D(\tilde{A})$, and $R(\tilde{A}) = H$. They therefore enable a decomposition into an orthogonal sum of two subspaces $H_1 \otimes H_2$ of $D(\tilde{A})$. Regarding non-linear problems we mention, that for a vector space H, the empty set, the space H itself, and any linear subspace of H are convex cones.

A decomposition of a Hilbert space H into an orthonal sum of two spaces H^1 and H^2 with corresponding projection operators P^1 and P^2 enables a definition of a „*potential*" and a related „*potential operator*":

for x being an element of H its "*potential*" is about an indefinite metric given by ((VaM) (11.1))

$$\varphi(x) \coloneqq \big((x)\big)^2 = \|P^1x\|^2 - \|P^2x\|^2$$

with a related potential operator $\boldsymbol{W}(x)$ in the form (VaM) (11.4)

$$\boldsymbol{W}(x) := \frac{1}{2}\operatorname{grad}(\varphi(x)) \coloneqq P^1(x) - P^2(x).$$

The potential criterion $\varphi(x) = c > 0$ defines a manifold, which represents a hyperboloid in the Hilbert space H with corresponding hyperbolic and conical regions.

The theory of Hilbert spaces with an indefinite metric is provided in e.g. (DrM), (AzT), (DrM), (VaM). Following the investigations of Pontrjagin and Iohvidov on linear operators in a Hilbert space with an indefinite inner product, M. G. Krein proved the Pontrjagin-Iohvidov-Krein theorem (FaK). The tool set for an appropropriate generalization of the above "grad" definition in case of non-linear problems is about the (homogeneous, not always linear in h) Gateaux (weak) differential $\boldsymbol{VF}(x,h)$ of a functional $\boldsymbol{F}$ at a point x in the direction h ((VaM) §3)).

If there exists an operator A with $D(A) = H_1$, $R(A) = H_0$ and $\|x\|_1 = \|Ax\|_0$, whereby the operator A is positive definite, self-adjoint and A^{-1} is compact, the corresponding eigenvalue problem $A\varphi_i = \sigma_i\varphi_i$ has infinite solutions $\{\sigma_i, \varphi_i\}$ with $\sigma_i \to \infty$ and $(\varphi_i, \varphi_k) = \delta_{i,k}$. For each element $x \in H_1 = A^{-1}H_0$ it holds the representation

$$x = \sum_{i=1}^{\infty}(x,\varphi_i)\,\varphi_i.$$

Inner products with corresponding norms of a distributional Hilbert scale can be defined based on the eigen-pairs of an appropriately defined operator in the form

$$(x,y)_\alpha := \sum_i^{\infty} \lambda_i^\alpha (x,\phi_i)\,(y,\phi_i) = \sum_i^{\infty} \lambda_i^\alpha x_i y_i \text{ .}$$

Additionally, for $t > 0$ there can be an inner product resp. norm defined for an additional governing Hilbert space with an "exponential decay" behavior in the form $e^{-\sqrt{\lambda_i}t}$ given by

$$(x,y)_{(t)}^2 := \sum_{i=1} e^{-\sqrt{\lambda_i}t}(x,\phi_i)(y,\phi_i) \quad , \|x\|_{(t)}^2 := (x,x)_{(t)}^2 \quad .$$

It enables an approximation theory for (distributional) Hilbert scales $H_{-\alpha}$, $\alpha > 0$, (NiJ), (NiJ1), (*). The essential applied estimate is given by

$$\|x\|_{\rho-\alpha}^2 \leq \delta^{2\alpha}\|x\|_\rho^2 + e^{t/\delta}\|x\|_{\rho.(t)}^2,$$

which follows from the inequality $\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{t(\delta^{-1}-\sqrt{\lambda})}$, being valid for any $t, \delta, \alpha > 0$ and $\lambda \geq 1$. The special choises $\alpha = 1/2$, $\rho = 0$ lead to the above $H_{-1/2}$ related inequality. We note the similarity of the above inner product $(x,y)_{(t)}^2$ to general solution of the Schrödinger equation given by $\phi(\vec{x},t) = \sum_{\boldsymbol{n}} c_n e^{-i\lambda_n(\frac{h}{2\pi})t}\phi_n(\vec{x})$.

(*) Theorem: Let $\alpha < \beta < \gamma$. Then to $t > 0$ and $x \in H_\beta$ there is an approximation $y \in H_\gamma$ according to $\|x-y\|_\alpha \leq t^{\beta-\alpha}\|x\|_\beta$, $\|x-y\|_\beta, \|y\|_\beta \leq \|x\|_\beta$ and $\|y\|_\gamma \leq t^{-(\gamma-\beta)}\|x\|_\beta$.

Regarding the above Hilbert space decompositions there is an analogy to Robinson's hyper-real (ideal) numbers $\sim\ {}^{*}r = r + i$, which can be decomposed into real numbers r and infinitesimal numbers i. Robinson's Non-Standard-Analysis „*marks a new stage of development in several famous and acient paradoxes about infinitely small and large numbers*", started with Euclid (who deliberately excluded both, the infinite and the infinitesimal), until Weierstrass' „limit" concept, building the foundation of current Standard Analysis $^{(*)}$.

The extended field of hyper-real numbers is still an ordered field, but the additional infinitesimal numbers violate the Archimedean axiom. Roughly speaking, the Archimedean axiom is the property of having no infinitely larger or infinitely smaller elements. This axiom can be physically interpreted, as the capability to surpass any distance between zero and a real number $y > x$ by $nx > y$. In simple words, a distance measure is possible.

(DaP) Nonstandard Analysis, pp 237 ff:

> *„Robinson revived the notion of the „infinitesimal", a number that is infinitesimal small yet greater than zero…. In the nineteenth century infinitesimals were driven out of mathematics once and for all, or so it seemed. To meet the demands of logic the infinitesimal calculus of I. Newton and G. W. Leibniz was reformulated by K. Weierstrass without infinitesimals. Yet today it is mathematical logic, in its contemporary sophistication and power, that has revived the infinitesimals and made it acceptable again. …. In Euclid both the infinite and the infinitesimal are deliberately excluded. We read in Euclid that a point is that has a position but no magnitude. …The atomism of Demokrit had been meant to refer not only to matter but also to time and space. But then the arguments of Zenon had made untenable the notion of time as a row of successive instants, ot the line as a row of successive „indivisibles". Aristotle, the founder of systematic logic, banished the infinitely large or small from geometry. …*
>
> *The full flower of infinitesimal reasoning came with the generation after Pascal: Newton, Leibniz, the Bernoulli brothers and I. Euler. The fundamental theorems of the calculus were found by Newton and Leibniz in the 1660s and 1670s. The first textbook on the calculus was written by L'Hospital, …. Here it is state as the outset as an axiom that two quantities differing by an infinitesimal can be considered to be equal. In other words, the quantities are at the same time considered to be equal to each other and not equal to each other! A second axiom states a curve is „the totality of an infinity of straight segments, each infinitesimal small." ….*
>
> *Leibniz did not claim that infinitesimals really existed, only that one could reason without error as if they did exist. … Newton tried to avoid the infinitesimal. ….*
>
> *… Dynamics had become as important as geometry in providing questions for mathematical analysis. The leading problem was the connection between „fluents" and „fluxions," what would today be called the instantaneous position and the instantaneous velocity of a moving body. …*
>
> *… We let dt stand for the infinitesimal increment of time and ds for the corresponding increment of distance. … thus the ratio $\frac{ds}{dt}$ which is the quantity we are trying to find, is equal to $32 + 16dt$. …Since the answer should be a finite quantity, we should like to drop the infinitesimal term $16dt$, and get the asnwer, 32 feet per second, for the instantaneous velocity.*
>
> *… Berkeley declared that the Leibniz produre, simply „considering" $32 + 16dt$ to be „the same" as 32, was unintelligible. „Nor will is avail," he wrote, „to say that (the term neglected) is a quantity exceedingly small, since we are told, that if something neglected, to matter how small, we can no longer claim to have exact velocity but only in approximation. …*
>
> *… To find an instantaneous velocity according to the Weierstrass method we abandon any attempt to compute the speed as a ratio. Instead we define speed as a limit, which approximated the ration of finite increments. … The approach succeeded, ….. We do however, pay a price. The intuitively clear and physically measurable quantity, the instantaneous velocity, becomes subject to the surprisingly subtle notion of „limit". ….*
> *… The reconstruction of the calculus on the basis of the limit concept and its epsilon-delta definition amounted to a reduction of the calculus to the arithmetic of real numbers. … Leibniz had thought of infinitesimals as being infintely small positive or negative numbers that still had „the same properties" as the ordinary numbers of mathematics. On its face the idea seems self-contradictory. …. It was by using a formal language that Robinson was able to resolve the paradoxon. Robinson showed how to construct a system containing infinitesimals that was identical with the system of „real" numbers with respect to all those properties expressible in a certain language".*

Summary

pp. 12 ff.

Braun K., Looking back, part B, (B1)-(B17)

July 30 2020

The Einstein field equations are classical non-linear, hyperbolic PDEs defined on differentable manifolds coming along with the concepts of „affine connexion" and „external product".

The Standard Model of Elementary Particles (SMEP) is basically about a sum of three Langragian equations, one equation, each for the considered three „Nature forces".

Quantum mechanics is basically about matter fields described in a $L_2 = H_0$ Hilbert space framework modelling quantum *„states"* (position and momentum).

Our proposed quantum gravity model is based on a distributional Hilbert scale framework (avoiding the Dirac „function" concept to model a „point" charge, modelled as element of the distributional Hilbert space $H_{-n/2-\varepsilon}$). Certainly, a Hilbert space based quantum gravity requires some goodbyes from current postulates of both theories, the Hilbert space based quantum theory and the metric space based gravitation theory.

The central changes to current quantum theory and gravity theory are :

- as the L_2 Hilbert space is reflexive, the current operator quantum mechanics/dynamics equations can be equivalently represented as variational equations with respect to the L_2 inner product; those variational representations are extended to a newly proposed quantum element Hilbert space $H_{-1/2}$; we note that the regularity of the Dirac function, as an element of the distributional Hilbert space $H_{-n/2-\varepsilon}$, is (in case of space dimension $n=1$) at most an element of $H_{-1/2-\varepsilon}$. In this context, we emphesis, that the main gap of Dirac's related quantum theory of radiation is the small term representing the coupling energy of the atom and the radiation field. Our proposed model omitts this additional „coupling" term

- current classical Partial Differential Equations (PDE) can be also equivalently represented as variational equations with respect to the L_2 inner product; also those variational representations are extended to a newly proposed quantum element Hilbert space $H_{-1/2}$. This extension is accompanied with reduced regularity requirements to the underlying domain oft he considered PDE. We note that the Einstein field equations and the wave equation are hyperbolic PDEs and that PDEs are only well defined in combination with approproiate initial and boundary value functions, a spart of a properly defined domain. From a physical modelling perspective, we note, that the main gap of the Einstein field equations is, that it does not fulfill Leibniz's requirement, that "*there is no space, where no matter exists*". The GRT field equations (usually also not with properly defined domain) provide also solutions for a vaccuum, i.e. the concept of "space-time" does not vanishes in a matter-free universe.

The Friedrichs extension of the classical Laplace operator in a L_2 Hilbert space framework defines the inner product of a related „energy" Hilbert space H_1. The extended Laplace operator in the newly proposed $H_{-1/2}$ framework leads to an extended energy Hilbert space $H_{1/2}$. The new energy Hilbert space $H_{1/2}$ is decomposed into the current "kinematical" energy Hilbert space H_1 (with its corresponding underlying (fermion elements) Hilbert space H_0) and its complementary "ground state" energy Hilbert space $H_1^{\perp}$ (with its corresponding underlying (boson elements) Hilbert space $H_0^{\perp}$), i.e. $H_{1/2} = H_1 \otimes H_1^{\perp}$. The kinematical Hilbert space H_1 can be further decomposed into repulsive and attractive kinematical energy spaces, in alignment with a corresponding underlying decomposition of the (fermion elements) Hilbert space H_0 into repulsive and attractive fermion element spaces.

Mathematically speaking, the decomposition $H_{1/2} = H_1 \otimes H_1^{\perp}$ is about a "coarse grained" Hilbert space H_1 (i.e. it is compactly and densely (with respect to the $H_{1/2}$ norm) embedded into $H_{1/2}$) and its complementary closed (in the sense of Cantor's cardinality measure, very much larger) sub-space $H_1^{\perp}$ of $H_{1/2}$. In the sense of Cantor, the decomposition corresponds to the "decomposition" of the field of real numbers R into rational (countable) numbers Q and irrational (non countable) numbers. We also mention that "distributions" are also called "ideal functions", (CoR) p. 766: the name "distributions" indicates that ideal functions, such that the Dirac delta function and its derivatives, may be interpested by mass distributions, dipole distributions, etc., concentrated in points, or along lines or on surfaces, etc.

The considered Hilbert scale is based on appropriately defined eigen-pair solutions of a problem adequate linear operator A with the properties (1) A selfadjoint, positive definite, (2) A^{-1} compact. The corresponding polyonmial decay norms are enriched by an "exponential decay" inner product resp. norm with parameter $t > 0$, given by (BrK5)

$$(x,y)_{\alpha.(t)} = \sum_k \sigma_k^\alpha e^{-\sqrt{\sigma_k}t}(x,\varphi_k)(y,\varphi_k) \quad , \quad \|x\|_{\alpha.(t)}^2 := (x,x)_{\alpha.(t)}.$$

An element $x = x_0 + x_0^\perp \in H_{-1/2} = H_0 + H_0^\perp$ with $\|x_0\|_0 = 1$ is governed by the norm of its (observation) subspace H_0 and the norm $\theta := \|x_0^-\|_{-1/2}^2$ by, (BrK3), (BrK5),

$$\|x\|_{-1/2}^2 \leq \theta\|x\|_0^2 + \sum_{k=1}^\infty e^{1-\sqrt{\sigma_k}\theta} x_k^2 \ .$$

This norm estimate is a special case of the general inequality ($\beta > 0$ be fixed)

$$\|x\|_{\alpha-\beta}^2 \leq \delta^{2\beta}\|x\|_\alpha^2 + e^{t/\delta}\|x\|_{\alpha.(t)}^2.$$

The proposed quantum gravity model is based on a Hilbert space framework. Wavelet analysis can be used as a mathematical microscope, looking at the details that are added if one goes from a scale "a" to a scale "$a + da$", where " da " is infinitesimally small. We mention that an alternative model for an "a" to a scale "$a + da$ model is the concept of the ordered field of ideal points, an extension to the ordered field of real numbers with same cardinality, but having additionally infinitesimal elements (also called non-Archimedean numbers).

The mathematical microscope wavelet tool 'unfolds' a function over the one-dimensional space R R into a function over the two-dimensional half-plane of "positions" and "details". This two-dimensional parameter space may also be called the position-scale half-plane. The wavelet duality relationship provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "microscope observations" of two wavelet (optics) functions f and g can be compared with each other by the "reproducing" ("duality") formula.

Physically speaking, the "*coarse grained*" (kinematical hyperbolic space-time, matter, action, Shannon entropy governed world) Hilbert space pair $(\mathrm{H}_0, \mathrm{H}_1)$, which is compactly and densely embedded into the (quantum element / quantum energy) Hilbert space pair $(\mathrm{H}_{-1/2}, \mathrm{H}_{1/2})$, and its complementary closed (ground „*state*" elliptic world) sub-space pair $(H_0^\perp, H_1^\perp)$ of $(\mathrm{H}_{-1/2}, \mathrm{H}_{1/2})$ allows to revisite the Hawking-Hartle interpretation of their „*wave function of the universe*" interpretation concerning a required physical initial state and a corresponding mathematically required measure on an initial state (DrW).

We note that the Fourier analysis based applied spectral analysis methods (e.g. cosmological distance measurement or the Doppler effect in combination with the Hubble diagram leading to the interpretations of moving apart galaxies from each other galaxies with superluminal velocity in an expanding universe) is only defined in the „*coarse-grained*" kinematical Hilbert space framework H_1, i.e. the proposed quantum gravity model allows an re-interpretation of the observed cosmological background radiation phenomenon (*).

(*) At the same point in time H. Weyl's requirement concerning a truly infinitesimal geometry ((WeH) p. 30), are fulfilled as well, because ...

… a truly infinitesimal geometry (wahrhafte Nahegeometrie) … should know a transfer principle for length measurements between infinitely close points only ... (WeH*)

The physical principle for the proposed kinematical Hilbert space H_1 is the (original) „*Leibniz least action*" principle, which is based on the „*Leibniz action element*" $w \cdot dt$ resp. $m \cdot v \cdot ds$ defined for any arbitrary system of arbitrary matter particles being subject to arbitrary forces. Leibniz's „*actio*" is defined as the action of the movement of a single matter particle during a certain time period. The least action principle in combination with Euler's variational calculus enabled multiple ODE or PDE models of physical laws, (KnA).

The good bye to current physical classical PDE model solutions is that those PDE are considered as approximation solutions to the underlying weak ($H_{-1/2}$-based) variational representations and not the other way around. The current Lagrange equations are only valid in the classical sense, whereby the weak variational models are governed by a common Hamiltonian ($H_{1/2}$-based) formalism.

Physically speaking, the currently modeled "forces" phenomena keep part of the specific corresponding classical PDE model, but are governed by the same (kinematical and ground state) energy field. In other words, there is only one single common kinetical and dynamical energy governing the several classical PDE physical (Lagrange formalism based) models; physically speaking, the current 3 Nature forces are model spacific phenomena, based on „elementary particle interactions", governed by a single common kinetical and dynamical energy model; all Lagrange formalism models (and its combinations) can be derived from a single common underlying (energy based) Hamiltonian formalism, where the physical model specific (force based) Lagrange formalism is only valid with additional regularity requirements to ensure the existence of the classical PDE solutions. In other words, the different (force based) Lagrange formalisms (and its related transformation groups combinations) provide only approximation models of the considered special physical situations to the underlying single quantum element & quantum energy „world".

The key ingredients of the proposed quantum gravity theory to integrate the Einstein field equations is about differential forms equipped with the inner product of the correspondingly defined distributional Hilbert space, with direct relationship to the Hilbert space $H_{1/2}$ and the mathematical concept of indefinite inner product spaces.

An immediate consequence of the extended energy Hilbert space concept is the solution of the 3D-NSE and Yang-Mills mass problems. The correspondingly extended Cauchy problems of the NSE and Maxwell equations become long term stable and well-posed, while the extended Maxwell equations also allows standing (stationary) waves, i.e. the Yang-Mills equations (coming along with the physical mass gap problem) are no longer required:

- regarding the 3D NSE problem the newly proposed "fluid element" Hilbert space $H_{-1/2}$ with corresponding extended energy („momentum", "velocity") space $H_{1/2}$ leads to Ricci ODE estimates of order $1/2$ enabling a corresponding bounded Sobolevskii (energy inequality) estimate. Regarding the second unknown term of the NSE, the pressure, we note that "pressure" corresponds to "energy density", ($\frac{N \cdot Meter}{Volume} \sim \frac{N}{Area}$).

- the variational representation of the Maxwell equations in the proposed quantum element/energy Hilbert space framwork $(H_{-1/2}, H_{1/2})$ conserves the two H_1-based progressive $(1-$ parameter (space or time variable)) electric and magnetic waves concept while also allowing additional standing (stationary) $H_1^{\perp}$ $-$based $(2-$parameter) wavelets. The vaccuum solution of the first ones conserves the linkage to the classical wave equations for the electric and magnetic field (while this transformation still requires additional, physical not relevant regularity requirements to the underlying solution), while the second ones provides additional information regarding the elementary particle dynamics.

With respect to „The large scale structure of space-time" and the role of gravity (Hawking S. W., Ellis G. F. R., Cambridge University Press, 1973) and the positive answer regarding „the global nonlinear stability of the Minkowski space" (ChD1) we note that the notions „matter, space-time, action, .." etc. are only defined in the H_1 energy Hilbert space with its underlying Minkowski space governed by hyperbolic PDEs, while the orthogonal Hilbert space $H_1^{\perp}$ is governed by elliptic PDE, only. The (nonlinear) stability of the Minkowski space framework requires initial data sets with finite energy and linear and angular momentum (ChD1).

From (CoR) p. 763, we recall the following conjecture for the wave equation, which would show that the four-dimensional physical space-time world of classical physics enjoys an essential distinction: "*families of spherical waves for arbitrary time-like lines exist only in case of two and four variables, and then only if the differential equation is equivalent to the wave equation (which includes also the radiation problem*)."

The proposed model is only about truly bosons w/o mass, modelled as elements of the H_1-complementary sub-space of the overall energy Hilbert space$\mathrm{H}_{1/2}$. Therefore, the main gap of Dirac's quantum theory of radiation, i.e. the small term representing the coupling energy of the atom and the radiation field, becomes part of the H_1-complementary (truly bosons) sub-space $H_1^{\perp}$ of the overall energy Hilbert space $\mathrm{H}_{1/2}$. It allows to revisit Einstein's thoughts on

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in the context of the space-time theory and the kinematics of the special theory of relativity modelled on the Maxwell-Lorentz theory of the electromagnetic field.

Einstein's field equations are hyperbolic and allow so called „*time bomb solutions*" which spreads along bi-characteristic or characteristic hyper surfaces. Actual quantum theories are talking about „inflations", which blew up the germ of the universe in the very first state. The inflation field due to these concepts are not smooth, but containing fluctuation quanta. The action of those fluctuations create traces into a large area of space. The existence of quantum fluctuations (in a „*world*" without a time arrow and without entropy) has been verified by the Casimir and the Lamb shift effects.

The standard „*big bang*" theory assumes that the creation of the first mass particle (fermion) was the „birthday" of the universe. This event was caused by an „*inflation*" energy field triggered by a „disturbance", called fluctuations, which needs to be valid before the gravity theory can „happen". In the proposed quantum gravity model the „birthday" of the „*coarse-grained*", compactly embedded fermion-energy Hilbert (sub-) space H_1 of $\mathrm{H}_{1/2}$ (coming along with the (kinematical) notions "space", "time", "action", etc.) is interpreted as first disturbance of the purely (pre-universe) boson energy field $H_1^{\perp}$ with not existing entropy. The latter one can be interpreted as the (in sync with the Casimir effect) not empty quantum vaccuum; its oscillation is the cosmic background radiation, which contains all features of dynamic energies.

With the „*birthday*" of fermions the correspondingly adapted variational representation of the wave equation is then governed by the purely kinematical (*fermions*) energy Hilbert space H_1, while its underlying initial values are purely (*undistorbed*) vacuum (CBR, bosons) energy data from $H_1^{\perp}$). As a consequence, the wave equation becomes time-asymmetric and the second law of (kinematical) thermodynamics (the entropy phenomenon coming along with the notions „mass", „time", „space" etc.) can be interpreted (and derived from this wave equation) as „action" principle of the ground state energy to damp and finally eliminate (remedy the deficiency) of any kinematical energy „disturbance".

(CoR) p. 763: "*Little is known about the scope of the concept of relatively undistorbed spherical waves relating sherical waves to the problem of transmitting with perfect fidelity signals in all directions. All we can do here is to formulate a conjecture … some support ...: - families of spherical waves for arbitrary time-like lines exist only in case of two or four dimensions if and only if the underlying differential equation is the wave equation -,*" (which includes the radiation problem, (CoR) p. 695). A proof of this conjecture would provide (additional) evidence of the below proposed integrated SMEP & gravity theory.

In the context with some relevance of the considered Kummer functions to plasma physics we refer to (KoV), (PaY):

- regarding the linear response of magnetized Bose plasmas at $T=0$ for large and small values of its parameter; the large parameter expansion plays a determining role in the behaviour of these Bose systems in the limit that the external magnetic field B approaches zero. This particular expansion is generalized for the Hurwitz zeta function, (KoV).

- regarding the linearized collision operator in the Boltzmann equation with repulsive intermolecular (inverse-power) potentials $V(r)=a\cdot r^{-\alpha}$ for $\alpha>2$; the collision operator has a purely discrete spectrum and its eigenfunctions are infinitely differentiable L_2 -functions which are complete in L_2 . The proof relies on the formalism of pseudo-differential operators; the special case $\alpha=2$ is about the Maxwell's molecules, (PaY).

In the context with the building of distributional Hilbert scales based on a linear operator with discrete spectrum and eigenfunctions, which are complete in L_2 , ist underlying approximation theory, and an „exponential decay" inner product resp. norm with parameter $t > 0$, given by

$$(x,y)_{\alpha.(t)} = \sum_k \sigma_k^\alpha e^{-\sqrt{\sigma_k}t}(x,\varphi_k)(y,\varphi_k) \quad , \quad \|x\|_{\alpha.(t)}^2 := (x,x)_{\alpha.(t)}$$

govering all „polynomial decay" Hilbert scale norms we refer to (NiJ), (NiJ1).

In the context with some relevance of the considered Kummer functions to the Navier-Stokes equation we refer to (PR1) regarding an integral representation of the Navier-Stokes equations for an incompresslble viscous fluid. „*Making use of standard integral transferm methods and considering the longitudinal components of the velocity field, thereby eliminating the pressure field, the Navier-Stokes equations are cast in integral form. The intrinsically non linear character of the equations has proved to be an unsurmountable difficulty that has severely restricted their practical use. The limited understanding of the turbulent motion of fluids and the lack of a comprehensive theory of turbulence is a consequence of this mathematical complication. … The final result is a non linear integral equation for the velocity fieId alone, involving a single convolution over the space and time variables*."

The convolution kernel of the integral representation of the Navier-Stokes equations is build on the functions

$$I_0(\vec{r},t) := \frac{1}{(4\pi\nu t)^{3/2}} e^{-\frac{\vec{r}^2}{4\nu t}} \quad , \quad I_1(\vec{r},t) := (\frac{\nu t}{\vec{r}^2}) \frac{1}{(4\pi\nu t)^{3/2}} \; {}_1F_1\left(\frac{1}{2},\frac{3}{2};-\frac{\vec{r}^2}{4\nu t}\right) .$$

Regarding the non-linear, non-stationary Navier-Stokes equations a change from a H_0 based weak variational framework to a $H_{-1/2}$ based framwork leads to reduced regularity assumptions to the initial and boundary value functions, the NSE problem becomes well posed, while at the same time the Serrin gap problem disappears.

From a physical modelling perspective the extended $H_{1/2}$ norm based energy measure of the non-linear term does not vanishes, in opposite to the current H_1 energy norm; at the same point in time the potential incompatibility of the initial boundary values of the NSE with the Neumann problem based prescription of the pressure at the bounding walls dissappears.

One can seek the harmonic function solution of the Neumann boundary value problem

$$\Delta u = 0 \qquad \text{in } R^3 - S$$
$$\frac{\partial u}{\partial n} = f \qquad \text{on } S$$

for a closed connected surface $S \subset R^3$ in the form $u(x) := \frac{1}{4\pi}\oiint_S v(y)\frac{\cos\phi_{xy}}{|x-y|^2}dS_y$, where ϕ_{xy} denote the angle between the vector $|x-y|$ and the normal n_y to the surface at the point y and $v(y)$ is the density of the double layer potential.

The unknown function $v(y)$ is obtained by the equation

$$(\prod v)(x) := \frac{1}{4\pi}\oiint_S v(y)\frac{\cos\phi_{xy}}{|x-y|^2}dS_y = f(x).$$

The operator $\prod$ is called the Prandtl operator.

With respect to the considered newly proposed energy Hilbert space $H_{1/2}$ we note the following (LiI),

Theorem: The Prandtl operator $\prod \; : H_{1/2} \to H_{-1/2}$ is bounded, the function

$$u(x) := \frac{1}{4\pi}\oiint_S v(y)\frac{\cos\phi_{xy}}{|x-y|^2}dS_y$$

is an element of $H_1(R^3 - S)$ and the exterior Neumann problem admits one and only on generalized solution.

The above theorem summarizes the following properties, (LiI) (4.1.40), proposition 4.2.1, Theorem 4.2.2, proposition 4.3.1:

i) the Prandtl operator $\prod \quad : H_r \to H_{r-1}$ is bounded for $0 \le r \le 1$

ii) there is a representation $\prod = A + K$ with

$$(Av)(x) := \frac{1}{4\pi} \oiint_S \frac{v(y)}{|x-y|^3} dS_y \quad \text{and} \quad (Kv)(x) := \frac{1}{4\pi} \oiint_S k(x,y)v(y)\, dS_y$$

whereby

$$\left|k(x,y)dS_y\right| \le \left|\frac{|x-y|^2\big((n_x,n_y)-1\big)-3(|x-y|,n_x)(|x-y|,n_y)}{|x-y|^5}\right| \le \frac{c}{|x-y|}.$$

iii) For $0 < r < 1$ the Prandtl operator is Noetherian, i.e. it has a right regularizer R with $R\prod = RL + RN$, whereby $\boldsymbol{RN}$ is a compact operator in H_r, R is bounded from H_{r-1} to $\boldsymbol{H_r}$ and the operator N is bounded from H_r to H_0,The operators $\boldsymbol{NR}$ and $\boldsymbol{LR}$ are a compact operators in H_{r-1}.

iv) For $v \in H_r$, $r \ge 1/2$, the function

$$u(x) := \frac{1}{4\pi} \oiint_S v(y) \frac{\cos \phi_{xy}}{|x-y|^2} dS_y$$

is an element of $H_1(R^3 - S)$.

v) For $1/2 \le r < 1$ the exterior Neumann problem admits one and only on generalized solution.

Plasma physics as a „proof of concept" of the proposed Hilbert space based quantum gravity model

Plasma is an inonized gas consisting of approximately equal numbers of positively charged ions and negatively charged electrons. One of the key differentiator to neutral gas is the fact that its electrically charged particles are strongly influenced by electric and magnetic fields, while neutral gas is not. There are two nonlinear equations that have been treated extensively in connection with nonlinear plasma waves: The Korteweg-de Vries equation and the nonlinear Schrödinger equation. .... "*When an electron plasma wave goes nonlinear, the dominant new effect is that the ponderomotive force of the plasma waves causes the background plasma to move away, causing a local depression in density called caviton. Plasma waves trapped in this cavity then form an isolated structure called envelope soliton or envelope solitary wave. Considering the difference in both the physical model and the mathematical form of the governing equations, it is surprising that solitons and evelopes solitons have almost the same shape*", (ChF) 8.8.

(MiK) : „Charge Neutrality *is one of the fundamental property of plasma: it is about the shielding of the electric potential applied to the plasma. When a probe is inserted into a plasma and positive (negative) potential is applied, the probe attracts (repulses) electrons and the plasma tends to shield the electric disturbance.*

Landau Damping *is the other fundamental process of plasma: it is about collective phenomena of charged particles. Waves are associated with coherent motions of charged particles. When the phase velocity of wave or perturbation is much larger than the thermal velocity of charged particles, the wave propagates through the plasma media without damping or amplification. However when the refractive index N of plasma media becomes large and plasma becomes hot, the phase velocity c/N (c is light velocity) of the wave and the thermal velocity become comparable (c/N ~ vT), then the exchange of energy between the wave and the thermal energy of plasma is possible. The existence of a damping mechanism of wave was found by L. D. Landau. The process of Landau damping involves a direct wave-particle interaction in collisionless plasma without necessity of randamizing collision. This process is fundamental mechanism in wave heatings of plasma (wave damping) and instabilities (inverse damping of perturbations)*".

Plasma physics modelling is basically about statistical evolution of a large number of particles interacting through „collisions". The mathematical models are the Boltzmann and Landau equations, where the unknown function f corresponds at each time t to the density of particles at the point x with velocity v, (LiP): „If the related non-local, quadratic operator $Q(f,f)$were zero, the kinetic Boltzmann and Landau equations would simply mean that the particles do not interact an the density f would be constant along particle paths". The operator $Q(f,f)$ was introduced by Maxwell and Boltzmann for the case, that collisions occur.

"*In case the described particles of the Boltzmann equation interact with a two-body force (collisions case), this leads to a Vlasov-like force (or self-consistent force, or mean field...)* F", (LiP1). Its underlying potential function $V(x)$ is governed by the Laplace operator "$\Delta = div(grad)$" based potential equation, given by $-\Delta = \nabla F$. In (NiJ*) corresponding unusual (Sobolev and Hölder) norm estimates are provided, enjoying appreciated shift theorems for the Landau damping phenomenon critical Coulomb potential case; the shift theorems also well fit to the proposed $H_{-1/2}$ Hilbert space framework. The provided proofs are all based on standard estimates for the Newtonian potential.

In case the Boltzmann collision kernel B of the non-local, quadratic operator $Q(f(t,x,°),f(t,x,°))$ presents singularities of an arbitrarily high order, it is about so-called grazing collisions, (LiP): when almost all collisions are grazing this leads to Landau collision operator resp. the Landau equation (also called the Fokker-Landau equation).

The microscopic kinetic description of plasma fluids leads to a continuity equation of a system of (plasma) "particles" in a phase space (x,v). In case of a Lorentz force the equation reduces to the so-called collisions-less (kinetic) Vlasov equation (ShF) (28.1.2)), where the force F of the baseline Boltzmann equation, acting on the particles, is entirely electromagnetic (ChF) 7.2. Physically speaking, collisions are neglegted in case of sufficiently hot plasma, i.e. in case of sufficiently high plasma energy.

We note that the related Vlasov formula for the plasma dielectric for the longitudinal oscillators

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{}'(v)dv}{\frac{\omega}{k}-v} \quad ,$$

is not well defined, from a mathematical and from a physical point of view. Mathematically speaking, it is not well defined, even (as Vlasov suggested) if the integral is interpreted as a principle-value integral, (ShF) p. 93. Physically speaking, the integral is divergent in case of the important physical phenomenon of electrons travelling with exactly the same material speed $\frac{\omega}{k}$ and the wave speed v. The underlying „*erroneous assumption is, that longitudinal oscillations set up initially in a plasma with nonpathological electron distribution function should be able to persist forever in the absence of dissipative collisions. In other words, it should be possible to considerer real values for both* ω *and* k*. Mathematically speaking, what should be done about electrons that travel at a material speed exactly equal to the wave speed?*" (ChF) p. 393.

One of the probably most important physical aspects of the considered Kummer functions are in the context of (quantum theory related) Schrödinger operators with a Coulomb potential, (DeJ). The self-adjoint Schrödinger operators with a Coulomb potential correspond to Whittaker equations with parameter $m = 1/2$. Therefore, corresponding variational representations of the self-adjoint Whittaker equations (especially the one with the parameter $m = 1/2$) based on the extended $\mathrm{H}_{1/2}$ (energy) Hilbert space result into convergent energy norm estimates governing also 3D-Newton/Coulomb potential singularities.

The current abstract, functional analysis framework to model physical processes as neutron transport, radiative transfer, rarified gas dynamics, lectron scattering is about a single abstract transport equation in the form $\frac{d}{dx}\left(T[\varphi](x)\right) = -A[\varphi](x))$, where the left hand side describes the free streaming and the right hand side describes the collisions (GaA). Krein space methods (going along with the theory of „linear operators in space with an indefinite metric", (AzT), (AzT1), (BoJ)) can be used to derive unique solvability of such abstract linear, kinetic equations, like Landau (Fokker-Planck) type equations, (GaA).

The eigenvalue equations of the (hyperbolic-type) Whittaker self-adjoint operators $\mathrm{H}_{\beta.\mathrm{m}}$ (on the domain of functions, that behave properly near zero) for the eigenvalue (energy) $-1/4$, is given by the Whittaker equations. The (hyperbolic-type) Whittaker equations can be reduced to the confluent hypergeometric (Kummer) equations. The Kummer function related Bessel functions are annihilated by the general Whittaker operator; the asymptotics of zero-energy eigenfunctions near zero of the Whittaker operator with value $m = 1/2$ is $c \cdot (1 + O(xlog(x))$.

We note that the hyperbolic-type Whittaker equations get trigonometric (or elliptic)-type Whittaker equations by replacing the z variable by $\pm iz$ and by replacing the parameter β by $\mp i\beta$ of the oncerned Whittaker operato $\mathrm{H}_{\beta.\mathrm{m}}$. The trigonometric-type Whittaker equations is related to the eigenvalue (energy) $+1/4$, (DeJ).

Vlasov's mathematical argument for the Vlasov equation as a proper microscopic kinetic description of hot plasma fluids, alternatively to the Landau equation was, that "*this model of pair collisions is formally (!) not applicable to Coulomb interaction due to the divergence of the kinetic terms*". The proposed Hilbert space based quantum gravity model ensures the convergence of the Coulomb potential related singularity.

The Landau damping phenomenon is about "*wave damping w/o energy dissipation by collisions in plasma*", because electrons are faster or slower than the wave and a Maxwellian distribution has a higher number of slower than faster electrons as the wave. As a consequence, there are more particles taking energy from the wave than vice versa, while the wave is damped ((BiJ)).

The Landau damping property is complementary to the properties of electro-magnetic forces, which weaken themselves spontaneously over time w/o increase of entropy or friction. "*It involves coupling between single-particles and collective aspects of plasma behavior. ..this topic is related to one of the main unsolved questions in physics. .... Landau damping involves a flow of energy between single particles on the one hand side, and collective excitations of plasma on the other side*", (DeR) p. 94.

"*In its purest form, Landau damping represents a phase-space behavior peculiar to collisionless systems. Analogs to Landau damping exist, for example, in the interactions of stars in a galaxy at the Lindblad resonances of a spiral downsity wave. Such resonances in an inhomogeneous medium can produce wave absorption (in space rather than in time), which does not usually happen in fluid systems in the absence of dissipative forces. An exception in the behavior of corotation resonances for density waves in a gaseous medium*", (ShF) p. 402. In other words, the Landau damping phenomenon can be interpreted as the capability of stars to organize themselves in a stable arrangement.

In (MoC) a proof is provided for the Landau damping phenomenon based on the Vlasov equation using analytical norm estimates. Neither the Vlasov equation itself (a collisions-less equation to model wave damping w/o energy dissipation by collisions in plasma) nor the application of analytical norm estimates (a hammer being used for nuclear fission) are appropriate to model or „to prove‟ hot plasma physical phenomena. Alternatively, we propose a weak variational PDE representation of the Coulomb force based Landau equation in the proposed distributional Hilbert space framework. The counterpart of the analytical norms in (Moc) are given by the related norm of the exponential decay inner product

$$(x,y)_{\alpha.(t)} = \sum_k \sigma_k^{\alpha} e^{-\sqrt{\sigma_k}t}(x,\varphi_k)(y,\varphi_k) \quad , \quad \|x\|^2_{\alpha.(t)} := (x,x)_{\alpha.(t)}$$

accompanied by wavelet analysis capabilities. The alternative approach also avoids the (physically not relevant) Penrose stability criterion assumption. In case of grazing collisions, the kernel function B of the collision integral operator presents singularities with bounded orders, e.g. the Coloumb (Newtonian) potential related singularity. In case of non-grazing collisions, the kernel function B presents singularities of an arbitrarily high order, being governed by the $\|x\|^2_{\alpha.(t)}$-norm. Mathematically speaking, the proposed $\mathrm{H}_{1/2}$-based (strong and weak) Landau-Poisson-Maxwell PDO systems cover all types of PDE, which are parabolic-elliptic-hyperbolic PDE, while the differentiated (!) standard Maxwell equations result into the (hyperbolic) wave equation, defining the principle of maximal electro-magnetic information exchange by the speed of light and all other related special and general relativity theory aspects.

Conceptually speaking the parabolic Landau (evolution) equation connects the elliptic and hyperbolic (space-time) quantum world.

We further note, that the elliptic vs. hyperbolic "worlds" are very much in line with D. Bohm's notions of implicate and explicate order, (BoD):

With respect to the elliptic "world" we recall from (BoD) A.2:

"*Rather, an entirely different sort of basic connection of elements is possible, from which our ordinary notions of space and time, along with those of separately existent material particles, are abstracted as forms derived from the deeper order. These ordinary notions in fact appear in what is called the explicate or unfolded order, which is a special and distinguished form contained within the general totality of all the implicate orders… Explicate order arises primarily as a certain aspect of sense perception and of experience with the content of such sense perception. It may be added that, in physics, explicate order generally reveals itself in the sensibly observable results of functioning of an instrument. … „What is common to the functioning of instruments generally used in physical research is that the sensibly perceptible content is ultimately describable in terms of a Euclidean system of order and measure, i.e., one that can adequately be understood in terms of ordinary Euclidean geometry. … The general transformations are considered to be the essential determining features of a geometry in a Euclidean space of three dimensions; those are displacement operators, rotation operators and the dilatation operator*."

The hyperbolic world in the standard statistics (reflexiv) Hilbert space framework L_2 is about statistical thermodynamics and related Shannon (discrete) entropy, based on the countable spectrum of the considered differential operators with range L_2.

The norm of the quantum $\mathrm{H}_{-1/2}$ elements is governed by the sum of the corresponding "observables" L_2 norm and the exponential decay norm, while both summands are interwoved by a parameter, which can be appropriately choosen to model the influencing & balancing contribution of underlying "ground state" energy effect ("time-independent "action""), see also below.

With regards to the proposed integrated SMEP and gravity model, the "*between bodies interacting*" force in the Boltzmann equation is decomposed into two "forces" defined by a corresponding (Hamiltonian formalism based) integrated (kinematical & dynamical) energy concept. This is achieved by considering the Landau integral operator equation in a weak $\mathrm{H}_{-1/2}$ Hilbert space framework. The Coulomb force (Poisson equation based) force and the Lorentz (electro-magnetic) force (Maxwell equation based) are replaced by the concept of underlying related kinematical and (complementary, not only electro-magnetic) dynamical energy, modelled as decomposition of the (energy) Hilbert space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp}$).

With regards to the Maxwell equations we recall that the components of the electric and magnetic field forces E, H build the 4-dimensional electromagnetic field force tensor $\mathrm{F}_{\mathrm{i,k}} = (E, H)$. The Maxwell stress tensor s(i,k) $\sigma_{\mathrm{i,k}}$ is built on the field force tensor in combination with the Dirac function. The standard Maxwell operator is not coercive. For the time-harmonic Maxwell equations, (KiA), there is a coercive bilinear form provided, containing tangential derivatives of the normal and tangential components of the field on the boundary, vanishing on the subspace H_1, (CoM) below. In the proposed $\mathrm{H}_{-1/2}$ framework the Dirac function is replaced by $\mathrm{H}_{-1/2}$ distributions to model point/surface densities. The Laplace operator of the Poisson equation also defines a coercive bilinear form (see also (WeP) below. Thus, in the proposed new framework standard and complementary variational methods can be applied, based on coercive bilinear forms.

With regards to the changes coming along with the above proposed quantum element/quantum energy distributional Hilbert space framework we further note :

- normal and tangential derivatives, mass density, and „flow through a surface" are replaced by Plemelj's Stieltjes' integral based concept of the notions „mass" and „flux" at each point of a surface (PlJ); the definitions require less regularity assumptions to the underlying potential function; we mention that the Vlasov-Poisson-Boltzmann system is about the Poisson potential function defining the forces term F in the general Boltzmann equation, (LiP1)

- the extended Maxwell equations (making the Yang-Mills equation superfluous & enabling an unique stabil 3D-NSE Cauchy problem solution with appropriately defined distributional initial value function) define a coercive bilinear form in the related variational equation representation; we mention that the Vlasov-Maxwell-Boltzmann system is about the (collision-free) Lorentz potential function defining the forces term F in the general Boltzmann equation, (LiP1)

- the role of the Gaussian density function to measure the statistics in the observable space H_0 can be extended by the mathematical microscope wavelet tool 'unfolding' a function over the one-dimensional space R into a function over the two-dimensional half-plane of "positions" and "details"

- in general, the usage of a $\mathrm{H}_{-1/2}$ Hilbert space framework allows a variational calculus with differentials and related pseudo-differential equations, including Gateaux and Frechet differentials, (VaM).

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