A Bessel function based proof that the Euler-Mascheroni constant γ is irrational

Klaus Braun

Dedicated to my son Mario
on the occasion of his 30th birthday

Dec 2, 2021

We shall consider generalized Bessel functions in the forms (**)

$$B_\nu(x) := \frac{J_\nu(2x)}{x^\nu} - \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}} = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{x^k}{k!}\frac{(1-x^k)}{\Gamma(\nu+k+1)}$$

where

$$\frac{J_\nu(2x)}{x^\nu} := \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{x^{2k}}{\Gamma(\nu+k+1)} .$$

Lemma:

i) $\gamma = \int_0^\infty [J_0(2x) - J_0(2\sqrt{x})]\frac{dx}{x} = \int_0^\infty B_0(x)\frac{dx}{x}$

ii) for $0 < Re(\mu) < Re(\nu) + \frac{3}{2}$ it holds $\int_0^\infty x^\mu B_\nu(x)\frac{dx}{x} = \frac{1}{\mu}\left[\frac{\Gamma\left(1+\frac{\mu}{2}\right)}{\Gamma\left(1+\nu-\frac{\mu}{2}\right)} - \frac{\Gamma(1+\mu)}{\Gamma(1+\nu-\mu)}\right]$

iii) for $0 \le \alpha < \beta$, $0 \le x < 1$ it holds $B_\beta(x) - B_\alpha(x) > 0$.

Proof: i) - iii) see lemmata 1-3 below.

Putting $\mu =: \nu = \frac{1}{n}$ and

$$a_n := \int_0^\infty x^{1/n} B_{1/n}(x)\frac{dx}{x} = \int_0^1 \left[x^{1/n}B_{1/n}(x) + x^{-1/n}B_{1/n}\left(\frac{1}{x}\right)\right]\frac{dx}{x}$$

it follows

i) $a_1 = 0$, $a_\infty = \int_0^\infty B_0(x)\frac{dx}{x} = \gamma$

ii) $a_n = n\left[1 - \Gamma\left(1+\frac{1}{n}\right)\right] < a_{n+1}$ (*)

iii) for $n > 1$ all a_n are transcendental.

(*) (AbM) p. 255: for $1 \le x \le 2$ it holds

i) $max\{\Gamma(x)\} = \Gamma(1) = \Gamma(2) = 1$,

ii) $\Gamma(x)$ is convex with minimum at $\Gamma(3/2) \sim 0{,}88622.. = 1 - 0{,}11378$ ...

iii) $1/\Gamma(x)$ is concave with maximum at $\frac{1}{\Gamma\left(\frac{3}{2}\right)} \sim 1 + 0{,}11378$ ...

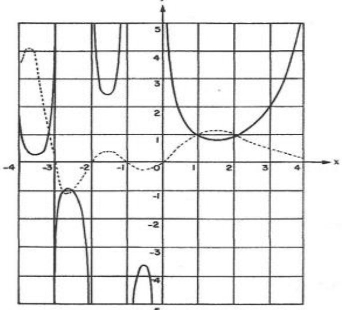

FIGURE 6.1. *Gamma function.*
———, $y=\Gamma(x)$, - - - -, $y=1/\Gamma(x)$

(**) Remark: for $\nu \in C$, and α algebraic with $\alpha \neq \pm\frac{1}{2}, -1, \pm\frac{3}{2}, -2$ ... , $J_\nu(\alpha)$ are transcendental, (SiC).

Lemma 1: It holds $\gamma = \int_0^\infty \left[J_0(2x) - J_0(2\sqrt{x})\right] \frac{dx}{x}$.

Proof: In (BrR) it is proven that $\int_0^\infty [e^{-t} - J_0(2t)] \frac{dt}{t} = 0$. The formula is derived from the related Mellin transforms formula, (WaG) 13-24, (WeH),

$$\int_0^\infty t^{\mu-1} J_0(2t)dt = \frac{\frac{\mu}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(1-\frac{\mu}{2}\right)} \quad \text{for } 0 < Re(\mu) < 3/2$$

in combination with the asymptotics

$$\Gamma(\mu) - \frac{\frac{1}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(1-\frac{\mu}{2}\right)} = \frac{1}{\mu}\left[\Gamma(1+\mu) - \frac{\Gamma\left(1+\frac{\mu}{2}\right)}{\Gamma\left(1-\frac{\mu}{2}\right)}\right] = \frac{1}{\mu}\left[(1-\gamma\mu) - \frac{\left(1-\frac{\gamma\mu}{2}\right)}{\left(1+\frac{\gamma\mu}{2}\right)} + O(\mu^2)\right] = O(\mu) \to_{\mu\to 0^+} 0 \ .$$

Analogue it can be proven that $\int_0^\infty \left[e^{-t} - J_0(2\sqrt{t})\right] \frac{dt}{t} = \gamma$. In this case the formula is derived from the corresponding asymptotics

$$\Gamma(\mu) - \frac{\Gamma(\mu)}{\Gamma(1-\mu)} = \frac{1}{\mu}\left[\Gamma(1+\mu) - \frac{\Gamma(1+\mu)}{\Gamma(1-\mu)}\right] = \frac{1}{\mu}\left[(1-\gamma s\mu) - \frac{(1-\gamma\mu)}{(1+\gamma\mu)} + O(\mu^2)\right] = \gamma + O(\mu) \to_{\mu\to 0^+} \gamma,$$

which proves lemma 1.

The Mellin transforms of $B_\nu(x)$ are given by

Lemma 2: for $0 < Re(\mu) < Re(\nu) + \frac{3}{2}$ it holds

$$\int_0^\infty x^\mu B_\nu(x) \frac{dx}{x} = \frac{1}{\mu}\left[\frac{\Gamma\left(1+\frac{\mu}{2}\right)}{\Gamma\left(1+\nu-\frac{\mu}{2}\right)} - \frac{\Gamma(1+\mu)}{\Gamma(1+\nu-\mu)}\right] .$$

Proof: For $0 < Re(\mu) < Re(\nu) + \frac{3}{2}$ it holds, (BrR), (WaG) 13-24, (WeH),

$$\int_0^\infty x^\mu \frac{J_\nu(x)}{x^\nu} \frac{dx}{x} = \frac{\Gamma\left(\frac{\mu}{2}\right)}{2^{\nu-\mu+1}\Gamma\left(1+\nu-\frac{\mu}{2}\right)} ,$$

from which it follows

i) $$\int_0^\infty x^\mu \frac{J_\nu(2x)}{x^\nu} \frac{dx}{x} = \frac{\frac{1}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(\nu+1-\frac{\mu}{2}\right)} = \frac{1}{\mu}\frac{\Gamma\left(1+\frac{\mu}{2}\right)}{\Gamma\left(1+\nu-\frac{\mu}{2}\right)}$$

ii) $$\frac{1}{2}\int_0^\infty x^{\mu/2} \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}} \frac{dx}{x} = \frac{\frac{1}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(1+\nu-\frac{\mu}{2}\right)} \quad \text{resp.} \quad \int_0^\infty x^\mu \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}} \frac{dx}{x} = \frac{\Gamma(\mu)}{\Gamma(1+\nu-\mu)} = \frac{1}{\mu}\frac{\Gamma(1+\mu)}{\Gamma(1+\nu-\mu)} .$$

From the representation

$$B_\nu(x) = \sum_{k=0}^\infty (-1)^{k+1} \frac{x^k}{k!} \frac{(1-x^k)}{\Gamma(\nu+k+1)}$$

one gets

Lemma 3: for $0 \le \alpha < \beta$, $0 \le x < 1$ it holds $B_\beta(x) - B_\alpha(x) > 0$.

Proof: The lemma follows from the equation

$$B_\beta(x) - B_\alpha(x) = \sum_{k=0}^\infty (-1)^{k+1} \frac{x^k}{k!} \frac{(1-x^k)}{\Gamma(\beta+k+1)} - \sum_{k=0}^\infty (-1)^{k+1} \frac{x^k}{k!} \frac{(1-x^k)}{\Gamma(\alpha+k+1)}$$

$$= \sum_{k=0}^\infty (-1)^k \frac{x^k(1-x^k)}{k!}\left[\frac{1}{\Gamma(\alpha+k+1)} - \frac{1}{\Gamma(\beta+k+1)}\right] > 0 \ .$$

References

(AbM) Abramowitz M., Stegun I. A., Handbook of mathematical functions, Dover Publications, Inc., New York, 1965(BeB) Berndt B. C., Ramanujan's Notebooks, Part I, Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1985

(BeI) Belogrivov I. I., On the transcendence and algebraic independence of the values of Kummer's functions, Sibirsk. Mat. Zh., 12 (1971), 961–982

(BrR) Brent R. P., An asymptotic expansion inspired by Ramanujan, The Australian Mathematical Society Gazette, 20 (1993), 149-155

(SiC) Siegel C. L., Transcendental Numbers, Annals of Math. Studies No. 16, Princeton University Press, Princeton, 1949

(WaG) Watson G. N., A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1922

(WeH) Weber H., Über einige bestimmte Integrale, Journal für Math. 69 (1868), 222-237