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A NOTE ON NYMAN'S EQUIVALENT FORMULATION OF THE RIEMANN HYPOTHESIS

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A certain subspace of $L^2((0,1),dt)$ has been considered by Nyman, Beurling, and others, with the result that the constant function **1** belongs to it if and only if the Riemann Hypothesis holds. I show that the product

$$\prod_{\zeta(\rho)=0,\mathrm{Re}(\rho)>\frac{1}{2}} \left|\frac{1-\rho}{\rho}\right|$$

is the norm of the projection of **1** to this subspace. This provides a quantitative refinement to Nyman's theorem.

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1

Let $\rho_\alpha(t) = \left\{\frac{\alpha}{t}\right\} - \alpha\left\{\frac{1}{t}\right\}$, for $0 < \alpha < 1$, and $t \in (0,1)$ (with $\{u\}$ the fractional part of the real number u). Let K be the closed span in $L^2((0,1),dt)$ of the functions ρ_α. We also consider both K and $L^2((0,1),dt)$ as closed subspaces of $L^2((0,\infty),dt)$. Note that K is invariant under the semi-group of unitary contractions defined as $U(\lambda) : f(t) \mapsto \sqrt{\lambda} f(\lambda t),\ \lambda \geq 1,\ t > 0$. So, it will contain the constant function $\mathbf{1}$ if and only if it actually coincides with all of $L^2((0,1),dt)$.

Theorem 1.1 (Nyman, 1950 [1]) *The constant function* $\mathbf{1}$ *belongs to* K *if and only if the Riemann Hypothesis holds.*

This statement was extended by Beurling ([2]) to a characterization (where L^2 is replaced with L^p) of the absence of zeros with real part (strictly) greater than $\frac{1}{p}$. Bercovici and Foias ([3]) have shown how the L^2–properties of the contraction semi–group $U(\lambda),\ \lambda \geq 1$ (obviously central to the case considered by Nyman) can also be the basis of a proof in the L^p–case. The paper by Balazard and Saias ([4]), which motivates the present note, gives further emphasis to the use of Hardy spaces and of fundamental results of Beurling ([5]) and Lax ([6]) in this context. In this note I deduce from these methods of proof the following quantitative refinement of **1.1**:

Theorem 1.2 *Let* P *be the orthogonal projector of* $L^2((0,1),dt)$ *onto* K. *Then*

$$\|P(\mathbf{1})\| = \prod_{\zeta(\rho)=0,\ \mathrm{Re}(\rho)>\frac{1}{2}} \left|\frac{1-\rho}{\rho}\right|$$

Note 1.3 In this product, and others to follow, each zero is counted according to its multiplicity. The most basic estimates concerning the imaginary parts of the zeros imply that the product (perhaps an *empty* one ...) converges to a non-zero value. Also, note that the map $s \mapsto z = \frac{1-s}{s}$ is a conformal representation of the half-plane $\mathrm{Re}(s) > \frac{1}{2}$ onto the open unit disc, so that (**1.2**) clearly implies (**1.1**).

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As in the papers cited above on this subject the main tool is the Fourier Transform, here in its multiplicative version:

$$(\ \mathbf{2.1}\ ) \qquad f(t) \in L^2((0,\infty), dt) \ \mapsto \widehat{f}(s) = \int_0^\infty f(t)\, t^{s-1}\, dt$$

The integral is to be understood in the L^2–sense, with $\mathrm{Re}(s) = \frac{1}{2}$. Indeed we really want to look at $\sqrt{t} f(t)$ in $L^2((0,\infty), \frac{dt}{t})$ and at its transform in the dual group $\int_0^\infty \sqrt{t} f(t)\, t^{i\tau}\, \frac{dt}{t}$, $\tau \in \mathbb{R}$. With $s = \frac{1}{2} + i\tau$ we end up with the formula above.

The proof of (**1.2**) requires some classical results of harmonic analysis (Hardy spaces, the factorization theorem, the Beurling–Lax description of invariant subspaces) explained in the books by Dym–McKean ([7]) and Hoffman ([8]). As the paper by Balazard and Saias ([4]) gives a useful résumé, I will only give a brief discussion.

If $f(t)$ actually belongs to $L^2((0,1), dt)$, then

$$\widehat{f}(s) = \int_0^\infty f(t)\, t^{s-1}\, dt$$

makes sense as an analytic function in the half–plane $\mathrm{Re}(s) > \frac{1}{2}$ and the set of such $\widehat{f}$ is characterized by the Paley–Wiener Theorem ([7, Chapter 3]) as the Hardy space $\mathbb{H}^2$ of analytic functions $h(s)$ whose L^2–norms on vertical lines

$$\frac{1}{2\pi} \int_{\mathrm{Re}(s)=\sigma>\frac{1}{2}} |h(s)|^2\, |ds|$$

are bounded independently of σ. Such an analytic function $h(s) = \widehat{f}(s)$ has (pointwise almost everywhere) non-tangential limits $h(\frac{1}{2} + i\tau)$ which are obtained as the Fourier–Mellin Transform (**2.1**) of f of $L^2((0,1), dt)$. Furthermore, for $\mathrm{Re}(s) > \frac{1}{2}$:

$$h(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{h(\frac{1}{2} + i\tau)}{s - \frac{1}{2} - i\tau}\, d\tau \tag{2.2}$$

The zeta–function appears in this story thanks to

$$\int_0^1 \left\{\frac{1}{t}\right\} t^{s-1}\, dt = \frac{1}{s-1} - \frac{\zeta(s)}{s}$$

$$\int_0^1 \left\{\frac{\alpha}{t}\right\} t^{s-1}\, dt = \frac{\alpha}{s-1} - \alpha^s \frac{\zeta(s)}{s}$$

$$\widehat{\rho_\alpha}(s) = \frac{\alpha - \alpha^s}{s} \zeta(s) \tag{2.3}$$

We note in passing that

$$\widehat{\mathbf{1}}(s) = \frac{1}{s} \tag{2.4}$$

According to the general theory recalled in ([4]) the identity (**2.3**) already implies the (absolute) convergence for any s in the half-plane $\mathrm{Re}(s) > \frac{1}{2}$ of the Blaschke product

$$B(s) = \prod_{\zeta(\rho)=0,\ \mathrm{Re}(\rho)>\frac{1}{2}} \frac{s-\rho}{s-(1-\bar{\rho})} \frac{1-\bar{\rho}}{\rho} \left| \frac{\rho}{1-\rho} \right|$$

whose value at 1 is

$$B(1) = \prod_{\zeta(\rho)=0,\ \mathrm{Re}(\rho)>\frac{1}{2}} \left| \frac{1-\rho}{\rho} \right| \tag{2.5}$$

Note 2.6 It is known that such a product built from the zeros of an element of $\mathbb{H}^2$ is an *inner function*, that is an analytic function bounded by 1 in the half-plane whose non-tangential limits on the critical line have modulus 1 (almost everywhere). The expression as an infinite product might cease to make sense (pointwise) for the boundary values, but in the case at hand it is absolutely convergent for all s in $\mathbb{C}$, except at the possible poles $1-\bar{\rho}$.

Furthermore Balazard and Saias ([4], section 4) prove that the Mellin Transform of the subspace K considered in the Introduction is $B(s)\mathbb{H}^2$. At this stage, adapting an argument of Beurling ([5, Theorem II]) enables us to show that the orthogonal projection of $\frac{1}{s}$ to $B(s)\mathbb{H}^2$ is simply $B(1)\frac{B(s)}{s}$. Hence its norm is given by (**2.5**), as stated in (**1.2**). The argument runs as follows:

Proof 2.7 We check that $\frac{1}{s} - B(1)\frac{B(s)}{s}$ is perpendicular to $B(s)h(s)$, for any $h(s) \in \mathbb{H}^2$:

$$\begin{aligned}
& \frac{1}{2\pi}\int_{\mathrm{Re}(w)=\frac{1}{2}} \overline{\left(\frac{1}{w} - B(1)\frac{B(w)}{w}\right)}\, B(w)h(w)\,|dw| \\
= \;& \frac{1}{2\pi}\int_{\mathrm{Re}(w)=\frac{1}{2}} \frac{(B(w)-B(1))\,h(w)}{\overline{w}}\,|dw| \\
= \;& \frac{1}{2\pi}\int_{\mathrm{Re}(w)=\frac{1}{2}} \frac{(B(w)-B(1))\,h(w)}{1-w}\,|dw| \\
= \;& (B(1)-B(1))\,h(1) = 0
\end{aligned}$$

where $|B(w)| = 1$ on $\mathrm{Re}(w) = \frac{1}{2}$ and then (**2.2**) were used. With this the proof is complete •

To conclude let us mention the following closely related theorem:

Theorem 2.8 (Balazard, Saias and Yor [9])

$$\frac{1}{2\pi}\int_{\mathrm{Re}(s)=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^2}\,|ds| = \sum_{\zeta(\rho)=0,\ \mathrm{Re}(\rho)>\frac{1}{2}} \log\left|\frac{\rho}{1-\rho}\right|$$

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