Time and complex numbers in canonical quantum gravity

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It is noted that not only time but also the imaginary unit is absent from the Wheeler-DeWitt equation (WDE), the basic equation of canonical quantum gravity. This leads to a severe problem in an attempt to recover the time-dependent Schrödinger equation from the WDE. It is suggested that the problem will be solved if the WDE is actually *complex* and not real, as in most models hitherto considered.

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I. INTRODUCTION: THE *i* PROBLEM

Both time and the imaginary i are absent from the basic equation of quantum gravity, the Wheeler-DeWitt equation (WDE). The absence of time is widely regarded as a serious problem, but the absence of i does not seem to cause any concern. It is clearly believed that quantum gravity will be complex just like ordinary quantum theory even if the WDE is real.

This may be complacent, since *all* specifically quantum-mechanical uses of complex numbers (or structures equivalent to them) can be attributed to and associated with the use in quantum mechanics of an external time (Sec. II). If, as in quantum gravity, there is no external time, the rationale for complex numbers may have disappeared too (Sec. III). However, they are certainly needed to describe laboratory physics. The attempts to recover ordinary quantum mechanics from the WDE are examined (Secs. IV and V) and found to be unsatisfactory precisely because i is not present in the WDE and has to be introduced artificially at some stage. It is argued (Sec. VI) that this is a serious problem and may suggest that the true Wheeler-DeWitt equation of the Universe may actually be complex for reasons that have not yet been recognized. In a separate paper, Kiefer [1] has considered physical reasons that could force the wave function of the Universe to be complex.

II. TIME AND COMPLEX NUMBERS IN STANDARD QUANTUM THEORY

Complex numbers, or mathematical structures equivalent to them, occur in quantum mechanics for several different reasons [2]: for mathematical convenience, in representations of the rotation group (spinors), and in complex phases associated with local gauge transformations. However, there appears to be one use of complex numbers that represents an essential integral part of quantum theory present under all circumstances. It is manifested in one of two ways.

In the Schrödinger representation, it appears in the form of the i that multiplies the time derivative in the dynamical equation

$$i\frac{\partial\Psi}{\partial t}=H\Psi\ , \tag{1}$$

where H is the Hamiltonian of the system (I set $h/2\pi=1$).

As Pauli [3] noted in response to a query of Ehrenfest [4], who had asked why the use of complex numbers is unavoidable in quantum theory, Eq. (1) reveals that wave mechanics has a very specific two-tier structure. The deeper level is represented by the fact that Ψ satisfies a linear wave equation on the complete configuration space Q of the system. Many of the most characteristic features of quantum mechanics follow directly from this, whatever the form of the wave equation (number of components of the wave function, orders of the various derivatives): the superposition principle, interference, the possibility of forming wave packets, Bohr's correspondence principle (through the geometrical-optics limit), and the association of energy with frequency and momentum with wave number.

The higher level is expressed in the characteristic doubling (compared with a real equation) of the number of components of the complex equation (1) and the use of a first time derivative multiplied by i. This has the important consequence that from the wave function Ψ at a given instant (i.e., without the use of its time derivative) one can form the positive quantity $\Psi\Psi^*$, which can be interpreted as a probability density whose space integral is automatically conserved in time. As a consequence, the dynamics is unitary. A further very characteristic property is the complete uniformity (translational invariance) of the probability density of a free particle in a momentum eigenstate [5].

The essential thing here is that a real function such as $\cos\omega t\cos kx$ leads to nodes in its square $\cos^2\omega t\cos^2 kx$, but if $\Psi=e^{-i\omega t}e^{ikx}$, then $\Psi^*\Psi=1$. What the i in the dynamical equation achieves is *phase-matched pairing* of components: Any $\cos^2$ term in the probability density is automatically paired with a $\sin^2$ term of the same argument, so that together they give 1.

The use of complex numbers brought in by Eq. (1) may be called *dynamic complexity*. It is a very remarkable property.

In alternative, more abstract forms of quantum theory, complex numbers are not introduced in the first place through a dynamical equation but rather through "kinematics," i.e., through the use of a *complex* Hilbert space in which analogs of the classical dynamical variables are represented by operators [6]. An equivalent possibility often employed is a real space but with, as an essential element in the formalism, a *complex structure* [7], i.e., a real linear operator J that satisfies $J^2=-1$.

In this case one may say that *kinematic complexity* is employed.

For the topics raised in this paper, it is important to note that in all cases kinematic complexity appears to be merely a convenient alternative to dynamic complexity.

Consider, for example, the quantum treatment of the real Klein-Gordon equation [8]. The real vector space V of its real solutions can be transformed into a complex quantum Hilbert space by introducing a complex structure J. Although J is a real operator and maps real solutions to real solutions, it is in essence complex and exploits dynamics since it is based on decomposition of a real solution into positive- and negative-frequency parts.

Indeed, let

$$\psi=\psi^{+}+\psi^{-}$$

be the decomposition of the real ψ into its positive- (ψ^{+}) and negative- (ψ^{-}) frequency parts. Then one defines

$$J\psi=i\psi^{+}+(-i)\psi^{-}\ .$$

Thus, if, say,

$$\psi=e^{i\omega t}\cos kx+e^{-i\omega t}\cos kx=2\cos\omega t\cos kx\ ,$$

then $J\psi=-2\sin\omega t\cos kx$.

This J, which creates from one real solution another with very special *phase matching* (as the $\cos\omega t$ and $\sin\omega t$ above), is then used as an integral part in the construction of the inner product and norm of a solution. In fact, if ψ_1 and ψ_2 are two real solutions, their inner product is constructed (using the symplectic form in the classical theory) from ψ_1 and $J\psi_2$ (or, equivalently, from $J\psi_1$ and ψ_2), and the norm of any ψ is constructed using ψ and $J\psi$.

As a consequence, the characteristic phase-matched pairing of components that is enforced dynamically in the Schrödinger representation can be implemented "kinematically" in the Hilbert-space context.

Note that J is a nonlocal operator, and its use to make the one-particle Klein-Gordon equation fit the standard quantum pattern appears counterintuitive and without an *a priori* justification. However, if one assumes that there is a fundamental theory described by the functional Schrödinger equation for the Klein-Gordon *field*, that will of necessity introduce coupled components (and perfect phase matching in plane-wave situations) in the single-particle limit of the theory represented by the Klein-Gordon equation. This then explains why "odd" things must be done to make a proper quantum theory out of the real Klein-Gordon equation.

Jackiw [9] has argued that the noncovariant functional Schrödinger equation has not hitherto been widely used since it was found to be much easier to perform many calculations and carry out regularization procedures in a Poincaré-invariant form. However, now that the techniques are better understood, the reasons for not using the Schrödinger representation largely disappear.

In fact, a very general argument indicates that the essential "complexity" of quantum theory is most readily understood in such a context. Any nonrelativistic mechanical system or relativistic field theory [10], expressed in a given frame of reference, can be cast in the parametrized form [11,12] in which the dynamical trajectories are labeled, not by the time t, but by an arbitrary label parameter λ. Simultaneously, the time t is promoted to the status of a dynamical variable. In such a form, the theory is reparametrization invariant and in the Hamiltonian formulation contains a constraint

$$\pi_t+H=0\ , \tag{2}$$

where π_t is the momentum conjugate to time and H is the ordinary Hamiltonian.

To obtain a quantum theory, one simply turns the classical constraint (2) into a wave equation by the usual substitutions, in accordance with which, in particular, π_t becomes $-i\partial/\partial t$. Then (2) becomes the time-dependent Schrödinger equation (1). One then sees that the very special form of (1) arises because π_t occurs linearly in (2) and represents the energy. Moreover, the form of the constraint (2) and the form of (1) must arise for *all* theories (both nonrelativistic and relativistic) formulated with respect to an external time.

To summarize, *standard quantum theory is inherently complex, and its specific universal complexity is inseparably associated with translations and evolution in an external time.*

III. DISAPPEARANCE OF TIME AND COMPLEX NUMBERS IN QUANTUM GRAVITY

In striking contrast to (1), the equation that is obtained if one attempts to quantize a closed universe contains no reference to time and has the basic form

$$H\Psi=0\ , \tag{3}$$

where the operator H, in virtually all cases considered hitherto [13], is *real*. An equation of the form (3) (strictly, an infinity of such equations) was obtained by De Witt [14] in 1967 when he applied the standard Dirac rules for the quantization of constrained systems [11] to general relativity in the case of closed universes. In fact, an equation of such form must always be obtained if one attempts to describe any universe without reference to an external time [15]. An equation of the generic form (3) is often called a Wheeler-DeWitt equation (WDE).

There already exists an extensive (but as yet inconclusive) literature [16–24] on the fact that (3) seems to describe a completely static situation: the "wave function of the Universe," Ψ, depends on the possible *three-dimensional configurations* of the Universe and nothing else. There does not appear to be any evolution in time. The absence of t in (3) has not escaped notice.

However, the absence of i has attracted remarkably little attention. I am not aware of *any* comment made in the literature which would suggest that this is a potentially serious issue, i.e., that not only time, but also the "complexity" of quantum theory has disappeared from quantum gravity. It seems to be assumed that "complexity" is so deeply embedded in quantum theory that it simply cannot disappear.

However, the analysis of Sec. II indicates that the specific quantum complexity is inseparably associated with time. Thus, if time goes, surely there must be at the least uncertainty about the complexity? That such doubt is justified follows from one of the few attempts [25] made to take the reality of the Wheeler-DeWitt equation seriously and treat it by analogy with the real Klein-Gordon equation in the manner described in Sec. II.

The heart of the problem is to define a complex structure on the space of real solutions of the WDE by means of which a proper Hilbert space with inner product and standard statistical interpretation can be constructed. Kuchař [25] shows that the attempt must fail, and he traces the failure to the nonexistence of a suitable symmetry in quantum gravity. The fact is that the decomposition into positive and negative frequencies works in Poincaré-invariant field theory because of time translational symmetry, as a result of which the decomposition (and, hence, the associated complex structure) can be uniquely defined. In the absence of symmetry, this is not possible. Thus the most consistent attempt to solve the Hilbert-space problem of quantum gravity directly at the level of the exact theory fails.

In recent years, especially as a result of failures of attempts made along more conventional lines, several authors [16–24] have explored the possibility that time and the standard form of quantum theory can be recovered only in a certain limiting form of quantum gravity, namely, in its semiclassical limit. I believe that this approach is very promising and am keen to see it succeed, but have long felt that it suffers from a potentially serious weakness, which is this: Given that the WDE (3) is real, how is the complexity of standard quantum theory to be recovered?

The following review of the semiclassical program aims to show that this is a serious and hitherto neglected problem.

IV. SEMICLASSICAL PROGRAM

The semiclassical approach to the interpretation of quantum gravity has three main aims: to understand how a classical world, evolution in time, and the need to describe laboratory quantum physics by the (complex) time-dependent Schrödinger equation (TDSE) (1) can arise from the seemingly static world described by the WDE (3). As there is already an extensive literature on the semiclassical program, including a clear and relatively full treatment by Vilenkin [21] and reviews by Kuchař [23] and Isham [24], I shall merely consider the parts relevant to the two above questions.

Aside from the inevitable quantum interpretational dilemma (collapse or many worlds), the first two aims of the semiclassical program are achieved in a convincing manner by exploiting the deep (and inescapable) connection between wave mechanics (which treats a wave function spread out over the complete configuration space Q) and classical mechanics (which treats one-dimensional trajectories in the same Q) that was established by Hamilton and Schrödinger. Namely, in any regime in which the amplitude A of a function that satisfies a linear wave equation (which may have a very general form) varies much more slowly than the phase S of the wave function, this phase S must be a solution to the Hamilton-Jacobi equation for a certain classical mechanical system whose action principle is uniquely determined through the eikonal approximation of the original wave equation (geometrical-optics limit or WKB regime).

Thus each wave equation has associated with it, at least formally, a certain classical mechanical system, the dynamical trajectories of which are, as it were, "highlighted" whenever the wave function of the original system enters the WKB regime. They are the trajectories perpendicular to the surfaces of constant S (light rays in geometrical optics). These classical trajectories, which in general appear in complete sets as extended congruences, play a crucial role in the semiclassical program, and are the basis of the claim that classical worlds can be recovered, even though there is still much discussion of why we observe just one classical world. It is certainly the case that classical trajectories are potentially present as mathematical constructs deeply embedded in the physical wave formalism.

Moreover, the classical trajectories permit the introduction of "time" in a very satisfactory manner. As recovered from the formalism of quantum gravity, these trajectories are in the first place merely curves in a timeless configuration space Q. However, it turns out that they are the curves in Q corresponding to zero-energy trajectories in the standard formulation of dynamics with an external time. This means that one can introduce on any such trajectory a label parameter that exactly mimics "time"; i.e., with respect to this parameter, the curve in Q is traversed at exactly the rate one would expect with respect to a standard external time—there appears to be classical evolution with respect to a conventional time. This is, in fact, precisely the way in which astronomers define ephemeris time [26]—it is the time parameter in accordance with which the celestial bodies pass through their successive configurations at the rate required by Newton's laws (with the necessary small relativistic corrections). This matter is discussed in more detail in Refs. [15,27–29]; for the moment, I merely wish to note that the second desideratum of the semiclassical program—a notion of time—appears simultaneously with the first, classical trajectories.

I have discussed the first two steps of the semiclassical program in slightly more than customary detail to emphasize that they are achieved by *dynamical necessity* once a WKB regime is allowed. In contrast, the third aim—the recovery of the TDSE—is not achieved so convincingly. The problem arises because there is no i in Eq. (3).

V. RECOVERY OF THE TIME-DEPENDENT SCHRÖDINGER EQUATION

Now it is obvious that if the TDSE is to be recovered, complex numbers must be introduced at some stage. In the literature, this has been done in two ways. In the first (e.g., in Refs. [17,21]), the solutions of the WDE are assumed to be complex from the outset; in the second (e.g., in Refs. [18–20]), the solutions are assumed to be real, but a decomposition into real and imaginary parts is then made, and it is in the decomposed components that the TDSE is recovered. Both approaches encounter the same difficulty; for brevity, I shall consider only the first method [17,21].

The first assumption in this approach is unproblematic; it is that only some of the degrees of freedom of the Universe enter the WKB regime. Let these be denoted collectively by c (for classical). Let the remainder, which stay essentially quantum mechanical, be denoted by q. The next step, also unproblematic, is to assume that the total wave function of the Universe has the form

$$\Psi=\psi(c)\phi(c,q)\ , \tag{4}$$

where ψ has the form of a WKB solution for the c system and the variation with respect to c is much slower in ϕ than in ψ. I shall come to the precise form of the WKB factor ψ, *which is problematic*, in a moment.

The ansatz (4) is substituted in (3), the basic equation of the theory; if ψ is regarded as a given function, an equation for the "quantum" function ϕ results, which, for appropriate boundary conditions, will determine it in the region R of Q in which the WKB regime holds.

However, this is more than is needed. Through the first two steps described above, we know that ψ defines a congruence of classical zero-energy trajectories of the c system in R, which, moreover, can each be parametrized by a "time" variable t. One may therefore ask how ϕ varies with respect to this t as one moves through R along one of the classical trajectories. The claim of the semiclassical approach is that the equation satisfied by ϕ has, to a good approximation, the form of a TDSE.

This an be illustrated by the simplest of systems: a nonrelativistic particle in two dimensions in an external potential V described by the time-independent Schrödinger equation (TISE)

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+k^2\Psi-V(x,y)\Psi=0\ , \tag{5}$$

where $k \gg 1$ is a constant. Since (5) contains neither time nor an i, it is like a WDE. Now *suppose* that (5) has an exact solution of the form

$$\Psi=e^{ikx}\phi(x,y)\ . \tag{6}$$

The first factor here is an exact WKB-type solution of the TISE obtained by retaining in (5) only the first and third terms. Substitution of (6) in (5) yields the exact equation

$$2ik\frac{\partial\phi}{\partial x}=-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}+V\phi\ . \tag{7}$$

This can be regarded as a TDSE if (1) x is regarded as "time"; (2) the first term on the right-hand side is sufficiently small to be ignored. In essence, there is nothing more to the recovery of a TDSE from a WDE than this [30], except that in the multidimensional case the time variable is not obtained directly but by finding the distinguished parameter along the trajectories orthogonal to the surfaces of constant phase S (in the above model, kx plays the role of S, and x, of course, already increases orthogonally to the level surfaces of kx).

There are, however, two problems with the above result. One, to which I shall return in a separate paper [31], concerns the permissibility of ignoring the second "time" derivative $\partial^2\phi/\partial x^2$ in Eq. (7). The other is the issue with which I am concerned in this paper: At the general level, what is the justification for choosing *complex* solutions to a *real* equation such as (5) or (3)? The justification that we have in the Klein-Gordon case—that it is really only the limiting case of a field theory described by a complex functional Schrödinger equation—is not available here [32]. At the level of specifics, why, even if complex solutions are reasonable, is the WKB part taken in the particular form e^{ikx} [for Eq. (5)] and $e^{iS(c)}$ (for general multidimensional cases)?

The specific question is serious, because the claimed recovery of the TDSE is not, unlike the first two steps of the WKB program, dynamic and generic but very special and an artifact of finely tuned boundary conditions. For, as pointed out [33], a general complex ansatz in (6) for the WKB part, say, $ae^{ikx}+be^{-ikx}$, where a and b are arbitrary constants, wreaks havoc in the end result—the coefficient $2ik$ in (7) becomes

$$2ik(ae^{ikx}-be^{-ikx})/(ae^{ikx}+be^{-ikx})\ , \tag{8}$$

and the crucial specific form of the TDSE is lost. Indeed, if $a=b=1$, then the coefficient (8) becomes real and equal to $-2k\tan kx$.

Thus, only if the semiclassical factor ψ has real and imaginary parts exactly matched in phase by $\pm\pi/2$, as in the pure e^{ikx} or e^{-ikx}, does one obtain (approximately) a TDSE. But how can this striking *phase matching* arise? There is no dynamical reason for it. Moreover, in the multidimensional case the solution S of the classical Hamilton-Jacobi equation can be quite different in the real and imaginary parts and correspond to entirely different congruences of classical trajectories. This makes attempted recovery of the TDSE even more hopeless.

It is clear that the claimed recovery of the TDSE is a product of nondynamical phase matching. Since phase matching is the very essence of the TDSE—it is just what the $i\partial/\partial t$ term enforces—it is evident that an ansatz of the form e^{iS} for a real equation simply imposes a TDSE-type structure on all solutions by brute force. Once the phase matching has been put in, it stays there, mimicking "recovery" of the TDSE.

It is obvious from examination of the literature that the proponents of the semiclassical program have not felt this to be a problem. The reason is also rather obvious. Complexity is so deeply rooted in standard quantum theory that the adoption of complex solutions appears entirely natural. Indeed, the nondynamical phase matching described above (which would have seemed utterly artificial to theoretical physicists in the prequantum era) has exactly the same nature and effect as the decomposition into positive and negative frequencies as is habitually used in the quantization of classical fields.

However, as I pointed out in Sec. II, in that context a natural dynamical justification for phase matching through complex structures is always available in the form of the corresponding time-dependent functional Schrödinger equation. Since that, in its turn, arises solely because a classical external time is present, there must be doubt about the use of "kinematic complexity" in a situation in which the dynamical origin of the complexity—the external time—has disappeared.

Moreover, we have already noted that the direct attempt [25] to introduce "kinematic complexity" by means of a complex structure at the level of the exact solutions of the WDE fails because of the absence of a suitable symmetry permitting unique definition of positive and negative frequencies. I shall now show that a somewhat similar problem arises in the semiclassical approach.

The approach we have been considering assumes the existence of a complex solution of the WDE that has the form

$$\Psi = Ae^{iS}\phi \ , \tag{9}$$

where the WKB prefactor A may be assumed to be real.

Now since the WDE is a real equation, the real and imaginary parts of (9) must be solutions quite independently of each other. Thus postulating the existence of (9) is equivalent to requiring the existence of *two* real solutions that have the special form

$$A\phi_{\mathrm{Re}}\cos S - A\phi_{\mathrm{Im}}\sin S \tag{10}$$

and

$$A\phi_{\mathrm{Re}}\sin S + A\phi_{\mathrm{Im}}\cos S \ . \tag{11}$$

Note that the natural WKB form for a real equation is

$$\Psi = A(c)\phi(c,q)\cos S(c) \ , \tag{12}$$

with all functions on the right-hand side of (12) real.

In comparison with (12), we see that (10) and (11) form an extremely special solution pair, since each consists of two rapidly oscillating components in which the dominant oscillations are produced by the perfectly phase-matched factors $\cos S(c)$ and $\sin S(c)$. There is a triple "miracle" here. First, in each of (10) and (11) the function S is the same in both trigonometric factors. Second, the phases of the two factors differ by exactly $\pi/2$. Third, there is not just one such solution but two. It is hard to see why *any* of these three things should be found in the solutions of a real equation, even allowing that the wave function of the Universe does get into a WKB regime.

There is yet another problem. Do solution pairs of the form (10) and (11) even exist? If, as is often conjectured, the WDE is like a TISE, we must expect *extremely* strong restrictions on its admissible solutions (such as Schrödinger had to impose to obtain the discrete spectrum of the hydrogen atom). Thus, even if *one* solution of the form (10) exists, what guarantee do we have that *another* of the form (11) exists [34]?

Once the problem is posed explicitly in terms of the existence of paired real solutions of the WDE of the form (10) and (11), the implausibility and incongruity of the exercise becomes manifest. One takes *four* WKB-like terms that in the pairs (10) and (11) are actual exact solutions and then adds the two pairs with relative weight i. It is all very delicate. Is this really the way our most fundamental physical law, the TDSE, comes into existence?

It may also be noted that many authors introduce complex numbers into quantum gravity by means of Feynman path integrals. However, since such integrals are, in conventional theory, equivalent to standard Schrödinger theory, it is obvious that the i in the Feynman exponent $\exp iS$ is the same i as appears in Eq. (1). It is, at the least, a major assumption to presume the Feynman i will still be present in the timeless context.

Finally, on the question of whether decoherence can ease this problem [35,36], I would say this. The generic solution to the WDE can hardly be a unique WKB solution: One must expect it to be a superposition of such solutions. Therefore, to understand why nevertheless only one classical world is observed, we shall probably have to appeal to a decoherence-type argument (like the majority of quantum cosmologists, I am here accepting some kind of many-worlds [37] interpretation of quantum gravity), though I would like to propose a slightly more explicit account of how decoherence works in practice (see Ref. [31]). But on the question of how classical worlds, time, and the TDSE arise, that I think must first happen cleanly and honestly within *one* WKB component. I do not see how that can ever be achieved if we start from a real WDE and simply attempt to impose "kinematic complexity" by brute force.

This leads me to the following suggestion.

VI. IS THE TRUE WHEELER-DeWITT EQUATION COMPLEX?

Since all the problems described above stem from the fact that the WDE is real, should we not consider the possibility that the basic equation of quantum gravity is not, after all, real *but complex?* There is, in fact, a large body of opinion (for reviews, see Refs. [23,25]) which holds that a complex dynamical equation should be obtained in quantum gravity by solving (before quantization) the basic *quadratic* constraint in the conjugate momenta that leads to the WDE for one of the dynamical degrees of freedom (say, the volume of the Universe or the trace of the extrinsic curvature), which is then declared to be a distinguished variable that plays the role of "time" ("internal time"). In such an approach, the "time" variable then appears in the quantum form of the theory exactly like the ordinary time in the TDSE (1). *I am not thinking of such a resolution to the i problem.* Kuchař [23] has identified many problems with this approach, and I believe there are several arguments that can be added to his [15,29].

My suggestion, which I advanced in Ref. [38], is that all the forms of the WDE that have so far been considered may be physically incomplete. This would be the case if the true wave equation of the Universe contains not only second functional derivatives (as in existing WDE's) but also first derivatives, which would then appear multiplied by an i. This does, of course, happen in ordinary nonrelativistic quantum mechanics for a charged particle in an external field, which, as I noted in Ref. [38], contains among others the interaction term

$$2ieA_k\frac{\partial\Psi}{\partial x^k}\ , \tag{13}$$

where e is the electric charge and A_k is the vector potential at the position x^k at which the particle is situated. This, of course, is in the first-quantized theory, and the i that appears in (13) is not the "quantum i" but the i associated with the gauge group of quantum electrodynamics; in the more fundamental second-quantized theory this i does not appear in the wave function but only in the transformations of the classical fields on which the wave functional is defined.

However, *a priori* there is reason why the WDE should not contain linear derivatives (not with respect to an external time but with respect to some or all of the dynamical variables) multiplied by i.

The attraction of a term such as (13) is that if ever a WKB regime for the corresponding WDE is established (and the term with i that produces the dynamical coupling between the real and imaginary parts is involved in the generation of the regime; i.e., the term is sufficiently large), then it must have a pure phase-matched form of the type e^{iS} or e^{-iS} alone, just as does happen for the TDSE.

In the presence of such a WKB regime, an approximate TDSE for the non-WKB variables would indeed arise as a dynamical necessity, and the third and final part of the semiclassical program would have been realized in as convincing a manner as the first two parts.

Indeed, consider the very simple model described by

$$\frac{\partial^2\Psi}{\partial x^2}+ai\frac{\partial\Psi}{\partial x}+k^2\Psi+\frac{\partial^2\Psi}{\partial y^2}+W(x,y)=0\ , \tag{14}$$

in which, compared with (5), the term with the i has been added. If we now seek a WKB solution of the equation obtained by omitting the last two terms, this must have the phase-matched form $Ae^{i\nu x}$. Indeed, $e^{i\nu x}$ is an exact such solution if $\nu^2+\alpha\nu-k^2=0$. Substitution of $e^{i\nu x}\phi(x,y)$ into (17) then leads to the exact equation

$$i(2\nu+a)\frac{\partial\phi}{\partial x}=-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-W\phi=0\ , \tag{15}$$

which again has the form of an approximate time-dependent Schrödinger equation, though now obtained by dynamical necessity once a WKB solution for the x system has been assumed. It is interesting to note that in this case there is no need to assume the presence of the first term in (14) with the second derivative. The second term with the i is sufficient, in the presence of the potential term $k^2\Psi$, to enforce the phase matching. Then in (15) the first term on the right-hand side is absent (the coefficient on the left-hand side is also changed but that is immaterial); since this term rather spoils the recovery of the TDSE (especially in view of the fact that higher derivatives can completely change the qualitative behavior of solutions even though they may be small), this could represent an attractive possibility.

I believe that this issue warrants a thorough reexamination in the light of the disappearance of both time and i in quantum gravity. Surely this must prompt as serious a review of the status of complex numbers in quantum theory as has been made for time? In such a review, which should update Pauli's response [3] to Ehrenfest to take into account all the many developments in the last 60 years, it will be important to distinguish the specifically quantum-mechanical use of i from the other uses of complex numbers, e.g., to represent charged fields, spinors, etc.

VII. CONCLUSIONS

If it turns out that the Wheeler-DeWitt equation is complex (or, alternatively, that there is some independent physical mechanism which forces its solutions to be complex and phase matched), this must have far reaching implications for quantum gravity. For one of the great problems of quantum gravity is the seeming remoteness of any possibility of experimental verification. All the interesting effects are held to occur at utterly unattainable Planck scales. But we are talking here about things all around us, for the TDSE governs all laboratory phenomena. If the emergence of the TDSE is a genuine physical consequence of specific terms in the WDE, this should be reflected in observational relationships, say, between the dimensionless numbers of physics.

The need to resolve the i problem could also serve as a guide to new physics. Is it altogether too fanciful to see a possible repetition of history, in which resolution of the i problem (in which the complex TDSE of laboratory physics stands opposed to the timeless real WDE) could have as dramatic consequences as Dirac's reconciliation of special relativity with the needs of quantum mechanics? In fact, the protagonists are the same in both cases: time and complex numbers.

Successful recovery of the TDSE within the semiclassical approach could show that the problem of quantizing general relativity is much simpler than has hitherto been assumed. Indeed, according to two recent reviews [13,39], the conditions that have to be met before one can say that general relativity has been quantized are extraordinarily severe. The reason for this severity is that a conservative attitude to the problem is adopted: Quantized general relativity is expected to fit the mold of existing quantum theory and have a well-defined Hilbert-space structure, inner product, self-adjoint operators, etc. It is an understatement to say that the task of finding all these things is daunting.

In fact, I suspect that the full quantum framework that Dirac [6] formulated with such lucidity is fine for microscopic systems in the background of a laboratory anchored in a classical world, but a Procrustean bed into which a theory of the Universe such as general relativity *just will not go.*

This is where the issue of convincing recovery of the TDSE is so relevant. If this can only be done by making the WDE complex by some physical element (rather than by assumed "kinematic complexity"), it will then be obvious that much of what is currently taken to be essential formal structure of the quantum mechanics of the world is nothing of the sort. The essence of quantum gravity would be simply a wave equation on the configuration space of the Universe. The rest would then be an effective theory that results from a potentially very complicated amalgam: other physics (which would make the WDE complex), the fact that we consider microscopic systems in a WKB regime of the Universe, and our inability to interpret quantum mechanics satisfactorily.

If this view is correct, general relativity *has already been quantized.* That happened in 1967 when DeWitt [14] turned the Hamiltonian and momentum constraints of general relativity into wave equations on the timeless configuration space of the Universe. According to this view, the part of canonical quantum gravity that is currently regarded as supremely difficult—the introduction of a complex Hilbert space with proper conserved inner product and statistical interpretation—would simply fall away as premature. That structure need only appear in a WKB regime—where it will happen automatically if a genuine dynamical recovery of the time-dependent Schrödinger equation is once achieved.

I hope this paper will stimulate work in this direction.

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