The three Millennium problem solutions, RH, NSE, YME, and a Hilbert scale based quantum geometrodynamics

Overview

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The homepage www.fuchs-braun.com provides solutions to the following Millennium problems

- the Riemann Hypothesis
- well-posed 3D-nonlinear, non-stationary Navier-Stokes equations
- the mass gap problem of the Yang-Mills equations.

A common underlying distributional Hilbert space framework provides an answer to Derbyshine's question ((DeJ) p. 295)$^{(*)}$: „ *What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles*?". It enables quantum gravity theory based on an only (energy related) Hamiltonian formalism, as the corresponding (force related) Lagrange formalism is no longer defined (due to the reduced regularity assumptions to the domains of the concerned operators).

The Bagchi Hilbert space reformulation of the Nyman, Beurling and Baez-Duarte RH criterion provides the link between the two solution areas above (BaB). The Zeta function on the critical line is an element of the distributional Hilbert space H_{-1}. Therefore, in order to verify the Hilbert-Polya conjecture any (weak) eigenfunction solution of a self-adjoint operator equation to verify the Hilbert-Polya conjecture needs to be an element of a $H_{-1/2}$. The imaginary part values ω_n of the zeros of the considered Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ (alternatively to $e^{2\pi inx}$) with its corresponding Mellin transform

$$M\left[\,{}_1F_1\left(\tfrac{1}{2},\tfrac{3}{2};-x\right)\right](s)=\int_0^\infty x^s\,{}_1F_1\left(\tfrac{1}{2},\tfrac{3}{2};-x\right)\frac{dx}{x}=\frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)}\quad,\quad 0<Re(s)<1$$

enjoy appropriate properties (SeA), e.g. $2n-1<2\omega_n<2n<\omega_n+\omega_{n+1}<2n+1$ satisfying the "Hadamard" gap" condition.

The corresponding analysis of the 2D-NSE for the 3D-NSE fails due to not appropriate Sobolev norm estimates. This is called the Serrin gap.

The classical Yang-Mills theory is the generalization of the Maxwell theory of electromagnetism where chromo-electromagnetic field itself carries charges. As a classical field theory it has solutions which travel at the speed of light so that its quantum version should describe massless particles (gluons). However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. This is the Yang-Mills mass gap.

The current quantum state Hilbert space $H_0=L_2$ is extended to $H_{-1/2}$ to enable a Hilbert space based quantum gravity theory.

PART I:
Braun K., *A Kummer function based alternative Zeta function theory to solve the Riemann Hypothesis and the binary Goldbach conjecture*

PART II:
Braun K., *3D-NSE and YME mass gap solutions in a distributional Hilbert scale frame enabling a quantum gravity theory*

$^{(*)}$: ... "*The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles*?"

PART I

A Kummer/cot-function based alternative Zeta function theory to solve the Riemann Hypothesis

The Riemann Hypothesis states that the non-trivial zeros of the Zeta function all have real part one-half. The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self-adjoint operator. It is related to the Berry-Keating conjecture that the imaginary parts of the zeros of the Zeta function are eigenvalues of an „appropriate" Hermitian operator $H = \frac{1}{2}(xp + px)$ where x and p are the position and conjugate momentum operators, respectively, and multiplicity is noncommunative. The operator H is symmetric, but might have nontrivial deficiency indices (W. Bulla, F. Gesztesy, J. Math. Phys. 26 (1), October 1985), i.e. in a mathematical sense H is not Hermitian.

The key ingredients of the Zeta function theory are the Mellin transforms of the Gaussian function and the fractional part function. To the author´s humble opinion, the main handicap to prove the RH is the not-vanishing constant Fourier term of both functions. The Hilbert transform of any function has a vanishing constant Fourier term.

Let H and M denote the Hilbert and the Mellin transform operators. Replacing the Gaussian function $f(x) := e^{-\pi x^2}$ and the fractional part function by its Hilbert transforms enables an alternative Zeta function theory.

The Mellin transform of the Gaussian function is given by

$$M[f](s) = \frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) \;,\quad M[-xf'(x)](s) = \frac{s}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) = \frac{1}{2}\pi^{-s/2}\Pi(\frac{s}{2}) \;.$$

The related Theta function properties (based on the Poisson summation formula) of

$$G(x) := \theta(x^2) := \sum_{-\infty}^{\infty} e^{-\pi n^2 x^2} = 1 + 2\sum_{1}^{\infty} e^{-\pi n^2 x^2} =: 1 + 2\psi(x^2) = \frac{1}{x}\sum_{-\infty}^{\infty} e^{-\pi\frac{n^2}{x^2}} = \frac{1}{x} G\frac{1}{x})$$

leads to the Riemann duality equation in the form (EdH) 1.8)

$$\xi(s) := \frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) = (1-s)\cdot\zeta(s)M[-xf'(x)](s) = \zeta(s)\cdot M[-x(xf'(x))'](s) = \xi(1-s).$$

The Mellin transform for Riemann's auxiliary function

$$H(x) := -\frac{d}{dx}(x^2\frac{d}{dx})G(x)$$

is well defined and it holds

$$\int_0^{\infty} x^{1-s} H(x)\frac{dx}{x} = \int_0^{\infty} x^{s} H(x)\frac{dx}{x} \;.$$

(*) The Hilbert transform of the Gaussian function is given by the Dawson function

$$F(x) := e^{-x^2}\int_0^x e^{t^2}dt = \int_0^{\infty} e^{-t^2}\sin(2xt)\,dt = x\;{}_1F_1\left(1,\frac{3}{2};-x^2\right) = xe^{-x^2}\;{}_1F_1\left(\frac{1}{2},\frac{3}{2};x^2\right).$$

The appropriate related Mellin transform formulas are given by ((GrI) 7.612)

$$\int_0^{\infty} x^s\;{}_1F_1(\alpha,\beta;-x)\frac{dx}{x} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\Gamma(s)\frac{\Gamma(\alpha-s)}{\Gamma(\beta-s)} \;,\; 0 < Re(s) < Re(\alpha) \;,$$

$$\int_0^{\infty} x^{\mu-1}e^{-\beta x^2}\sin(\gamma x)dx = \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}}\;{}_1\Gamma(\frac{1+\mu}{2})F_1(1-\frac{\mu}{2};\frac{3}{2};\frac{\gamma^2}{4\beta}) \;,\; Re(\beta) > 0 \;,\; Re(\mu) > -1$$

leading to e.g., $\frac{1}{2}\int_0^{\infty} x^{s/2}\;{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\frac{dx}{x} = \frac{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)}{1-s} = \frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)} = \frac{\Pi\left(\frac{s}{2}\right)}{s(1-s)}$, $0 < Re(s) < 1$. It indicates a replacement of the Gauss „Gamma" function definition ((EdH) p.8)

$$\Pi\left(\frac{s}{2}\right) := \Gamma\left(1+\frac{s}{2}\right) = \frac{s}{2}\Gamma\left(\frac{s}{2}\right) \qquad \rightarrow \qquad \Gamma^*\left(1+\frac{s}{2}\right) := \Gamma\left(\frac{s}{2}\right)tan\left(\frac{\pi}{2}s\right) = \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{\Gamma\left(1+\frac{s-1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma\left(1+\frac{s}{2}\right)}{(k-\frac{1}{2})^2-(\frac{s}{2})^2}.$$

We note the formula ((GRI) 3.511, 8.332) $\frac{2}{\pi}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2 dt = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\pi}{cosh\,(\pi t)}dt = 1$, i.e. $\Gamma(\frac{1}{2}+it) \in L_2(-\infty,\infty)$.

Formally it also holds

$$\int_0^\infty x^{-s}\left[(-\frac{d}{dx}x^2\frac{d}{dx})G(x)\right]dx = s(1-s)\int_0^\infty x^{-s}G(x)dx.$$

It implies that the invariant operator $x^{-s} \to \int_0^\infty x^{-s}G(x)dx$ is formally self-adjoint with the transform $2\xi(s)/(s(s-1))$. But this operator has no transform at all as the integrals do not converge, due to the not vanishing constant Fourier term of the Poisson summation formula ((EdH) 10.3). Replacing $f(x) \to \ f_H(x) := M[f](x)$ leads to an alternative entire Zeta function $\xi^*(s)$ in the form

$$\xi^*(s) := \frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\,tan(\frac{\pi}{2}s)\cdot\zeta(s) = \zeta(s)\cdot M\left[\frac{d}{dx}[-x\cdot f_H(x)]\right](s)$$

with same zeros as $\xi(s)$, as it holds $s(1-s)\xi^*(s)\xi^*(1-s) = \pi\xi(s)\xi(1-s)$.

A similar situation is valid, if the duality equation is built on the fractional part function ([TiE] 2.1).

The Mellin transforms in the critical stripe for the distributional Fourier series representation of the cot −function in a distributional H_{-1} −sense are given by (*)

$$M[Cot^*](s) = \zeta(1-s)\cdot tan\left(\frac{\pi}{2}s\right) = \zeta(1-s)\cdot cot\left(\frac{\pi}{2}(1-s)\right)$$

$$M\left[\frac{1}{x}Cot^*(\frac{1}{x})\right](s) = M[Cot^*](1-s) = \zeta(s)\cdot cot\left(\frac{\pi}{2}s\right).$$

(*) The Bagchi Hilbert space based RH criterion is dealing with the fractional part function. Its Hilbert transform is given by

$$g(x) := \ln\left(2\sin\left(\frac{x}{2}\right)\right) = -\sum_{n=1}^\infty \frac{\cos(nx)}{n},$$

which is an element of H_0. Therefore, its related Clausen integral ((AbM) 27.8) is an element of H_1, and its first derivative, $\frac{1}{2}\cot\left(\frac{x}{2}\right)$ resp. $\cot(\pi x)$, joins the Zeta function on the critical line as an element of H_{-1}. The H_{-1} Hilbert space corresponds to the weighted l_2^{-1} −space as considered in (BhB). As $g(x) \in H_0 = H_0^*$, it holds

$$(g,v)_0 \cong (g',v)_{-\frac{1}{2}} = (S^1[g],v)_{-\frac{1}{2}} = (Cot,v)_{-1/2} < \infty,\ \ \forall v \in H_0$$

i.e. the formally derived Fourier series representation of

$$Cot(x) = \sum_{n=1}^\infty \sin(nx) \quad \text{resp.} \quad Cot^*(x) = 2\sum_{n=1}^\infty \sin(2\pi nx)$$

is defined in a distributional H_{-1} −sense (see also (BeB) (17.12) (17.13)). For $a > 0$ and $0 < |Re(s)| < 1$ it holds ((GrI) 3.761)

$$\int_0^\infty x^s \sin(a)\frac{dx}{x} = \frac{\Gamma(s)}{a^s}\sin(\frac{\pi}{2}s) \quad , \quad \int_0^\infty x^s \cos(a)\frac{dx}{x} = \frac{\Gamma(s)}{a^s}\cos(\frac{\pi}{2}s).$$

Therefore the Mellin transforms of the H_{-1} – distributional Fourier series representation of the $Cot^{(*)}$- resp. $G_H(x)$ – functions are given by

$$M[Cot](s) = \Gamma(s)\sin(\frac{\pi}{2}s)\zeta(s) \text{ resp. } M[Cot^*](s) = 2(2\pi)^{-s}\Gamma(s)\sin(\frac{\pi}{2}s)\zeta(s)$$

$$M[G_H(x)](s) = 2\sqrt{\pi}\sum_1^\infty\int_0^\infty\int_0^\infty x^s e^{-\pi t^2}\sin(2\pi nxt)\,dt\frac{dx}{x} = 2M[\sum_1^\infty f_H(nx)](s)$$

$$= \sqrt{\pi}\int_0^\infty e^{-\pi t^2}\int_0^\infty x^s Cot^*(tx)\frac{dx}{x}dt = \sqrt{\pi}\left[\int_0^\infty t^{1-s}e^{-\pi t^2}\frac{dt}{t}\right]\cdot\left[\int_0^\infty x^s Cot^*(x)\frac{dx}{x}\right] = \pi^{\frac{s}{2}}\Gamma(\frac{1-s}{2})\cdot M[Cot^*](s)$$

In combination with the functional equation of the entire Zeta function in the form $\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\zeta(1-s)$ ((TiE) (2.1.1)) this leads to

$$M[Cot^*](s) = \zeta(1-s)\cdot tan\left(\frac{\pi}{2}s\right) \ \ , M\left[\frac{1}{x}Cot^*(\frac{1}{x})\right](s) = M[Cot^*](1-s) = \zeta(s)\cdot cot\left(\frac{\pi}{2}s\right).$$

On the critical line $s = \frac{1}{2} + it$ holds $M[Cot^*](s)\cdot M[Cot^*](1-s) = \zeta(s)\cdot\zeta(1-s)\cdot[1+tanh^2(\pi t)]$ because of

$$\sin\left(\frac{\pi}{2}s\right) = \frac{1}{\sqrt{2}}\left[\cosh\left(\frac{\pi}{2}t\right) + i\cdot\sinh(\frac{\pi}{2}t)\right] \text{ and } |\Gamma(s)|^2 = \frac{\pi}{\cosh(\pi t)}$$

$$\cot\left(\frac{\pi}{2}s\right) = \tan\left(\frac{\pi}{2}(1-s)\right) = 1 - i\cdot\tanh(\pi t) = 1 - 2i\cdot\sum_{k=1}^\infty(-1)^k e^{-2kt} \ \ (\ t > 0\ )$$

$$\cot\left(\frac{\pi}{2}(1-s)\right) = \tan\left(\frac{\pi}{2}s\right) = 1 + i\cdot\tanh(\pi t) = 1 + 2i\cdot\sum_{k=1}^\infty(-1)^k e^{-2kt} \ \ (\ t > 0\ )$$

From (TiE) 4.14)), (ObF) p. 182, and (EsR) p. 139, we recall the formulas

$$\zeta(s) - \sum_{n<x} n^{-s} = \sum_{n>x} n^{-s} = -\frac{1}{2i}\int_{x-i\infty}^{x+i\infty} z^{1-s}\cot(\pi z)\frac{dz}{z}\ , \ \ Re(s) > 1;$$

$$M\left[\frac{1}{\pi}\frac{x^n}{1-x}\right](s) = cot(\pi s) \text{ (principle value) } \ -n < Re(s) < 1-n\ ,\ n = 0, \pm1, \pm2, \dots$$

$$F.p.(P.v.\int_0^\infty \frac{x^\alpha}{1-x}dx = \begin{cases} 0 & ,\alpha \in Z \\ \pi\cot(\pi\alpha) & , else \end{cases}.$$

They are related to the operator $x^{-s} \to \int_0^\infty x^{-s} G_H(x)dx$ (in a distributional H_{-1} –sense) by

$$M[G_H(x)](s) = \pi^{\frac{s}{2}}\Gamma(\frac{1-s}{2})M[Cot^*](s)$$

whereby it holds

$$M[-xG'_H(x)](s) = s[G_H(x)](s) \ , \ M[(xG_H)'(x)](s) = (1-s)[G_H(x)](s).$$

The Polya criterion is about the approximation of the Mellin transform integral over the half-line $(0,\infty)$ by integrals over *finite intervals* to obtain a theorem about zeros of the Mellin transforms ((EdH) 12.5), (PoG). The Mellin transform $M[G_H(x)](s)$ is a Müntz type representation, i.e. in a classical framework the Polya criterion cannot be applied.

We note the similar structure between the Polya RH criterion the automodel criterion ((EsR) p.57). The functions $k(x) := cot(x)$ resp. $h(x) := \frac{1}{x}cot\left(\frac{1}{x}\right)$ are slow varying functions (automodels) of order zero ((EsR) p.57) (*). Other slow varying functions are $-logx$ at $x = 0^+$ or $-\log(1-x)$ at $x = 1$ (SeE).

The functional analysis approach to prove the Prime Number Theorem (PNT) is based on Tauberian theorems, which are derived from the celebrated Wiener Tauberian theorem, that „*the closed linear hull of translates of a function f is the whole space L_1 if and only if its Fourier transform never vanishes*" (**).

In (PiS) Tauberian theorems for integral transforms are provided, which are of Mellin convolution type and whose kernels belong to suitable test function spaces. The result is based on the Wiener-Tauberian theorems for distributions as proven in (PiS1). In (ViJ) a corresponding functional analysis scheme for Tauberian problems is provided to *(prove)* the prime number theorem based on the Dirac delta measure δ_a ($(\delta_a,\varphi) = \varphi(a)$). It is built on the Delta function representation of

$$\psi'(x) = \sum_{n\le x}\Lambda(n)\delta(x-n) \in H_{-\frac{1}{2}-\varepsilon} \qquad \left(\psi(x) = \sum_{n\le x}\Lambda(n) = \int_{a-i\infty}^{a+i\infty}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]x^s\frac{ds}{s} \approx x\right)$$

whereby the generalized Mellin transform of $\sum_{n=1}^\infty \delta(x-n)$ ($Re(s) < 0$) is given by $\zeta(1-s)$ ((ZeA) 4.3). It is proposed to replace the formal delta series $(f_x,\varphi) = \sum_{n=0}^\infty c_n\,\delta_{\frac{n}{x}}$ to the numerical series $\sum_{n=0}^\infty c_n$ by $(f_x,\varphi)_{-1/2} < \infty$,$\forall\varphi \in H_{-1/2}$. Conceptually this goes along with a replacement of the „dual" relationship $L_1 \leftrightarrow L_\infty$ by $H_{-1/2} \leftrightarrow H_{1/2}$(***). The latter Hilbert spaces are the appropriate framework for central functions in current Zeta function theory (****). For a corresponding generalized Mellin (integral) transformation in the form $F(s) = (f(x),x^s)_{-1/2}$ we refer to (ZeA). For $\vartheta(x) = \sum_{n\le x}\Lambda(n)\,log(\frac{x}{n})$ we note the related asymptotics (KoJ) (ViJ)

$$\lim_{\lambda\to\infty}\vartheta'(\lambda x) = \frac{d}{dx}\left[\sum_{n=1}^\infty \Lambda(n)\,log(\frac{\lambda x}{n})\right] = \lim_{\lambda\to\infty}\frac{\psi(\lambda x)}{\lambda x} = 1 \ .$$

(*) For $k(x) := cot(x)$ resp. $h(x) := \frac{1}{x}cot\left(\frac{1}{x}\right)$ it holds $\frac{xk'(x)}{k(x)} = -\frac{2x}{\sin(2x)}$ resp. $\frac{xh'(x)}{h(x)} = -1 + \frac{2/x}{\sin(2/x)}$;

(**) It is about the behavior of the function f, where the limit for the convolution integral $K[f](x)$ when $x\to\infty$ corresponds to $\hat{k}(0)$ ($\hat{k}$ denotes the Fourier transform of the kernel function $k(x)$);

(***) There is a similar differentiator between a proof of the PNT (****) (from which the convergence of the series $\sum_{n=1}^\infty\frac{\mu(n)}{n}$ can be derived) and a proof of the convergence of the series $\sum_{n=1}^\infty\frac{\mu(n)}{n}log(\frac{1}{n}) = 1$. Ikehara showed a Tauberian theorem for Dirichlet series in a L_1 – framework, which is equivalent to the statement that $d\psi(x)\sim dx$ as a Cesaro average.

„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived" ((LaE) §160).

(****)

$$\rho(x) = x - [x] = \frac{1}{2} + \sum_{n=1}^\infty\frac{sin(2\pi nx)}{\pi n},\ \rho_H(x) = \sum_{n=1}^\infty\frac{cos(2\pi nx)}{\pi n} = -\frac{1}{\pi}log(2\,sin(\pi x)), log\left(tan\left(\pi\frac{x}{2}\right)\right) \in L_2^\#(0,1),$$

$$\rho'_H(x) = -cot(\pi x) = -2\sum_{n=1}^\infty sin(2\pi nx), log'\left(tan\left(\pi\frac{x}{2}\right)\right) = \frac{\pi}{sin(\pi s)} \in H_{-1}^\#(0,1),$$

$$\|\Xi\|_{-1}^2 = \sum_{n=1}^\infty\frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} = \int_0^1\frac{log\,x}{x-1}dx = \left[\sum_{n=1}^\infty\frac{\mu(n)}{n^2}\right]^{-1}, \text{ i.e. } \Xi \in l_2^{-1}$$

(****) (EdH) 12.7: „The PNT is about the asymptotics equivalence of $\psi(x) = \sum_{n<x}\Lambda(n)\sim x$, which is equivalent to the statement that $d\psi(x)\sim dx$ as a Cesaro average in the context of Tauberian theorems. Hardy-Littlewood were able to prove the PNT by showing $d\psi(x)\sim dx$ as an Abel average, where a significant amount of work is done by a Tauberian theorem."

Alternatively to the usage of the Hardamard distribution function $\psi'(x)$ with the Dirac function domain $H_{-\frac{1}{2}-\varepsilon}$ we shall use distribution functions with a $log(\frac{x}{n})$ structure in combination with point measures enabling integer subsets with Snirelmann density ½.

The considered Hilbert space in (BaB) is about of all sequences $a = \{a_n | n \in N\}$ of complex numbers such that $\sum_{n=1}^{\infty} \theta_n |a_n|^2 < \infty$ with $\frac{c_1}{n^2} \leq \theta_n \leq \frac{c_2}{n^2}$, which is isomorph to the Hilbert space $H_{-1} \cong l_2^{-1}$. The real part values of the zeros of the considered Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ (alternatively to $e^{2\pi inx}$) enjoy appropriate behaviors [(*)]. The linkage to convergent Dirichlet series

$$f(s) := \sum_1^{\infty} a_n e^{-s\log n} \quad g(s) := \sum_1^{\infty} b_n e^{-s\log n} \text{ , for } s > 0$$

to the (distributional) Hilbert spaces $H_{-1/2} \cong l_2^{-1/2}$ resp. $H_{-1} \cong l_2^{-1}$ is given by the inner products [(**)]

$$\left((f,g)\right)_{-1/2} := \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(1/2+it) g(1/2-it) dt = \sum_1^{\infty} \frac{1}{n} a_n b_n$$
$$\left((f,g)\right)_{-1} := \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(1+it) g(1-it) dt = \sum_1^{\infty} \frac{1}{n^2} a_n b_n .$$

For the Zeta function on the critical line $\Xi(t) \coloneqq \zeta\left(s = \frac{1}{2} + it\right) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ it holds

$$\Xi \in H_{-\frac{1}{2}-\varepsilon} \quad \text{resp.} \quad \|\Xi\|_{-1/2}^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\Xi(t)|^2 dt = \sum_{n=1}^{\infty} \frac{1}{n} = \zeta(1) = \infty.$$

Putting $w(t) \coloneqq \sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right) \frac{\mu(n)}{n^s}$ it holds $\left((\Xi, w)\right)_{-1/2} = 1$ from which it follows that $w \in H_{-\frac{1}{2}+\varepsilon}$.

The Hilbert space $H^{\#}_{-\frac{1}{2}} \cong l_2^{-1/2}$ enables a distributional form of the Snirelmann density $\lim_{n\to\infty} \frac{A(n)}{n}$ given by $\sum_{n=1}^{\infty} \frac{1}{n} a_n^2 = \|A\|_{-1/2}^2$ with $A = (a_n)_{n\in N} \in l_2^{-1/2}$. It puts another light on the dispersion method in binary additive number theory problems, where the binary Goldbach problem is inaccessible in the given form (LiJ).

What can derived from the PNT is the convergence of $\sum_{n=1}^{\infty} \frac{\mu(n)}{n}$. What cannot derived from the PNT is the convergence of the series $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} log(\frac{1}{n}) = 1$ [(**)]. *„This theorem goes deeper than the PNT"* ((LaE2) §159). For the corresponding arithmetical function $\sigma(x) := \sum_{n\leq x} \frac{\mu(n)}{n} \log\left(\frac{x}{n}\right)$ and $A(x) \coloneqq \sum_{n\leq x} \frac{\mu(n)}{n}$ it holds $A(x) = o(1)$ ((ApT) p. 71) and for $x \geq 1$

$$\sigma(xy) + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} log(\frac{1}{n}) = \sigma(xy) + 1 = \sigma(x) + \sigma(y) \;, \quad \sigma'(x) = \frac{1}{x} \sum_{n\leq x} \frac{\mu(n)}{n} \sim \frac{1}{x} \quad .$$

Its inverse mapping is given by

$$\sigma^{-1}(x) = \sum_{n\leq x} \frac{1}{n} log\left(\frac{x}{n}\right).$$

[(*)] For the real part values ω_n of the zeros of ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ it holds (SeA) $2n-1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n+1 < 2\omega_{n+1} < 2(n+1)$ and the sequences $2\omega_n$ and $\omega_n + \omega_{n+1}$ fulfill the Hadamard gap condition

$$\frac{\omega_{n+1}}{\omega_n} > \frac{n+\frac{1}{2}}{n} = 1 + \frac{1}{2n} > q > 1 \quad \text{resp.} \quad \frac{\omega_{n+1}+\omega_{n+2}}{\omega_n+\omega_{n+1}} > \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > q > 1.$$

We mention the theorem of Kakeya (HuA) from which is follows that all zeros of $\sum_{k=1}^{n} s_k x^k = 0$ lie in the circular disk $\frac{1}{2} < |x| < 1$. We further mention the relationship to the uniform distribution of numbers mod 1(WeH).

[(**)] The average orders of $d(n), \varphi(n), \mu(n), \Lambda(n)$ orders are $D(x) \coloneqq \frac{1}{x}\sum_{n\leq x} d(n) = \log x + (2\gamma - 1) + \frac{1}{x}\delta(x) \sim log(x)$ (with $\delta(x) = O(\sqrt{x})$), $\sum_{n\leq x} \varphi(n) = \frac{3}{\pi^2} + x^2 + O(x\log x)$, zero and one. The latter two average order results are both equivalent to the PNT. Some weighted average orders of those two artithmetic functions are given by $\sum_{n\leq x} \mu(n) \left[\frac{x}{n}\right] = 1$, $\sum_{n\leq x} \Lambda(n) \left[\frac{x}{n}\right] = \sum_{p\leq x} \left[\frac{x}{p}\right] logp = x\log x - x + O(\log x)$, (ApT) pp. 57, 66, 68. Dirichlet's asymptotics $\delta(x) = O(\sqrt{x})$ has been improved, but the exact order is still undetermined. The problem is closely related to that of the Riemann Zeta function: for $c > 0$ and x not an integer it holds $D(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^2(z) x^z \frac{dz}{z}$ and therefore (for $0 < c' > 1$) $\delta(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \zeta^2(z) x^z \frac{dz}{z}$ (TiE) 12.1. It further holds $\sum_{n=1}^{\infty} d(n) n^{-s} = \zeta^2(s)$ resp. $(1-s)\int_1^{\infty} D(x) x^s \frac{dx}{x} = \sum_{n=1}^{\infty} \frac{d(n)}{n} n^s = \zeta^2(1-s)$ (NaC), and $d(n) = O(x^{\varepsilon})$, $\lim_{T\to\infty} \frac{1}{T} \int_1^T |\zeta(\sigma+it)|^2 dt = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^{2\sigma}}$ for $\sigma > 0$ (TiE) p. 171, 148.
The RH is true iff $\frac{1}{x}\sum_{n\leq x} \mu(n) = O(x^{-\frac{1}{2}+\varepsilon})$ iff $\psi(x) - x = O(x^{-\frac{1}{2}+\varepsilon}$, (TiE) 14.25.

[(**)] (ApT): „mean value formulas for Dirichlet series", p. 240, (BiN1): „orthogonal polynomials on the unit circle; $l_2^{1/2}$ as subspace of l_2, and Szegö's theorem and its probabilistic descendants, new definition of long range dependence", (NaS): „$h(x) \coloneqq \frac{1}{x}\sum_{n\leq x} d(n) - [\log x + (2\gamma - 1)] \in L_2(0,\infty)$, $H(s) \coloneqq M[h](s) = \frac{\zeta^2(1-s)}{1-s} \in L_2(-\infty,\infty)$ on the critical line, and a formula involving sums of the form $\sum d(n) f(n)$".

In the context of additive number theory problems the asymptotics of some related arithmetical functions for $x \geq 2$ (!) are given by ((LaE1) (*), (ApT) (**), (ScW) p. 216),

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} \sim \log x = \int_1^x d(\log t) \sim \Phi(x) = \sum_{n \leq x} \frac{1}{\varphi(n)} \sim \sum_{p \leq x} (1 - \frac{1}{p})^{-1}$$

$$\sum_{n \leq x} \frac{\mu(n)}{n \log n} \sim \log(\log x) = \int_1^{\log x} d(\log t) \sim \sum_{p \leq x} \frac{1}{p} \quad .$$

Putting $0 < \omega_0 := 1 - \omega_1 < \frac{1}{2}$ we consider the sequences $s_n^{(1)} := \frac{2\omega_n}{2n}$, $s_n^{(2)} := \frac{\omega_{n-1}+\omega_n}{2n-1}$ fulfilling

$$\min\left\{\frac{2n-1}{2n}, \frac{2n}{2n+1}\right\} = 1 - \frac{1}{2n} < s_n^{(1)}, s_n^{(2)} < 1 \ .$$

The related integer subsets

$$F_{1;2n} := \{\{[\omega_{n-1} + \omega_n]\} | n \in N\} = \left\{\{1, \{2n\}\} \middle| n \in N\right\}, \ F_{2n-1} := \{\{[2\omega_n]\} | n \in N\} = \{\{2n-1\} | n \in N\}$$

do have the Snirelmann density $\sigma(F_{2n-1}) = \sigma(F_{2n}) = \frac{1}{2}$. They enable the definition of (binary additive) distributional functions for $x \geq 1$ (!) in the form

$$\sum_{\substack{n \leq x \\ n \in F_{1;2n}}} a_n \log(s_n^{(1)} \frac{x}{n}) + \sum_{\substack{n \leq x \\ n \in F_{2n-1}}} a_n \log(s_n^{(2)} \frac{x}{n}) \quad \text{resp.} \quad \sum_{\substack{n \leq x \\ n \in F_{1;2n}}} a_n \log(\frac{x}{2\omega_n}) + \sum_{\substack{n \leq x \\ n \in F_{2n-1}}} a_n \log(\frac{x}{\omega_{n-1}+\omega_n}) \ .$$

We note that the Snirelmann density is sensitive to the first values of a set. This is why the subset of even integers has a Snirelmann density zero, while the subset of odd integers has Snirelmann density $\frac{1}{2}$. Putting

$$\sigma^*(x) := \sum_{\substack{n \in F_{1;2n} \\ n \leq x}} \frac{\mu(n)}{n} \log\left(s_n^{(2)} \frac{x}{n}\right) + \sum_{\substack{n \in F_{2n-1} \\ n \leq x}} \frac{\mu(n)}{n} \log\left(s_n^{(1)} \frac{x}{n}\right)$$

$$= \log x + \sum_{\substack{n \text{ odd} \\ n \leq x}} \frac{\mu(n)}{n} \log\left(s_n^{(2)} \frac{x}{n}\right) + \sum_{\substack{n \text{ even} \\ n \leq x}} \frac{\mu(n)}{n} \log\left(s_n^{(1)} \frac{x}{n}\right) \ .$$

it follows $(x \geq 1)$

$$\sigma^*(xy) = \sigma^*(x) + \sigma^*(y) \quad , \quad \sigma^{*\prime}(x) = \frac{1}{x} + \sigma'(x) = \frac{1}{x} + \frac{1}{x} \sum_{n \leq x} \frac{\mu(n)}{n} \sim \frac{1}{x}$$

whereby $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$, $\left|\sum_{n \leq x} \frac{\mu(n)}{n}\right| \leq 1$ with equality holding only if $x < 2$ ((ApT) p.66, p. 97).

Let G_n denote the number of decompositions of an even integer n into the sum of two primes (whereby $p+q$ and $p+q$ are counted seperately); let $\tilde{G}_n := \frac{1}{2C} \frac{1}{\varphi(n)} \frac{n^2}{\log^2(n)}$ denotes the Stäckel approximation formula with $C := \frac{105 \cdot \zeta(3)}{2\pi^4} \sim 0{,}648 \ldots \sim \frac{1}{2}$. Then it holds (LaE1)

$$\frac{1}{2} \frac{x^2}{\log^2(x)} \sim \sum_{n=1}^{[x/2]} G_{2n} \sim \sum_{n=1}^{[x/2]} \tilde{G}_{2n} = \frac{1}{2C} \sum_{n=1}^{[x/2]} \frac{1}{\varphi(n)} \frac{n^2}{\log^2(n)}$$

whereby (*)

$$\frac{1}{2C} \sum_{n=1}^{[x/2]} \frac{\sigma_1(n)}{\log^2(n)} \leq \frac{1}{2C} \sum_{n=1}^{[x/2]} \frac{1}{\varphi(n)} \frac{n^2}{\log^2(n)} \leq \frac{\zeta(2)}{2C} \sum_{n=1}^{[x/2]} \frac{\sigma_1(n)}{\log^2(n)} .$$

Based on the above concept we propose the following alternative arithmetical function (**)

$$\sum_{n=1}^{[\omega_n + \omega_{n+1}]} \frac{\sigma_1(n)}{\log(\omega_n)} \frac{\sigma_1(n+1)}{\log(\omega_{n+1})} \ .$$

(*) With $C := \frac{105 \cdot \zeta(3)}{2\pi^4} \sim 0{,}648 \ldots \sim \frac{1}{2}$ and the Euler constant γ the asymptotics of

$$\Phi(x) := \sum_{n \leq x} \frac{1}{\varphi(n)} = \Phi_1(x) + \Phi_2(x) := \sum_{\substack{n \leq x \\ n \text{ odd}}} \frac{1}{\varphi(n)} + \sum_{\substack{n \leq x \\ n \text{ even}}} \frac{1}{\varphi(n)} = C\left[\log x + \gamma - \sum_p \frac{\log p}{p^2 - p + 1}\right] + \delta(x)$$

is given by $\lim_{x \to \infty} \delta(x) = o$ and (LaE1) $\Phi_1(x) = C \cdot \log x + c_1 + O(\frac{\log x}{x})$, $\Phi_2(x) = 2C \cdot \log x + c_2 + O(\frac{\log x}{x})$.

The related estimate to the sum of the divisors of n function $\sigma(n) = \sigma_1(n)$ is given by ((ApM) pp. 38, 57, 71), $\frac{\sigma(n)}{n^2} \leq \frac{1}{\varphi(n)} \leq \frac{\pi^2}{6} \frac{\sigma(n)}{n^2} = \zeta(2) \frac{\sigma(n)}{n^2}$, $n \geq 2$. The proof of the inequality is based on the formula $\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$ and the relation $1 + x + x^2 + \cdots = \frac{1}{1-x} = \frac{1+x}{1-x^2}$ with $x = \frac{1}{p}$. The average order of the divisor function $d(n)$ is given by $\log n$. The partial sums of the divisor function is given by $D(x) := \frac{1}{x} \sum_{n \leq x} d(n) = \log(x) + (2\gamma - 1) + O(1) \sim \log(x)$. It further holds $\sum_{n \leq x} \frac{d(n)}{n} \sim \frac{1}{2} \log^2(x)$, $\sum_{2 \leq n \leq x} \frac{1}{n \log n} \sim \log(\log((x))$.

(**) We note the two inequalities $\frac{1}{ab} > \frac{2}{a^2+b^2}$ & $\log 2 > \log \omega_1$ and $d(n) = O(x^{\varepsilon})$ (TiE) p. 171. *„The odd integers can be disregarded, as every odd integer n can be represented as sum of two primes, if $n-2$ is a prime number only, otherwise not“* (LaE5).

The current tool trying to prove the tertiary and binary Goldbach conjecture is about the Hardy-Littlewood circle method. It is about a dissection of the circle $x = e^{2\pi i\alpha}$, or rather a smaller concentric circle, into „Farey arcs". The major arcs, or basic intervals, provide the main term in the asymptotic formula for the number of representations. Their treatment does not give rise to any very serious difficulties compared to the problems presented by the *„minor arcs"*, or *„supplementary intervals"*. The latter ones are analyzed by estimates of the Weyl (trigonometrical) sums

$$S(x) \coloneqq \sum_n e^{2\pi inx}$$

without taking any (Goldbach) problem relevant information into account. We note that an asymptotic behavior in the form $O(N^{\frac{1}{2}+\varepsilon})$ of the Farey series is equivalent to to the Riemann Hypothesis (LaE5).

The Cesàro summable Fourier series representation (ZyA) VI-3, VII-1)

$$cot(\pi x) = 2\sum_{n=1}^{\infty} sin(2\pi nx)) \in H_{-1}^{\#}(0,1)$$

is related to the eigenfunctions $e^{2\pi inx} = e^{i\pi(2n)x}$. The proposed alternative Abel summable functions

$$cot^{(*)}(x) := \sum_{n=1}^{\infty} sin(\pi(2\omega_n)x) + sin\,(\pi(\omega_n + \omega_{n+1})x \in H_0^{\#}(0,1)$$

is related to the eigenfunctions pair $e^{i\pi(2\omega_n)x}$ and $e^{i\pi(\omega_n+\omega_{n+1})x}$ with corresponding alternative Weyl sums in the form

$$S_1^*(x) \coloneqq \sum_n e^{i\pi(2\omega_n)x},\ \ S_1^*(x) \coloneqq \sum_n e^{i\pi(\omega_n+\omega_{n+1})x}.$$

For the „weighted" $cot^{(*)}$ –function with the „alternative" harmonic numbers

$$2h_n := \sum_{k=1}^{n} \frac{2}{2k-1} = 2H_{2n} - H_n$$

the series

$$\sum \frac{2h_n}{n}(sin(2\pi\omega_n x) + sin\,(\pi(\omega_n + \omega_{n+1})x)$$

converges almost everywhere (*) (**) (***) (****).

The H_n are always fractions (except for $H_1 = 1$, $H_2 = 1.5$, $H_6 = 2.45$), the series is divergent, but the number n that the sum H_n past 100 is in the size of 10^{43}, i.e. a computer which takes 10^{-9} seconds to add each new term to the sum will have been completed in not less than 10^{17} (American) billion years (HaJ).

The extremly slow nondecreasing property on the interval $[1, n]$ might motivate the definition of an appropriate function to enable the corresponding Polya criterion (EdH) 12.5, (PoG)).

(*) For $T(x) \coloneqq -\frac{\pi}{2} log\left(tan\left(\pi\frac{x}{2}\right)\right)$ the following series representation holds true (ElL) $T(x) = \sum\frac{2h_n}{n} sin\,(\pi(2n)x) = \sum c_n\, sin\,(2\pi nx)$, whereby $\sum_{n=1}^{\infty} c_n^2 < \infty$ i.e. $T(x) \in L_2^{\#}(0,1)$ resp. the formal Fourier series representation of its first derivative $log'\left(tan\left(\pi\frac{x}{2}\right)\right) = \frac{\pi}{sin\,(\pi x)} \in H_{-1}^{\#}(0,1)$. The convergent series $\sum_{n=1}^{\infty} c_n^2 = \frac{\pi^4}{32} < \infty$ in combination with the

Lemma (KaM1): Let $\{n_k\}$ be a sequence of integers satisfying the "Hadamard gap" condition, i.e. $\frac{n_{k+1}}{n_k} > q > 1$. Then the trigonometric gap series $\sum_{k=1}^{\infty} c_k \sin\,(2\pi n_k x)$ converges almost everywhere, if and only if, $\sum_{k=1}^{\infty} c_k^2 < \infty$

then proves that the series $\sum\frac{2h_n}{n}(sin(2\pi\omega_n x) + sin\,(\pi(\omega_n + \omega_{n+1})x)$ converges almost everywhere.

(**) We note the related potency series in the form (ChH) $\frac{1}{2}log^2\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty}\frac{2h_n}{n}x^{2n}$.

(***) Alternatively to $\frac{\sin\,(\pi x)}{\pi x}$ the Fourier theory of cardinal functions enables a correspondingly absolute convergent cardinal series in the form $C(x) := \frac{\sin\,(\pi x)}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{H_{2n}-\frac{1}{2}H_n}{x-n}$ (WhJ2).

(****) (RiB) p. 11: „„.. *denn so gross auch unsere Unwissenheit darüber ist, wie sich die Kräfte und Zustände der Materie nach Ort und Zeit um Unendlichkleinen ändern, so können wir doch sicher annehmen, dass die Functionen, auf welche sich die Dirichlet'sche Untersuchung nicht erstreckt, in der Natur nicht vorkommen*"

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PART II

3D-NSE and YME mass gap solutions in a distributional Hilbert scale frame enabling a quantum gravity theory

A common mathematical model of unified quantum and gravity theories requires a truly infinitesimal geometric framework. The Hilbert space based framework in quantum theory is certainly the more suitable geometric framework compared to Weyl's manifold based ones. At the same point in time both theories need to leave something out as they are not compatible. In quantum theory already in the simple quantum harmonic oscillator model the eigenvalues converge equidistant to infinity, i.e. the total energy is infinite as well (*). A similar situation is given by the concept of „wave packages" with other (less regular) domain as the H_1 domain for standard Fourier waves. A related concept is a about wavelets leading to the extended Hilbert space $H_{1/2}$ (**). The standard quantum theory Hilbert space is $H_0 = L_2$ in order to enable to full statistical analysis, which is basically statistical thermodynamics.

In a general Hilbert scale framework H_α ($\alpha \in R$) a convergent momentum operator in a variational (quantum mechanical) representation would be given by $(u^{\prime}, v)_\alpha = (u, v)_{\alpha+1/2} < \infty$, $\forall\, v \in H_\alpha$. One of the current handicaps of quantum mechanics is the purely mathematically conditioned Dirac „point mass density" Hilbert space $H_{-n/2-\varepsilon}$, with $\varepsilon > 0$, where $n =$ denotes the space dimension (i.e. a point mass density in a one dimensional world (like the harmonic quantum oscillator) is different from a similar situation in a three dimensional (or even four-dimensional) model. The choice $\alpha \coloneqq -1/2$ is proposed as new quantum state Hilbert space with its corresponding energy space $H_{1/2}$.

The GRT is built on Riemann´s mathematical concept of „manifolds"; we note that the mathematical model of the GRT even requires *„differentiable*" manifolds, whereby only *continuous* manifolds are required by physical GRT modelling aspects, w/o taking into account any appropriate quantum theoretical modelling requirements. Therefore, challenging the „continuity" concept, taking into account also its relationship to the quantum theory Hilbert space framework H_a and the related Sobolev embedding theorem, supports to the proposed replacement of the Dirac function concept by an alternative $H_{-1/2}$ −quantum state Hilbert space also from a GRT perspective.

The Lagrange formalism is related to the concept of „force", while the Hamiltonian formalism is related to the concept of „energy". Both formalisms are equivalent *only* (!) in case the Legendre (contact) transform can be applied. Our proposed „alternative energy (Hilbert space) concept" goes along with reduced regularity assumptions of the concerned operators (similar to the regularity reduction when moving from standard potential function (*„mass density*") definition to Plemelj's *„mass element*" concept ($\sim C^1 \to C^0$)), (PlJ).

The *„mass generation process*" is modelled as a „selfadjoint property" break down by the orthogonal projection $H_{1/2} = H_1 \otimes H_1^{\urcorner} \to H_1$, i.e. the closed subspace $H_1^{\urcorner}$ is the model for the ground state (vacuum) energy, which is and can be neglected in all („less granular") Lagrange formalism based physical models.

(*) With respect to the Kummer functions from the part I we note that the eigenvalue problem of the Schrödinger equation with a Coulomb potential is solved by confluent hypergeometric series

(**) ((BiN): „Traditionally, the subject of time series seemed to consist of two non-intercommunicating parts, „time domain" and „frequency domain" (known to be equivalent to each other via the Kolmogorov Isomorphism Theorem. The subject seemed to suffer from schizophrenia …. This unfortune schism has been healed by the introduction of wavelet methods." Other relations to the Hilbert space $H_{1/2}$ are given by $H_{1/2} \subset VMO \subset BMO$, $H_{1/2}$ occurs in work on topological degree and winding number, conformal mapping, analytical continuability of the Szegö function beyond the unit disc and scattering theory; (NaS): the $H_{1/2}$ space as first cohomology is fundamental to explain the properties of period mapping on the universal Teichmüller space (NaS).

We further note that the exterior Neumann problem admits one and only one generalized solution in case the related Prandtl operator of order one $P\colon H_r \to H_{r-1}$ is defined for domains with $1/2 \leq r < 1$ (LiI).

In a nutshell from a mathematical modelling perspective

in the proposed model the (standard) „calculus in the small" meets the „calculus in the large" (MoM) in a Hamiltonian formalism for classes of non-linear equations, where the kinetic (matter, Lagrange formalism) energy part is based on a Krein space setting/decomposition (GaA) of (H_0, H_1) within a $(H_{-1/2}, H_{1/2} = H^*_{-1/2})$ Hilbert space framework (*).

The Hilbert space subspaces (H_0, H_1) are *compactly* embedded into the Hilbert spaces $(H_{-1/2}, H_{1/2})$. This is about the same cardinality relationships as for the embededness of the set of *rational* numbers into the fields of *real* or *hyper-real* numbers (**).

In a nutshell from a physical modelling perspective

the physical concepts of „*time*" and „*change*" are different sides of the same coin, i.e. there is no „time" w/o „change" and there is no „change" w/o „time". In other words, the concepts of „time" and „change" are and need to be in scope of the „matter/kinetic" energy model H_1, while its complementary ground state (vacuum) energy model $H_1^{\neg}$ is per definition independent from the *thermodynamical* concept of „*time*" (***) ((SmL), (PeR), (RoC1)).

As H_1 is compactly embedded into $H_{1/2}$, and given an initial universe w/o any thermodynamical „time" (i.e. $H_1 = \{\ \}$, with only existing ground state energy state for the whole mathematical model system) the probability for „symmetry break down" events to generate mass were and are zero; obviously those events happened and will go on to be happen. At the same point in time the generated and still being generated „matter world" H_1 is governed by e.g. the „least action principle" (KnA), and the principles of „statistical thermodynamic" (ScE), whereby the classical action variable of the system determines the „time" (HeW).

In a nutshell from a philosophical perspective we refer to (HaJ), (KaI) p. 67 (****), (ScE1), (WeH3) pp. 175, 177, 213.

(*) We note that the set of integers or rational numbers is „countable", while it is not for the fields of the real and hyper-real numbers. The corresponding (Cantor) cardinalities are given by $card(N) = card(Q) = \aleph$, $card(R) = card(\,^*R) = 2^{\aleph}$.

(**) The standard energy Hilbert space H_1 enables a differentiation of "elementary particles" with and w/o mass (modelled by the orthogonal decomposition of the Hilbert spaces $H_{-1/2} = H_0 \otimes H_0^{\perp}$ resp. $H_{1/2} = H_1 \otimes H_1^{\perp}$). The Hilbert space H_1 is proposed to be interpreted as „ferminons mass/energy" space; $H_1^{\perp}$ is proposed to be interpreted as the orthogonal „bosons energy" space. Both together build the newly proposed quantum energy space $H_{1/2} = H_1 \otimes H_1^{\perp}$. The sub-space $H_1^{\perp}$ may be interpreted as zero point energy space containing "wave package" resp. „eigen-differential" „elements".

The concept of an optical function is an essential tool in the strategy to overcome technical difficulties to overcome the problems of „coordinates", and the „strongly nonlinear hyperbolic features of the Einstein equations" for a global stability of the Minkowski space (*). It is basically about appropriately modified Killing and conformal Killing vectorfields in the definition of the basic norm.

(HoM) 1.2: „The idea of wavelet analysis is to look at the details are added if one goes from scale a to scale $a - da$ with $da > 0$ but infinitesimal small. … Therefore, the wavelet transform allows us to unfold a function over the one-dimensional space R into a function over the two-dimensional half-plane $\boldsymbol{H}$ of positions and details (where is which details generated?). … Therefore, the parameter space $\boldsymbol{H}$ of the wavelet analysis may also be called the position-scale half-plane since if g localized around zero with width Δ then $g_{b,a}$ is localized around the position b with width $a\Delta$. The wavelet transform itself may now be interpreted as a mathematical microscope where we identify $b \leftrightarrow$ position; $(a\Delta)^{-1} \leftrightarrow$ enlargement; $g \leftrightarrow$ optics.
The continuous wavelet transform with the complex Shannon wavelet can be considered via solutions of Cauchy problems for PDE in the context of the construction of wavelets for an analysis of non-stationary wave propagation in inhomogeneous media ((PoE).

(***) The macroscopic and microscopic state of quanta relate to corresponding frequencies of its vibrations. The corresponding action variables of the system ((HeW) II.1.c) define the related kinematical (physical) and thermodynamical concept of "time" (RoC), (SmL)), (RoC1), section 13)
(PeR): „one of the deepest mysteries of our universe is the puzzle of whence it came."
(RoC1), section 13: "Our interaction with the world is partial, which is why we see it in blurred way. To this blurring is added quantum indeterminacy. The ignorance that follows from this determines the existence of a particular variable - thermal time - and of an entropy that quantifies our uncertainty. Perhaps we belong to a particular subset of the world that interacts with the rest of it in such a way that this entropy is lower in one direction of our thermal time."

(****) (KaI) I. Convolut, V. Bogen, 4. Seite: *„Selbst der Gebrauch der Mathematik in Ansehung der Anschauungen a priori in Raum u. Zeit gehört zur Transc. Phil.. Es sollte nicht mit Newton heissen Philosophiae naturalis principia mathematica (den es gibt eben so wenig mathematische Principien der Phil. als philos. der Mathematik) sondern phil. transscend. princ. vel mathem. vel phil. als genus. Transc. Phil. ist das subjective Prinzip der vereinigt theoretisch//speculativen and moralisch//practischen Vernunft in einem System der Ideen von einem All der Wesen unter dem Prinzip synthetischer Sätze a priori worin es eben so wenig mathematische Principien der Philosophie als philosophische der Mathematik giebt. Transc: Phil. ist das Prinzip eines Systems der Ideen der synthetischen Erkenntnis a priori aus Begriffen wodurch das Subject sich selbst zum Objecte constituirt (Aenesidemus) und das Formale der Wahrnehmungen zum Behuf moglicher Erfahrung anticipirt"*

A common Hilbert space framework for all quantum gravity related physical mathematical models requires common conceptual building principles for problem specific mathematical-physical PDE system models.

The following changes to current building principles are proposed:

1. a classical PDE system is an „only" approximation model to its corresponding physical relevant variational representation, and not the other way around

2. only the Hamiltonian formalism is valid (*), but not the Lagrange formalism (both formalisms are equivalent, if the Legendre transform is valid), because of only physical (energy related) relevant, but no longer mathematically (force related) assumed regularity assumptions to the variational solution. In this context *we note that „continuity" is one of the commonsense notions, which should be dropped out of the assumptions list of ground principles of the Universe (KaM) p. 12*)); consequently, the physical concept of „*force*" stays to be a phenomenon of the considered PDE (problem specific physical model) system, but is no longer a conceptual element of the overall „physical world reality" (i.e. it is not a notion as part of the stage of theoretical physics).

3. The „Newton/Dirac" „point/particle *mass density*" concept (whereby the regularity of the Dirac „function" depends from the space dimension) ist being replaced by the „Leibniz/Plemelj" „ideal point/differential *mass element*" concept.

The proposed common Hilbert space framework enables variational methods for nonlinear operators (VaM) for the considered mathematical physics models. It overcomes the (claimed) common purely mathematical handicaps for problem adequate solutions in alignment with the *purpose* of physical models. From a *physical* modelling perspective it is about a replacement of Dirac's model of the „*density*" of an idealized point mass or point charge, which is called the Dirac or Delta „function". It is a distribution equal to zero everywhere except for zero, and whose integral over the entire line is equal to one. The Dirac model of the „*density*" of an idealized point mass is replaced by Plemelj's concept of a „*mass element*" (PlJ), with the essential consequence, that the regularity requirement for those distributions $d\mu$ are independent from the space-dimension in opposite to the Dirac function: *the regularity of Dirac's model of the point mass density of an idealized point mass is $\delta \in H_{-n/2-\varepsilon}$ ($\varepsilon > 0$, n = space dimension), while for Plemelj's mass element definition it holds $d\mu \in H_{-1/2}$.*

The „*mass generation process*" is modelled as a „selfadjoint property" break down by the orthogonal projection $H_{1/2} = H_1 \otimes H_1^{\neg} \to H_1$, i.e. the closed subspace $H_1^{\neg}$ is the model for the ground state (vacuum) energy, which is and can be neglected in all („less granular") Lagrange formalism based physical models.

(*) Non-linear minimization problems can be analyzed as saddle point problems on convex manifolds in the following form (VeW): $J(u): a(u,u) - F(u) \to min$, $u - u_0 \in U$. Let $a(\cdot,\cdot): V \times V \to R$ a symmetric bilinear form with energy norm $\|u\|^2 := a(u,u)$. Let further $u_0 \in V$ and $F(\cdot): V \to R$ a functional with the following properties:

i) $F(\cdot): V \to R$ is convex on the linear manifold $u_0 + U$, i.e. for every $u, v \in u_0 + U$ it holds $F((1-t)u + tv) \le (1-t)F(u) + tF(v)$ for every $t \in [0,1]$

ii) $F(u) \ge \alpha$ for every $u \in u_0 + U$

iii) $F(\cdot): V \to R$ is Gateaux differentiable, i.e. it exits a functional $F_u(\cdot): V \to R$ with $\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = F_u(v)$.

Then the minimum problem is equivalent to the variational equation $a(u,\phi) + F_u(\phi) = 0$ for every $\phi \in U$ and admits only an unique solution. In case the sub-space U and therefore also the manifold $u_0 + U$ is closed with respect to the energy norm and the functional $F(\cdot): V \to R$ is continuous with respect to convergence in the energy norm, then there exists a solution. We note that the energy functional is even strongly convex in whole V.

The proposed „energy" Hilbert space $H_{1/2}$ enables e.g. the method of Noble ((VeW) 6.2.4), (ArA) 4.2), which is about two properly defined operator equations, to analyze (nonlinear) complementary extremal problems. The Noble method leads to a "Hamiltonian" function $W(\cdot,\cdot)$ which combines the pair of underlying operator equations (based on the "Gateaux derivative" concept) $Tu = \frac{\partial W(\acute{u},u)}{\partial \acute{u}}$, $T^*\acute{u} = \frac{\partial W(\acute{u},u)}{\partial u}$ $u \in E = H_{1/2}$, $\acute{u} \in \acute{E} = H_{-1/2}$.
From a mathematical point of view this means that a Lebesgue integral is replaced by a Stieltjes integral. The corresponding $H_{-1/2}$ quantum state model (alternatively to the standard $L_2 = H_0$ model) goes along with a corresponding quantum energy Hilbert space model $H_{1/2}$. Its definition follows the same building principles as for the standard Laplace operator in a $L_2 = H_0$ framework with its corresponding Dirichlet (energy inner product) integral $D(u,v) = (\nabla u, \nabla v)_0 = (u,v)_1$. With respect to the Bianchi identities we emphasis that for the inner product $(u,v)_{-1/2}$ of $H_{-1/2}$ the following relationships hold true:
$(div(u), v)_{-\frac{1}{2}} \sim (u, \nabla v)_{-\frac{1}{2}} \sim (u,v)_0$.

The classical field equation of the Lagrange density of the Maxwell field is given by the wave equation. The quantized field theory with corresponding to be fulfilled commutator rule properties leads to retarding potentials generated by a „point source" modelled as Dirac function$^{(*)}$. In the context of field fluctuations (the „either or question" defining the states of the Maxwell fields (by their values or the number of quanta) we refer to (SmL1).

The decompositions $\mathrm{H}_{-1/2} = H_0 \otimes H_0^{\perp} = H_{1/2}^{*}$, $H_{1/2} = H_1 \otimes H_1^{\neg} = H_{-1/2}^{*}$ distinguish between elementary particle states & energy with or w/o „observed/measured mass". The *„symmetry break down"* model to „generate/explain" physical „mass" is replaced by a „projection of a *self-adjoint operator onto the observation/measure space* H_0" (**). In other words, the matter particles (fermions) are the manifestations of the vacuum energy (bosons).

The current quantum state Hilbert space $L_2 = H_0$ is extended to the Hilbert space $H_{-1/2}$ including „fluid, plasma, fermion, photon, boson" states. Its dual space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp}$ provides the corresponding quantum energy space, whereby the „mass-less EPs" (hot plasma) are (meta-physical, ground state (dark) energy) „elements" of the closed subspace $H_1^{\perp}$ of $\mathrm{H}_{1/2}$. The standard (variational) energy space H_1 is defined by the selfadjoint Friedrichs extension of the Laplacian operator in the standard $H_0 = L_2 -$ variational (statistics) framework. It keeps being valid for the quantum energy of the EPs *with* mass, including cold plasma. The corresponding mass/energy Hilbert space is given by the decomposition $H_{1/2} = H_1 \times H_1^{\neg}$ into the „fermions" space and the orthogonal „bosons" space. The latter one includes the Higgs boson. The Hilbert space framework enables a Cauchy problem representation of the Einstein-Vacuum field equation with an initial „inflation-field" with regularity $g_{Inflation} \in H_1^{\neg}$ without singularities for $t \to 0$, avoiding current early universe state model singularities.

In a physical world only field dimensions can be measured, which are averaged with respect to the space and time variables. Concerning the „energy" dimension in a Hilbert space framework this is about the measurement in the H_1 norm (strong topology, metric space), leading to the physical concept of matter and anti-matter. The Lamb shift occurs in the context of field fluctuations in a quantized electromagnetic (Maxwell) field. In the context of the inflaton theory those quantum fluctuations kicked off the birth of the matter/anti matter universe by the „symmetry break down" effect. The main arguments against this „Big Bang" explanation is, that the assumed mathematical quantum mechanical concept of a „field fluctuation" is given a priori, that the „thermodynamic time" clock started to run with the start of the inflation process after about $\sim 10^{-35} s$ and that the (not defined!) „time" duration until that starting point is explained as a „decision making time" of the „system" to generate (matter/anti-matter) kinetic energy w/o violating the (energy) conservation law principle (see also (HaS1). The proposed $H_{1/2}$ Hilbert space enables a weak topology based „self-adjoint property" break down process modelled by a projection operator on its compactly embedded Hilbert (sub) space H_1, i.e. the probability for such a measurable energy state generation process is zero. This model avoids the matter/anti-matter concept allowing an ongoing „fluctuation" based matter generation, which indeed happens with probability zero from a physical measurement perspective (similar to the process of picking a rational number out of the set of real numbers).

$^{(*)}$ *(CoR) p. 765: „Huygens' principle stipulates that the solution at a point does not depend on the totality of initial data within the conoid of dependence but only on data on the characteristic rays through that point … It is proven, that for the wave equation in 3,5,7,… space dimensions, and for equivalent equations, the Huygens' principle is valid. For differential equations of second order with variable coefficients Hadamard's conjecture states that the same theorem holds even if the coefficients are not constant. Examples to the contrary show that this conjecture cannot be completely true in this form although it is highly plausible that somehow it is essentially correct. … Altogehter, the question of Huygens's principle for second order equations should be considered in the light of the much more comprehensive problem of the exact domain of dependence and influence for any hyperbolic problem, a problem which is still completely open." Altogether, the question of Huygens' principle for second order equations should be considered in the light of the much more comprehensive problem of the exact domain of dependence and influence for any hyperbolic problem (see §5), a problem which is still completely open."*

$^{(**)}$ *The proposed common distributional Hilbert space framework $H_{-1/2}$ resp. its corresponding energy dual space $H_{1/2} = H_1 \otimes H_1^{\perp} = H_1^{repulsive} \otimes H_1^{attrac} \otimes H_1^{\perp}$ enables a common (Zeta function and quantum gravity) spectral theory providing an answer to Derbyshine's question ((DeJ) p. 295): "The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles*?"

The classical Yang-Mills theory is the generalization of the Maxwell theory of electromagnetism where chromo-electromagnetic field itself carries charges. As a classical field theory it has solutions which travel at the speed of light so that its quantum version should describe massless particles (gluons). However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. This is the mass gap. Another aspect of confinement is asymptotic freedom which makes it conceivable that quantum Yang-Mills theory exists without restriction to low energy scales. A variational Maxwell equations representation in a $H_{-1/2}$ Hilbert space framework includes also "gluon" bosons and corresponding "*self-adjointness break downs*", i.e. there is no mass gap anymore.

At the same point in time the „extended Maxwell equations" put the spot on H. Weyl's question of the existence of a substancial nucleus at the field center (*), being followed by pointing out G. Mie's modified Maxwell equations by explaining why the field possesses a granular structure and why the knots of energy remain intact in spite of the back-and-forth flux of energy and monentum (**). Such a model would deccribe the ether as required by the general theory of gravity (***).

The („physical") Hilbert space pairs (H_0, H_1) resp. the („meta-physical") closed subspaces $H_0^{\perp}, H_1^{\perp})$ of (H_0, H_1) are being governed by Fourier waves resp. Calderón´s wavelets (****). The current "*symmetry break down*" model to generate matter is replaced by a "*self-adjointness break down*" effect defined by the orthogonal projection from $H_{1/2}$ onto H_1. Consequently, the (kinetic) energy driven „inverse" is a kind of entropy operator with a „discrete/compactly embedded" Hilbert space domain to its complementary closed subspace of $H_{1/2}$ (*****).

(*) (WeH3) p. 171„*Since all physically important properties of an surrounding field rather than the substantial nucleus at the field center, the question becomes inevitable whether the existence of such a nucleus is not a presumption that may be completely dispensed with. This question is answered in the affirmative by the field theory of matter. …. Such an energy knot, which by no means is clearly delineated against the remaining field, propagates through empty space like water wave across the surface of a lake. There is no such thing as one and the same substance of which the electron consists at all times. Just as the velocity of a water wave is not a substantial but a phase velocity, so that the velocity with which an electron moves is only the velocity of an ideal „center of energy", constructed out of the field distribution. … This conception of the world can hardly be described as as dynamical any more, since the field is neither generated by nor acting upon an agent separate from the field, but following its own laws is in a quiet continuous flow. It is of the essence of the continuum.*"

(**) (WeH3) p. 171: „*G. Mie in 1912 pointed out a way of modifying the Maxwell equations in such a matter that they might possibly solve the problem of matter, by explaining why the field possesses a granular structure and why the knots of energy remain intact in spite of the back-and-forth flux of energy and monentum. The Maxwell equations will not do because they imply that negative charges compressed in an electron explode; to guarantee their coherence in spite of Coulomb's repulsive forces was the only service still required of the substance by H. A. Lorentz's theory of electrons. The preservation of the energy knots must result from the fact that the modified field laws admit only of one state of field equlibrium – or of a few between which there is no continuous transition (static, spherically symmetric solutions of the field equations). The field laws should thus permit us to compute in advance charge and mass of the electron and the atomic weights of the various chemical elements in existence. And the same fact, rather than the contrast of substance and field, would be the reason why we may decompose the energy or inert mass of a compond body (approximately) into the non-resolvable energy of its last elementary constituents and the resovable energy of their mutual bond.*"

(***) *(EiA) „Recapitulating, we may say that according to the general theory of relativity space is endowed with physical qualities; in this sense, therefore, there exists an ether. According to the general theory of relativity space without ether is unthinkable; for in such space there not only would be no propagation of light, but also no possibility of existence for standards of space and time (measuring-rods and clocks), nor therefore any space-time intervals in the physical sense. But this ether may not be thought of as endowed with the quality characteristic of ponderable media, as consisting of parts which may be tracked through time. The idea of motion may not be applied to it."*
The proposed common distributional Hilbert space framework $H_{-1/2}$ resp. its corresponding energy dual space $H_{1/2} = H_1 \otimes H_1^{\perp} = H_1^{repulsive} \otimes H_1^{attrac} \otimes H_1^{\perp}$ enables a common (Zeta function and quantum gravity) spectral theory providing an answer to Derbyshine's question ((DeJ) p. 295): "The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?"

(****) *(HoM) 1.2: „The idea of wavelet analysis is to look at the details are added if one goes from scale a to scale $a - da$ with $da > 0$ but infinitesimal small. … Therefore, the wavelet transform allows us to unfold a function over the one-dimensional space R into a function over the two-dimensional half-plane $\mathbf{H}$ of positions and details (where is which details generated?). … Therefore, the parameter space $\mathbf{H}$ of the wavelet analysis may also be called the position-scale half-plane since if g localized around zero with width Δ then $g_{b,a}$ is localized around the position b with width $a\Delta$. The wavelet transform itself may now be interpreted as a mathematical microscope where we identify $b \leftrightarrow$ position; $(a\Delta)^{-1} \leftrightarrow$ enlargement; $g \leftrightarrow$ optics.*

(*****) *The continuous wavelet transform with the complex Shannon wavelet can be considered via solutions of Cauchy problems for PDE in the context of the construction of wavelets for an analysis of non-stationary wave propagation in inhomogeneous media ((PoE).*

The Friedrichs extension of the Laplacian operator is a selfadjoint, bounded operator B with domain H_1. Thus, the operator B induces a decomposition of H into the direct sum of two subspaces, enabling the definition of a potential and a corresponding „grad" potential operator. Then a potential criterion defines a manifold, which represents a hyperboloid in the Hilbert space H_1 with corresponding hyperbolic and conical regions ((VaM) 11.2). The direct sum of the corresponding two subspaces of $H = H_1$ are proposed as a model to define a decomposition of the „fermions" space H_1 into

$$H_1 = H_1^{repulsive} \otimes H_1^{attrac} =: H_1^{(-)} \otimes H_1^{(+)}$$

whereby the potential criterion defines repulsive resp. attractive elementary mass particles (*). Then the corresponding proposed quantum energy Hilbert space (including attractive gravitons $\in H_1^{(+)}$) is given by (**), (***)

$$H_{1/2} = H_1^{(-)} \otimes H_1^{(+)} \otimes H_1^{\perp}$$

(*) we note that dark matter is only subject to gravity; the energy/matter distribution in the universe is: dark energy ~74%, dark matter 22%, atoms 4%, whereby 99.9% of all atoms are hydrogen and helium.

(**) The theory of Hilbert spaces with an indefinite metric is provided in e.g. ((DrM), (AzT), (DrM), (VaM)). Following the investigations of Pontrjagin and Iohvidov on linear operators in a Hilbert space with an indefinite inner product, M. G. Krein proved the Pontrjagin-Iohvidov-Krein theorem (FaK). In case of a Hilbert space H, this is about a decomposition of H into an orthonal sum of two spaces H^1 and H^2 with corresponding projection operators P^1 and P^2 (see also the problem of S. L. Sobolev concerning Hermitean operators in spaces with indefinite metric, (VaM) IV). We note, that for a vector space H, the empty set, the space H, and any linear subspace of H are convex cones. For x being an element of H this is about a defined "potential" ((VaM) (11.1))

$$\varphi(x) := \left((x)\right)^2 = \|P^1 x\|^2 - \|P^2 x\|^2$$

and a corresponding "grad" potential operator $\boldsymbol{W}(x)$, given by (VaM) (11.4) $\boldsymbol{W}(x) = \frac{1}{2} grad\varphi(x) := P^1(x) - P^2(x)$. The potential criterion $\varphi(x) = c > 0$ defines a manifold, which represents a hyperboloid in the Hilbert space H with corresponding hyperbolic and conical regions. It provides a model for „symmetry break down" phenomena by choosing $P^1 := P$, $P^2 := I - P$ for the orthogonal projections $P: H_{-1/2} \to H_0$, $P: H_{1/2} \to H_1$, leading to the decompositions $H_{-1/2} = H_0 \otimes H_0^{\perp}$, $H_{1/2} = H_1 \otimes H_0^{\perp}$.
The tool set for an appropropriate generalization of the above "grad" definition in case of non-linear problems is about the (homogeneous, not always linear in h) Gateaux differential (or weak differential) $\boldsymbol{VF}(x,h)$ of a functional $\boldsymbol{F}$ at a point x in the direction h ((VaM) §3)). If there exists an operator A with $D(A) = H_1$, $R(A) = H_0$ and $\|x\|_1 = \|Ax\|_0$, whereby the operator A is positive definite, self-adjoint and A^{-1} is compact, the corresponding eigenvalue problem $A\varphi_i = \sigma_i \varphi_i$ has infinite solutions $\{\sigma_i, \varphi_i\}$ with $\sigma_i \to \infty$ and $(\varphi_i, \varphi_k) = \delta_{i,k}$. For each element $x \in H_1 = A^{-1} H_0$ it holds the representation $x = \sum_{i=1}^{\infty} (x, \varphi_i)\, \varphi_i$. Inner products with corresponding norms of a distributional Hilbert scale can be defined based on the eigen-pairs of an appropriately defined operator in the form

$$(x,y)_\alpha := \sum_i^\infty \lambda_i^\alpha (x, \phi_i)\,(y, \phi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i .$$

Additionally, for $t > 0$ there can be an inner product resp. norm defined for an additional governing Hilbert space with an "exponential decay" behavior in the form $e^{-\sqrt{\lambda_i}t}$ given by $(x,y)_{(t)}^2 := \sum_{i=1} e^{-\sqrt{\lambda_i}t} (x, \phi_i)(y, \phi_i)$, $\|x\|_{(t)}^2 := (x,x)_{(t)}^2$.
The approximation "quality" of the proposed $H_{-1/2}$ –quantum state Hilbert space with respect to the „observable space" norm of H_0 is governed by the inequality

$$\|x\|_{-1/2}^2 \le \delta \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2 = \delta \|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}\delta} x_i^2 .$$

The estimate is valid for all $\alpha > 0$ in the form $\|x\|_{-\alpha}^2 \le \delta^{2\alpha} \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2$, which follows from the inequality $\lambda^{-\alpha} \le \delta^{2\alpha} + e^{t(\delta^{-1} - \sqrt{\lambda})}$, being valid for any $t, \delta, \alpha > 0$ and $\lambda \ge 1$. For a related approximation theory we refer to (BrK8), (NiJ), NiJ1). Applying the mathematical wavelet (microscopic view) tool is then about an analysis of a quantum state $x = x_0 + x_0^{\perp} \in H_0 \otimes H_0^{\perp}$. Putting $\sigma := \|x_0^{\perp}\|_{-1/2}^2$ the approximation "quality" of a quantum state with respect to the „observable space" norm of H_0 is governed by the inequality $\|x\|_{-1/2}^2 \le \sigma \|x\|_0^2 + e \|x\|_{(\sigma)}^2 = \sigma \|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}\sigma} x_i^2$.

(***) (NaS): „The Hilbert space $H_{1/2}$ can be interpreted as the first cohomology space with real coefficients of the „universal Riemann surface" – namely the unit disk – in a Hodge-theoretic sense."

(***) The $H_{1/2}$ space as first cohomology is fundamental to explain the properties of period mapping on the universal Teichmüller space. We note that a vector space and any linear subspace are convex cones, i.e. the tool „convex analysis and general vector spaces" can be applied. Morse's calculus of variations in the large enables a calculus of variations in the large e.g. on varifolds ((MoM), (SeH), (AlF)).

The quantum gravity model also addresses the dilemma, as pointed out by E. Schrödinger: "Since in the Bose case we seem to be faced, mathematically, with simple oscillator of Planck type, we may ask whether we ought not to adopt for half-odd integers quantum numbers rather than integers. Once must, I think, call that an open dilemma. From the point of view of analogy one would very much prefer to do so. For, the „zero-point energy" of a Planck oscillator is not only borne out by direct observation in the case of crystal lattices, it is also so intimately linked up with the Heisenberg uncertainty relation that one hates to dispense with it".
The formalism of 2-"spinors" as an alternative to the standard vector-tensor calculus (Penrose R., Rindler W.) is proposed to be physically re-interpreted and mathematically applied in the context of a H_1-space decomposition into repulsive and attractive fermions subspaces, whereby it holds $spin(4) = SU(2)xSU(2)$. „The two-component „spinor" calculus is a very specific calculus for studying the structure of space-time manifolds…. Space-time point themselves cannot be regarded as derived objects from spinor algebra, but a certain extension of it, namely the twistor algebra, can indeed be taken as more primitive than space-time itself. ... The programme of twistor theory, in fact, is to reformulate the whole of basic physics in twistor terms" (Penrose R., Rindler W. Volume II). The point of departure for the twistor theory is the (classical) twistor equation (with a similar form as the continuity equation). Its corresponding weak variational represention with respect to the proposed $H_{-1/2}$ quantum state inner product leads to the Friedrichs extension of the classical Dirac spinor operator with domain $H_{-1/2}$, which is about the square root operator of order one of the Laplacian operator. The corresponding singular integral operator representation is about the Calderón-Zygmund integrodifferential operator ((EsG) example 3.5).

The newly proposed energy Hilbert space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp} = H_1^{(-)} \otimes H_1^{(+)} \otimes H_1^{\perp}$ with its decompostions into two kinetic (repulsive & attractive collision particles) energy (sub-) spaces and a ground state energy (sub-) space $H_1^{\perp}$ is proposed for an alternative cosmic inflation model ((KaD1) (VeG)). The current model faces the following two major „problems":

- the „point in time", when the (physical) inflation process „starts" is the first „point in time" after the so-called Planck time $t_P \sim 10^{-43}\mathrm{s}$ ends. The „what-ever-before" „where" the „initial mover", the quantum fluctuation „happened", is not part of the mathematical model

- the main simple formula describing the inflation process (enabling physical interpretations of the unknown „phenomenea") is the energy conservation equations between the energy density $\rho = \rho(t)$ and the pressure density $p = p(t)$ (both just appearing from nowhere) given by $\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)^2 (\rho + 3p)$ with the scale factor $a = a(t)$ being the cosmic time derived from the Friedmann-Robertson-Walker (FRW) metric also defining the Hubble expansion rate $H = H(t) := \dot{a}(t)/(a(t)$. It is derived from the solution of the Einstein field equations based on the cosmological principle assumption, which states that the universe should look like the same for all observers. „*that tells us that the universe must be homogeneous and isotropic. This then tells us, which metric must be used, which is the FRW metric*" (KaD1) (*). Essentially, the imbalance of $\rho \sim 3p$ is claimed to be the model for the expansion of the universe.

In other words, (1) the „first mover" of „everything" is not part of the model, and (2) the model itself is basically about an ordinary differential equation derived from the Einstein field equations (requiring even the existence of $\dot{\rho}(t_P), \dot{p}(t_P)$) based on physical large scale „universe" assumptions (**), while the purpose of the model is to describe the most chaotic „energy & mass generation process" of the universe, which ends up after a short time period in a stable state during billions of years until today.

The physical concepts of „*time*" and „*change*" are different sides of the same coin, i.e. there is no „time" w/o „change" and there is no „change" w/o „time". In other words, the concepts of „time" and „change" are and need to be in scope of the „matter/kinetic" energy model H_1 (reflecting the *physical reality/theory* (EiA2)), while its complementary ground state (vacuum) energy model $H_1^{\neg}$ is per definition independent from the *thermodynamical* concept of „*time*" (***).

As H_1 is compactly embedded into $H_{1/2}$, and given an initial universe w/o any thermodynamical „time" (i.e. $H_1 = \{\ \}$, with only existing ground state energy state for the whole mathematical model system) the probability for „symmetry break down" events to generate mass were and are zero; obviously those events happened and, according to the newly proposed model, will go on to be happen. At the same point in time the generated and still being generated „matter world" H_1 is governed by e.g. the „least action principle" (KnA), and the principles of „statistical thermodynamic" (ScE), whereby the classical action variable of the system determines the „time" (HeW).

(*) we claim that Gödel's metric $a^2(dx_0^2 - dx_1^2 + \left(\frac{e^{2x_1}}{2}\right)dx_2^2 - dx_3^2 + 2e^{2x_1}dx_0dx_2)$ would be a better option for the standard model as the FRM metric (GöK).
(**) (KaD1): „It is most amazing that the dynamics of the universe as determined by the equations of GR can be derived from purely Newtononian considerations. … The only difference between the Newtonian and Einstein version of cosmology becomes apparent only by differentiating the Newton energy conservation equation taking into account the relation between and the pressure from local energy conservation"

(***) The macroscopic and microscopic state of quanta relate to corresponding frequencies of its vibrations. The corresponding action variables of the system ((HeW) II.1.c) define the related kinematical (physical) and thermodynamical concept of "time" (RoC), (SmL)), (RoC1), section 13). (DeR) p. 93: „In general the resonance between the wave phase velocity and the velocity of individuals electrons cannot be neglected. It involves coupling between single-particle and collective aspects of plasma behaviour, and give rise to an energy flow which is known as Landau damping. Before continuing, we should note that this topic is related to one of the main unsolved questions of physics. It has not yet been possible to resolve fully the contrast between the reversibility in time of microscopic phenomena – for example, the dynamics of a particle described by Newton's laws of motion – and the irreversibility in time of macroscopic phenomena, as desribed by the second law of the thermodynamics. Any thermodynamic system is in fact constructed from a large number of particles, all of which obey Newton's laws, so that this contrast is central to physics. A resolution of this contrast would be particulary helpful to a full understanding of Landau damping; this is because Landau damping involves a flow of energy between single particles on the one hand side, and collective excitations of the plasma on the other."

"The theoretical discovery of wave damping without energy dissipation by collisions is perhaps the most astonishing result of plasma physics research. Landau damping (spontaneous stabilization of plasma; return to an equilibrium w/o increase of entropy) is a characteristic of collisionsless plasmas, but it may also have application in other fields. For instance, in the kinetic treatment of galaxy formation, stars can be considered as atoms of a plasma interacting via gravitational rather than electromagnetic forces. Instability of the gas of stars can cause spiral arms to form but this process is limited by the Landau damping." (ChF) 7.5) (*).

The proposed alternative cosmic inflation model is about the newly proposed energy Hilbert space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp} = H_1^{(-)} \otimes H_1^{(+)} \otimes H_1^{\perp}$ (with its decompostions into two kinetic (repulsive & attractive collision particles) energy (sub-) spaces and a ground state (collision-free particles) energy (sub-) space $H_1^{\perp}$ and a corresponding weak variational representation of extended (with respect to the underlying operator domains) Landau-Boltzmann equations (**)(***).

In (BrK6) we provide a distributional Hilbert space framework to enable a proof of the non-linear Landau damping phenomenon based on the non-linear Landau collision operator. The eigen-pair solutions of the related Oseen operator is proposed to be applied to build the problem adequate Hilbert scale. The appropriate physical model of the nonlinear Landau damping is built by the *weak variational representation* of a (Pseudo) Differential operator equation with a correspondingly defined domain, including appropriate initial and/or boundary conditions. The current classical related PDE system representation is interpreted as the approximation solution to it and not the other way around.

(*) (ChF) 1.1: occurance of plasmas in nature, "It has often been said that 99% of the matter in the universe is in the plasma state; that is, in the form of an electrified gas with atoms dissociated into positive ions and negative electrons. This estimate may not be very accurate, but it is certainly a reasonable one in view of the fact that stellar interiors and atmospheres, gaseous nebulae, and much of the interstellar hydrogen are plasmas. In our own neighborhood, as soon one leaves the earth's atmosphere, one encounters the plasma comprising the Van Allen radiation belts and the solar wind. On th other hand, in our everyday lives encounters with plasmas are limited to a few examples: the flash of a lightning bolt, the soft glow of the Aurora Borealis, the conducting gas inside a fluorescent tube or neon sign, and the slight amount of inoization in a rocket exhaust. It would seem that we live in the 1% of the universe in which plasmas do not occur naturally."

(ChF) 7.5: the meaning of Landau damping, "The theoretical discovery of wave damping without energy dissipation by collisions is perhaps the most astonishing result of plasma physics research. That this is a real effect has been demonstrated in the laboratory. ... Landau damping (spontaneous stabilization of plasma; return to an equilibrium w/o increase of entropy) is a characteristic of collisionsless plasmas, but it may also have application in other fields. For instance, in the kinetic treatment of galaxy formation, stars can be considered as atoms of a plasma interacting via gravitational rather than electromagnetic forces. Instability of the gas of stars can cause spiral arms to form, but his process is limited by the Landau damping."
(**) The model fulfills the set of principles to build a new cosmology theory in (SmL) §10, wherely it confirms the believe that basically all current fundamental (Lagrange formalism based) models are approximations, only

(***) The Landau equation is a model describing time evolution of the distribution function of plasma consisting of charged particles with long-range interaction. It is about the Boltzmann equation with a corresponding Boltzmann collision operator where almost all collisions are grazing. The Landau damping phenomenon ("wave damping w/o energy dissipation by collision in plasma") is an observed plasma/quantum physical phenomenon. In (MoC) this phenomenon has been „proven" for the non-linear Vlasov equation based on analytical norm estimates, which is about differentiability requirements beyond C^{∞}; even the mathematical model of the GRT (which is not consistent to the quantum mechanics mathematical model of „discrete" „quantum leaps") works out with differentiable manifolds, only, whereby the differentiability requirement is already w/o any physical meaning (!); we claim, that the proof in (MoC) is not a proof of the physical phenomenon, but provides evidence, that the Vlasov equation is not the adequate mathematical model of the Landau phenomenon. This statement is in alignment with the critisism of Landau regarding Vlasov´s equation.

Vlasov's mathematical argument against the Landau equation (leading to the Vlasov equation) was, that "this model of pair collisions is formally (!) not applicable to Coulomb interaction due to the divergence of the kinetic terms". This argument is being overcome by the proposed distributions framework.

Vlasov's formula for the plasma dielectric for the longitudinal oscillators is based on the integral ((ShF) p. 392)

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{}'(v)dv}{\frac{\omega}{k}-v} \quad .$$

As Landau pointed out, this model overlooks the important physical phenomenon of electrons travelling with exactly the same material speed $v_{\varphi} = \frac{\omega}{k}$ and the wave speed v. In ((ShF) p. 395) the correct definition (as provided by Landau) for the Vlasov formula is given, which is basically a threefold integral definition depending from the value ω_I the imaginary part of $\omega = \omega_R + i\omega_I$:

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{}'(v)dv}{\frac{\omega}{k}-v} \qquad \text{for } \omega_I < 0$$

$$W\left(\frac{\omega}{k}\right) = -p.v.\int_{-\infty}^{\infty} \frac{F_0{}'(v)dv}{\frac{\omega}{k}-v} - \pi i\, F_0{}'\left(\frac{\omega}{k}\right) sgn(k) \qquad \text{for } \omega_I = 0$$

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{}'(v)dv}{\frac{\omega}{k}-v} - 2\pi i\, F_0{}'\left(\frac{\omega}{k}\right) sgn(k) \qquad \text{for } \omega_I > 0$$

If ω_I were to continue and become positive (damped disturbance), then analytical continuation yields, in addition to the integral along the real line (which also presents no difficulty of interpretation), a full residue contribution.

This extended Landau-Botzmann equation (CeC) is proposed as alternative cosmical inflation model with an only $H_1^{\perp}$ ground state energy relevant initial/radiation value condition (in the sense of ((CoR) VI.7 refering to (WeA)). As there is per definition no change in a „purely" „ground state energy" framework there is also no „time" „existing", while the probability (measured in a H_0 Hilbert space) for a „symmetry break down" (the first orthogonal projection from $H_1^{\perp}$ onto H_1) is zero. In (GaA) unique solvability of a class of abstract kinetic equations with accretive collision operators is derived using Krein space methods (AzT1) (BoJ). Semi-groups on non-linear contractions on closed convex subsets of a Hilbert space H are considered in (BrH). The Boltzmann-Landau (Fokker-Planck, (DeR 5.4)) equation describing the transport of charged particle in hot plasma is worked out as an application.

We note that the $\mathrm{H}_{1/2}$ space as first cohomology is fundamental to explain the properties of period mapping on the universal Teichmüller space. We further note that a vector space and any linear subspace are convex cones, i.e. the tool „convex analysis and general vector spaces" can be applied. Morse's „*calculus of variations in the large*" enables a calculus of variations in the large on „*varifolds*" (*)((MoM), (SeH), (AlF)).

The cosmological inflation model (which can be derived from both, the Newtonian and the Einstein version of cosmology (!) (**)). It is about the simple, classical ordinary differential equation $\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)^2 (\rho + 3p)$, with energy density $\rho = \rho(t)$, the pressure density $p = p(t)$ and the scale factor $a = a(t)$ concerning the cosmic time derived from the Friedmann-Robertson-Walker (FRW) metric. This ODE is not well defined due to e.g. the missing initial value condition at the Planck time. As the unit of measure of the "pressure" is identical to the unit of measure of the "energy density" this unbalanced (not well-posed (!)) conservation law is the mathemtical modelt o „explain" the „matter generation & „explosion" phenomena and to estimate the required expansion energy during the inflation period. In combination with the Planck thermodynamical „Hohlraum" radiation model it lead to the model of the observed „cosmic microwave background (CMB)" phenomenon ((WeS) p. 506). Planck's „Hohlraum" (black body radiation) model provides the asymptotics of the matter energy density in the form $\rho \propto T^4$ with the temperature $T(t)$. In combination with the inflation theory asymptotics $\rho(t) \propto a(t)^{-4}$ this leads to the CMB asymptotics $T(t) \propto a(t)^{-1}$ (NaP). We note that the scope of Planck's „Hohlraum" thermodynamical statistical model is about countable, infinite many „EP", which corresponds to the compact embededness of H_1 into $H_{1/2}$.

We note that the assumption of the above Einstein version of cosmology to derive the ODE is based on the cosmological principle assumption; it states that the universe should look like the same for all observers. It „*tells us that the universe must be homogeneous and isotropic. This then tells us, which metric must be used, which is the Friedmann-Robertson-Walker (FRW) metric*" (KaD1). In other words, the physical assumptions about the state of the universe during the inflation period is the stable one, which we currently have (even Gödel's cosmological solution would be a better option than the FRW metric (GöK)); we further note, that the same model can be also derived from the classical Newtonian version of cosmology. This reminds to the origin of one of the titles of the books from R. Penrose: "The Emperor's New Clothes".

(*) Varifolds geometry (AlF) is about integral varifolds. It is based on real valued functions (which can be the „norm" of differentials) on the space of differential forms. Because of analogy with electric currents such continuous linear functions are also called „currents".
(**) (KaD1): „It is most amazing that the dynamics of the universe as determined by the equations of GR can be derived from purely Newtononian considerations. … The only difference between the Newtonian and Einstein version of cosmology becomes apparent only by differentiating the Newton energy conservation equation taking into account the relation between and the pressure from local energy conservation". The Einstein operator is given by $G = R_{ik} - R\frac{g_{ik}}{2}$ with the corresponding gravity field equations $G = -\kappa T_{ik}$ and the corresponding motion equations $\frac{d}{d\tau}\left(g_{\mu,\nu}\frac{dx^\mu}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^\beta}{\partial \tau}$ for the path $x^\mu = x^\mu(t)$ of a particle. The change from the Newton model is about a change from the potential equation to the Einstein equation $-\Delta\Phi = -4\pi k\rho \rightarrow G = -\kappa T_{ik}$ and a change from the motion equations $\frac{d^2\vec{x}}{dt^2} = -grad\Phi \rightarrow \frac{d}{d\tau}\left(g_{\mu,\nu}\frac{dx^\mu}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^\beta}{\partial \tau}$. Instead of one potential equation we now have 10 equations with 10 potentials Φ_{ik}; instead of a linear operator, we now have a non-linear operator, i.e. the gravity potential is no longer the sum of single gravitation potentials. Additionally there is a circle structure, i.e. the potentials are a functions of the T_{ik} ($\Phi_{ik} = f(T_{ik})$), while the space-time structure is a function of the potentials ($f(\Phi_{ik})$). The matter, as described by the energy-momentum tensor T_{ik}, reflecting the principles of energy and momentum conservation, generates a curvature of the space-time and particles move along of geodesics.

The extended Maxwell equations (with still valid Lorentz transform properties concerning the sub-space H_1 of $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp} = H_1^{(-)} \otimes H_1^{(+)} \otimes H_1^{\perp}$) and the proposed alternative cosmic inflation model puts the spot again on the special relativity theory (SRT) with its underlying 3-sphere model S^3. The least action principle provides the fundamental concept for the Hilbert-Einstein functional defining the Einstein field equations.

In the context of varifold geometry (AlF) and Morse theory (MiJ) for the underlying „action" integral a quite different physical model is possible, by which then an „action" integral in the form $\mathrm{E} = \int_0^1 \left\| \frac{d\omega}{dt} \right\|^2 dt$ is called „energy". It is about a rubber band which is stretched between two points of a slippery curved surface. If the band is described parametrically by the equation $\mathrm{x} = \omega(\mathrm{t})$, then the potential energy arising from tension will be proportional to our integral E (at least to a first order of approximation). For an equilibrium position this energy must be minimized, and hence the rubber band will describe a geodesic.

The collision operator of the Landau equation is given by

$$Q(f,f) = \frac{\partial}{\partial v_i} \left\{ \int_{R^N} a_{ij}(v-w) \left[f(w) \frac{\partial f(v)}{\partial v_j} - f(v) \frac{\partial f(w)}{\partial w_j} \right] dw \right\}$$

with

$$a_{ij}(z) = \frac{a(z)}{|z|} \left\{ \delta_{ij} - \frac{z_i z_j}{|z|^2} \right\} = \frac{a(z)}{|z|} P(z) := \frac{1-[1-a(z)]}{|z|} [Id - Q](z) \quad Q(z) := (R_i R_j)_{1 \le i,j \le N}$$

and $\mathrm{a(z)}$ symmetric, non-negative and even in z. Ist domain is given by an unknown function f corresponding at each time t to the „density of particle" at the point x with velocity v. It can be approximated by a linear Pseudo-Differential Operator (PDO) of order zero with symbol

$$b_{ij}(z) = z \cdot a_{ij}(z) = \frac{z}{|z|} \left\{ \delta_{ij} - \frac{z_i z_j}{|z|^2} \right\} = \frac{z}{|z|} P(z) := \frac{z}{|z|} [Id - Q](z)$$

whereby $a_{ij}(z)$ denotes the symbol of the Oseen kernel (LeN). The Riesz transforms operators are defined by

$$R_k u = -i c_n p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy \text{ (with } c_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \text{).}$$

They are related to the Caldéron- Zygmund operators $T(f) = S * F$ with a distribution S defined by a homogeneous function of degree zero satisfying a kind of average mean zero condition on the unit sphere with its underlying rotation invariant probability measure (MeY).

The search for conditions of minimal regularity in the context of the „pointwise multiplication" operator A is about an analysis of the commutator $[T, A]$. This leads to the „Caldéron operator"

$$(\Lambda u)(x) = (\sum_{k=1}^{n} R_k D_k u)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{n} p.v. \int_{-\infty}^{\infty} \sum_{k=1}^{n} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy$$
$$= -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy = -(\Delta \Lambda^{-1}) u(x)$$

with symbol $|\nu|$ and its inverse operator ((EsG) (3.15), (3.17), (3.35))

$$(\Lambda^{-1} u)(x) = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{u(y)}{|x-y|^{n-1}} dy \text{ , } n \ge 2 \text{ .}$$

In dimension 1, this is about $\Lambda = DH$ where H denotes the Hilbert transform and D the Schrödinger momentum operator in the form $P := D = -i \frac{d}{dx}$ ((MeY) p. 5). The Schrödinger momentum operator in dimension n, and its related Hamiltonian operator is given by $P := -i\hbar\nabla = \frac{\hbar}{i}\nabla$ resp. $H := -\frac{\hbar^2}{2m}\Delta = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2$. The corresponding generalization of the Schrödinger momentum operator is then given by $\Lambda = PR$.

The one-dimensional Hilbert space model is given by $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a 2π −periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β the Fourier coefficients

$$u_\nu := \frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx$$

enable the definitions of the norms (see e.g. (LiI) Remark 11.1.5)

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2 \ .$$

Then H is the space of L_2 −periodic function in R and the Fourier transform (denoted by $\hat{u}$) is an isomorphism between the Hilbert spaces $H_\beta := \{u | \|u\|_\beta^2 < \infty\}$ and its "dual" Hilbert space $H_{-\beta}$. The Fourier transforms $\hat{s}_i$ of the (kernel) functions s_i $(i = -1,0,1)$, (MuN) chapter 3, §28)

$$s_{-1}(x) := ln\frac{1}{2\sin\frac{x}{2}} \quad , \quad \hat{s}_{-1}(\nu) = sgn(\nu)/\nu$$
$$s_0(x) := \frac{1}{2}cot\frac{x}{2} = -s_{-1}'(x) \quad , \quad \hat{s}_0(\nu) = -i\, sgn(\nu)$$
$$s_1(x) := \frac{1}{4\sin^2\frac{x}{2}} = -s_0'(x) \quad , \quad \hat{s}_1(\nu) = -\nu\, sgn(\nu)$$

are the symbols of following (singular convolution integral) Pseudo-Differential operators (e.g. (LiI) (1.2.31)-(1.2.33), (LiI1)):

$$(S_{-1}u)(x) := \oint s_{-1}(x-y)\,u(y)dy$$
$$(S_0u)(x) := \oint s_0(x-y)\,u(y)dy$$
$$(S_1u)(x) := \oint s_1(x-y)\,u(y)dy.$$

The operators S_i $(i = -1,0,1)$ are isomorphisms $S_i : H_{\beta+i} \to H_\beta$ and self-adjoint with respect to the corresponding energy inner products $(u,v)_{i/2} = (S_iu,v)_0$. The operator S_0 is the Hilbert transform on the space of periodic functions for each Hilbert scale H_β. The operator S_1 can be considered as a generalization of the $\frac{d}{dx}$ operator being defined for each Hilbert scale H_β. It enables inner product definition of differentials (in the Plemelj integral sense (*)) in the form

$$\big((du,dv)\big) := (S_1[u],S_1[v]) \cong (u,v)_1 \quad , \quad \big((du,v)\big) := (S_1[u],v) \cong (u,v)_{1/2}.$$

The Friedrichs extension of the Laplacian operator $-\Delta: H_2 \to H_0$ is a selfadjoint, bounded operator B with domain H_1. The corresponding Friedrichs extension of the operator $d : H_1 \to H_0$ is a selfadjoint, bounded („*Hodge*" like) operator with domain $H_{1/2}$.

(*) J. Plemelj's suggestion ((PlJ) XV, p. 12, p. 17, is about a relationship between the differential form calculus and its application in physics (e.g. [HCa], [HFl]) and a modified representation of the potential in the form

$$(*)\ \ v(s) = -\frac{1}{\pi}\oint log|\zeta(s)-\zeta(t)|\,u(t)dt \qquad \to \qquad (**)\ \ v(s) = -\frac{1}{\pi}\oint log|\zeta(s)-\zeta(t)|\,du(t)$$

Plemelj's quote: "Bisher war es üblich, für das Potential die Form (*) zu nehmen. Eine solche Einschränkung erweist sich aber als eine derart folgenschwere Einschränkung, dass dadurch dem Potentiale der grösste Teil seiner Leistungsfähigkeit hinweg genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential nur in der Form (**) verwendbar."

In case of the harmonic quantum oscillator it holds in the L_2- *framework* $\bar{E}_0 = \frac{1}{2}\sum\hbar\omega_n \approx c\sum\hbar n = \infty$ *leading to the concept of "re-normalization" to ensure the existence of bounded Hermitian operators* $\bar{H}_{renorm}$, *with* $\bar{H} = \bar{H}_{renorm} + \bar{E}_0$.

The Hilbert space decomposition $H_{-1/2} = H_0 \otimes H_0^\perp$ *enables „mixed" discrete (*$\lambda_i \geq 1$*, eigen-function) based and continuous (*$\lambda,\mu \in (0,1)$*, eigen-differential) based „spectral" representations: let* $\{\phi_i\}$ *denote the ONS of the Hilbert space* H_0 *and* $[\phi_\mu] = H_0^\perp$*. The Dirac function set-up to build spectral function representations of Hermitian operator, whereby* $(\phi_n,\phi_m) = \delta_{n,m}$, $(\phi_\lambda,\phi_\mu) = \delta(\phi_\lambda - \phi_\mu$ *is replaced by the set-up* $(\phi_n,\phi_m)_{-1/2} = \delta_{n,m}$, $(\phi_\lambda,\phi_\mu)_{-1/2}$*. This leads to the representations*

$$x = \sum_1^\infty(x,\phi_i)_{-1/2}\phi_i + \int_0^1\phi_\mu(x,\phi_\mu)_{-1/2}d\mu\,,\ Hx = \sum_1^\infty\lambda_i(x,\phi_i)_{-1/2}\phi_i + \int_0^1\lambda\phi_\mu\big(x,\phi_\mu\big)_{-1/2}d\mu$$

resp.

$$(Hx,x)_{-1/2} = \sum_1^\infty\lambda_i\big|(x,\phi_i)_{-1/2}\big|^2 + \|\phi_\lambda\|_{-1/2}^2\int_0^1\lambda\big|(x,\phi_\lambda)_{-1/2}\big|^2 d\lambda.$$

Additionally, for $t>0$ *there can be an inner product resp. norm defined for an additional governing Hilbert space with an "exponential decay" behavior in the form* $e^{-\sqrt{\lambda_i}t}$ *given by* $(x,y)_{(t)}^2 := \sum_{i=1}e^{-\sqrt{\lambda_i}t}(x,\phi_i)(y,\phi_i)$, $\|x\|_{(t)}^2 := (x,x)_{(t)}^2$*. The related* $H_{-\beta}-$ *approximation theory is provided in (NiJ1). The proposed* $H_{-1/2}-$*quantum state Hilbert space with respect to the „observable space" norm of* H_0 *is governed by the inequality* $\|x\|_{-1/2}^2 \leq \delta\|x\|_0^2 + e^{\frac{t}{\delta}}\|x\|_{(t)}^2 = \delta\|x\|_0^2 + \sum_{i=1}^\infty e^{1-\sqrt{\lambda_i}\delta}x_i^2$.

The notion of „*time*" is strongly connected with the question of the origin of the universe. It became a central open question when the steady-state model has been discarded in the context of the discovery of „*the all-pervading electromagnetic radiation, coming from all directions, now referred to as cosmic microwave background (CMB). It was identified as a predicted implication of the „flash" of a Big-Bang origin to the universe*" (PeR). The current cosmic model (based on the FRW metric in combination with Planck's black body radiation model) starts at Planck time, while the reason for this very first action is not part of the model, and while the related hyperbolic wave equation is time-symmetric, starting at $t = 0$. Ideas about the origin of the *universe* are e.g. considered in (CiI) (*), (PeR) (**), (RoC1) (***), (SmL) (****). In the context of the newly proposed model (including newly the quantum theory, which is *somehow* a priori „*given*" in the current cosmic model) the notion „*universe*" needs to be defined with respect to physical and mathematical terms. „Our" physical world (including the „particle" interaction in the SMEP) is on this side of the light velocity border (which also is a key ingredient of the wave equation governed by „perfect" Fourier waves in the Hilbert space H_1). Regarding the cosmic inflation model the Planck time is the most prominent example of this kind of „borders" for physical „*observations*". This (Planck time) point in „*time*" is the birthday of „*our*" observed universe. In this sense, the CMB model, which is basically the wave equation starting at $t = 0$, is a not appropriate model. So, from a physical modelling perspective „our" current „universe model" starts at Planck time governed by hyperbolic (evolution) equations starting at $t = 0$. The observed CMB indicated „*something*" (*microwaves*) at the border of „*our*" universe with some impact on the „time" before the Planck time, concluded out of the current model, which could not be explained by a steady-state (elliptic) mathematical model. As a consequence, the steady-state (elliptic) mathematical model has been descarded and new ideas/principles like „time reborn" (SmL), and „cycles of time" (PeR) are currently discussed.

(*) (CiI) 5.2.1: „Thus we may present the following arguments against the concpetion of a space-infinite, and for the conception of a space-bounded, universe:
(1) From the standpoint of the theory of relativity, the condition for a closed surface is verv much simpler than the corresponding boundary condition at infinity of the quasi-Euclidean structure of the universe,
(2) The idea of Mach expressed inertia depends upon the mutual action of bodies, is contained, to a first approximation, in the equations of the theory of relativity; …,
(3) An infinite universe is possible only if the mean density of matter in the universe vanishes. Although such an assumption is logically possible, it is less probable than the assumption that there is a finite mean density of matter in the universe.

(**) (PeR) 2.6: We still need to understand the extraordinarily low-entropy start of the universe, and according to the arguments in §2.2 this lowness of entropy lay essentially in the fact that the gravitational degrees of freedom were not excited, at least not nearly to the extent that involved all other degrees of freedom.

(***) (RoC1) II, 6: Time, as Aristotle suggested, is the measure of change: …. The entire evolution of science would suggest that the best grammar for thinking about the world is that of change, not of performance. Not of being, but of becoming. We can think of the world as made up of *things*. Of *substances*. Of *entities*. Of something that *is*. Or we can think of it as made up of *events*. Of *happenings*. Of *processes*. Of something that *occurs*. Something that does not last, and that undergoes continual transformation, that is not permanent in time.

(****) (SmL) §10: What must we require of a true cosmological theory?
(1) *Any new theory must contain what we already know about nature*. We need to current theories,
(2) the Standard Model of Particle Physics - genral relativity, and quantum mechanics,
(3) to emerge as approximations to the unknown cosmological theory whenever we restrict our attention to scales of distance and time smaller than the cosmos
(4) *The new theory must be scientific*. …. There can be no just making things up because it makes a nice story. A real theory must imply specific testable predictions,
(5) *The new theory should answer the „Why these laws?" question*. It must give a substancial insight into how and why the particular elementary particles and forces described in the Standard Model were selected. In particular, it must explain the special and improbable values of the fundamental constants that obtain in our universe – the parameters, like masses of elementary particles and the strength of the various forces, that are specified by the Standard Model,
(6) The new theory should answer the „Why these initial conditions?" question, explaining why our universe has properties that seem unusual when compared to the possible universes that might be described by the same laws.

In our proposed model the birth"*day*" of the *physical universe* (which is the universe of the second law of thermodynamics additionally to the dynamical laws) is at Planck time; this is the very first interaction of created EP after „*symmetry break down*" onto the physical energy Hilbert space; from that point in time the radiation is being governed by (weak variational) evolution (hyperbolic) PDO in the proposed extended Hilbert space framework.

The *physical universe* model is part of the *mathematical universe* model, which is a steady-state model being governed by (weak variational) (elliptic) PDO equations. At the same point in time the integrated steady-state ground state energy (ether) model comes along with an explanation of the observed cosmic microwave background radiation. We note that the observed CMB is basically „only" about electromagnetic waves, which are a very specific phenomena of our planet.

Our proposed model is very much in line with Bohm's concept of „*hidden variables in quantum theory*" (*). It handles especially those physical problems dealing with extremely short distances (Planck length and shorter) and high energy ($\sim 10^9 eV$ and higher) ((BoD) p. 83). In our case the *first change* („*mover*") of the „system" happens/occurs at Planck (point in) „*time*"; the „*time*" *before* that „*point in time*" can be interpreted as a „*hidden variable*" in the sense of D. Bohm.

In (BoD1) Bohm shows (**), „*how many of our „self-evident" notions of space and of time are, in fact, far from obvious and are actually learnt for experience*, *starting to understand the importance of measure and the need to map the relationships of these objects on to a co-ordinate grid with time playing a unique role*".

Bohm's concept of *hidden variables* overcomes current challenging consequences of main features of the quantum theory, like the fact, that there is „*no wave function existing describing a state, where all physical relevant quantities are dispersionless, i.e. they are sharply defined and free from statistical fluctuations*" (***). Bohm himself challenged his alternative model with respect to the proposed notion of a „*quantum potential*" and its related „*many-dimensional field*" to describe the many-body problem (****).

We emphasis, that our proposed „*quantum potential*" model (the closed subspace $H_1^{\perp}$ of $H_{1/2}$) is complementary and therefore independent from the „physical world" Hilbert space H_1.

(*) David Bohm, Causality and Chance in Modern Physics, London, 1957

(**) (BoD1) Foreword: … *It is also shown, through perception and our activity in space, we become aware of the importance of the notion of relationship and the order in these relationships. Through the synthesis of these relationships, we abstract the notion of an object as an invariant feature within this activity which ultimately we assume to be permanent. It is through the relationship between objects we arrive at our classical notion of space. Initially, this relations are essentially topological but eventually we begin to understand the importance of measure and the need to map the relationships of these objects on to a co-ordinate grid with time playing a unique role.*"

(***) In (BoD) the main six features of quantum theory are recalled. As a consequence of three of those features (features 4-6) it follows that there is „*no wave function existing describing a state, where all physical relevant quantities are dispersionless, i.e. they are sharply defined and free from statistical fluctuations*". Then the corresponding interpretation of a quantum theory based on the proposed hidden variables concept is described by five bullet point. The model explicitly allows, that an electron do have more properties, than the so-called „observables" of the quantum theory is able to describe.

(****) (BoD) p. 102: First of all, it must be admitted that the notion of the „quantum potential" is not entirely a satisfactory one, for not only is the proposed form, $U = -\left(\frac{h}{2\pi}\right)^2 \frac{1}{2m}(\frac{\nabla^2 R}{R})$, rather strange and arbitrary but also (unlike other fields such as the electromagnetic) it has no visible source. …., we evidently cannot be satisfied with accepting such a potential in a definitive theory. Second, in a many-body problem, we are lead to introduce a many-dimensional ψ -field ($\psi(x_1, x_1, \dots . x_n, \dots xN)$) and a corresponding many-dimensional quantum potential $U = -\left(\frac{h}{2\pi}\right)^2 \frac{1}{2m} \sum_{i=1}^{N} \left(\frac{\nabla_i^2 R}{R}\right)$ with $\psi = Re^{-iS\frac{h}{2\pi}}$ as in the one-body case.

The central part to prove the well-posedness of the 2D non-linear, non-stationary Navier-Stokes equations is a proper energy norm inequality estimate. It does not lead to blow-up effects for $t = T$ and do not show a Serrin gap with respect to the corresponding Sobolev norm estimates. All attempts failed to extend the existing 2-D NSE problem solution technique to the 3D case (GiY). For the weak H_0 based representation of the 3D non-linear, non-stationary Navier-Stokes equations the non-linear part of the „energy"-term vanishes. This is a great thing from a mathematical perspective, but a doubtful thing from a physical modelling perspective.

Energy transport equations (e.g. radiation problems) need to deal with sometime „inappropriate" physical solution behaviors for $t \to 0$, as well as blow-up effects for existing global bounded solutions until a certain point in time ($t < T_{Blow-up}$), or no existing global bounded solution at all (e.g. 3D-NSE). Such singularity behaviors and blow-up effects are the mathematical consequence of corresponding Sobolev space (energy) norm estimates governed by corresponding Sobolev embedding theorems:

i) already the most simple, linear homogenous heat equation with non-regular initial value function $g \in H_0$ shows a singular solution behavior for $t \to 0$ in the form

$$\|z(t)\|_k^2 \le c t^{-(k-l)} \|g\|_l^2 \quad , \quad \int_0^T t^{-1/2} \|z'\|_{-1/2}^2 dt \le c \|g\|_0^2 \quad (*)$$

ii) the global boundedness of the solution of the *2D*-NSE is governed by the ODE $y'(t) = y^2(t)$, $y(0) = y_0$ with the solution $y(t) = y_0/(1 - t \cdot y_0)$ becoming infinite in finite time (blow-up effect)

iii) the 3D-NSE is governed by the ODE $y'(t) = y^3(t)$, $y(0) = y_0$, i.e. there is no global global boundedness at all (which is the 3D-NSE Millennium problem with the proposed solution in (BrK2).

The alternatively proposed "fluid state" Hilbert space $\mathrm{H}_{-1/2}$ with corresponding alternative energy ("velocity") space $\mathrm{H}_{1/2}$ avoids the blow-up effect due to Ricci ODE estimates in the form $y'(t) \le c \cdot y^{1/2}(t)$ (**), while enabling at the same time an „energy" norm inequality (including contributions from the non-linear term), based a corresponding Sobolevskii estimate. The newly proposed scale value $\alpha = -1/2$ fulfills also the requirement $0 < \alpha < n/2 + \varepsilon$. It therefore provides an alternative model to the Dirac (Delta) „function" for energy transport equations.

(*) From $z(x,t) = \sum z_\nu(t)\phi_\nu(x)$ *it follows* $\dot{z} - z'' = \sum(\dot{z}_\nu(t) + \lambda_\nu z_\nu(t))\phi_\nu(x) = 0$. *Therefore* $z_\nu(t) = z_\nu(0)e^{-\lambda_\nu t}$ *and* $z_\nu(0) = g_\nu = (g, \phi_\nu)$. *Putting* $C_{k,l}(t) := sup\left\{\lambda_\nu^{k-l} e^{-2\lambda_\nu t} \middle| \lambda_\nu \ge m > 0\right\}$ *it follows* $\|z(t)\|_k^2 = \sum \lambda_\nu^k z_\nu^2(t) = \sum \lambda_\nu^k e^{-2\lambda_\nu t} g_\nu \le C_{k.l}(t) \sum \lambda_\nu^l e^{-2\lambda_\nu t}$. *The conditions* $(k - l)\lambda^{k-l-1} e^{-2\lambda_\nu t} + \lambda^{k-l}(-2t)e^{-2\lambda_\nu t} = 0$ *resp.* $(k-l)\lambda^{k-l-1}e^{-2\lambda_\nu t} = 2t\lambda^{k-l}e^{-2\lambda_\nu t}$ *leads to (for the critical case* $k > l$ *)* $\lambda \approx t^{-1}$ *.*

For the orthogonal set $\{w_i, \lambda_i\}$ *of eigenpairs of the non-stationary Stokes operator*

$$\bar{A} := \dot{w} + Aw = f \ , \quad w(0) = 0 \ , \quad \tau \in [0,t]$$

one gets $w_i(\tau) = \int_0^\tau e^{-\lambda_i(\tau - s)} f_i(s) ds$. *By changing the order of integration it follows for* $\beta > -1$

$$\int_0^t \tau^\beta w_i^2(\tau) d\tau \le \int_0^t \left[\int_0^\tau e^{-\lambda_i(\tau-s)} ds\right]\left[\int_0^\tau s^\beta e^{-\lambda_i(\tau-s)} f_i^2(s) ds\right] d\tau$$

$$\le \lambda_i^{-1} \int_0^t s^\beta f_i^2(s) \left[\int_\tau^t e^{-\lambda_i(\tau-s)} d\tau\right] ds \le \lambda_i^{-2} \int_0^t s^\beta f_i^2(s) ds \ .$$

From this one gets $\left|\left\|t^{\beta/2} w(t)\right\|\right|_{\alpha+2}^2 \le c \left|\left\|t^{\beta/2} \bar{A} w(t)\right\|\right|_{\alpha}^2, \quad \beta > -1$, *with* $|\|v(t)\||_\alpha^2 := \int_0^t \|v(s)\|_\alpha^2 ds, \quad \alpha \in R$.

(**) Lemma of Gronwall (general form): Let $a(t)$ *and* $b(t)$ *nonnegative functions in* $[0, A)$ *and* $0 < \delta < 1$ *. Suppose a nonnegative function* $y(t)$ *satisfies the differential inequality*

$$y'(t) + b(t) \le a(t) y^\delta(t) \quad on \quad [0, A)$$
$$y(0) = y_0 .$$

Then for $0 \le t < A$

$$y(t) + \int_0^t b(\tau) d\tau \le (2^{\delta/(1-\delta)} + 1) y_0 + 2^{\delta/(1-\delta)} \left[\int_0^t a(\tau) d\tau\right]^{\delta/(1-\delta)}$$

The Stokes operator is a projector from $A: L_2 \rightarrow L_\sigma^2 := \{v | v \in L_2 \wedge div(v) = 0\}$. The Hilbert scale is built on the Stokes operator on $\Omega \subseteq R^n$ $(n \geq 2)$ in the form $A = \int_0^\infty \lambda dE_\lambda$. The Stokes operator enables the definition of a related Hilbert scale ($\alpha \in R$) with a corresponding norm $\|u\|_\alpha := \|A^{\alpha/2}u\|$ (**), enabled by the corresponding positive selfadjoint fractional powers ((SoH), IV15)

$$A^\alpha = \int_0^\infty \lambda^\alpha dE_\lambda \ , -1 \leq \alpha \leq 1$$

The corresponding Stokes semigroup family $\{S(t)\}$ is built on the everywhere bounded, positive selfadjoint operator

$$S(t) := e^{-tA} := \int_0^\infty e^{-t\lambda} dE_\lambda \, | \lambda \geq 0, t \geq 0.$$

Putting $B(u) := P(u, grad)u)$ in the NSE and assuming $Pu_0 = u_0$, the NSE initial-boundary equation is given by $\frac{du}{dt} + Au + Bu = Pf$, $u(0) = u_0$. Multiplying this homogeneous equation with $A^{-1/2}u$ leads to

$$(\dot{u}, u)_\alpha + (Au, u)_\alpha + (Bu, u)_\alpha = 0, \ (u(0), v)_\alpha = (u_0, v)_\alpha \text{ for all } v \in H_{-1/2}$$

We note that the the pressure p (which can be also interpreted as energy density) in the variational representation

$$(Au, v)_{-1/2} := (\nabla u, \nabla v)_{-\frac{1}{2}} + (\nabla p, v)_{-1/2} = (u, v)_{1/2} + (p, v)_0 \quad \text{for all } v \in H_{-1/2}$$
$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2} .$$

can be expressed in terms of the velocity by the formula

$$p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$$

with (R_1, R_2, R_3) is the Riesz transform.

In case of $\alpha = -1/2$ one gets from Sobolevskii-estimates ((*), (GiY) lemma 3.2) the corresponding generalized "energy" inequality, given by

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq |(Bu, u)_{-1/2}| \leq \|u\|_{-1/2}\|Bu\|_{-1/2} \cong \|u\|_{-1/2}\|A^{-1/4}Bu\|_0.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets $y'(t) \leq c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$, resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \leq \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s) ds \leq c\{\|u_0\|_{-1/2} + \|u_0\|_0^2\}.$$

This energy norm estimates ensures global boundedness provided that $u_0 \in H_0$.

(*) (GiY) lemma 3.2.: For $0 \leq \delta < 1/2 + n \cdot (1 - 1/p)/2$ *it holds* $|A^{-\delta}P(u, grad)v|_p \leq M \cdot |A^\theta u|_p \cdot |A^\rho u|_p$ *with a constant* $M := M(\delta, \theta, \rho, p)$ *if* $\delta + \theta + \rho \geq n/2p + 1/2, \ \theta, \rho > 0, \ \theta + \rho > 1/2$. *Putting* $p := 2$, $\delta := 1/4$,$\theta := \rho := 1/2$ *fulfilling* $\theta + \rho \geq \frac{1}{4}(n+1) = 1$ *it follows*

$$\|A^{-\delta}P(u, grad)u\| \leq c\|A^\theta u\| \cdot \|A^\rho u\| = c\|u\|_{2\theta} \cdot \|u\|_{2\rho} = c\|u\|_1^2$$

resp.

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq |(Bu, u)_{-1/2}| \leq c \cdot \|u\|_{-1/2}\|u\|_1^2.$$

Another rational for the more appropriate $H_{-1/2}$ based variational representation of the NSE is about the following Neumann problem representation for the pressure field $p(\vec{x},t)$ ($\vec{n}$ denotes the outward unit normal to the domain G)

$$\Delta p = \rho(\vec{v}\cdot\nabla\vec{v}-\vec{f}) \qquad \text{in } G$$
$$\frac{\partial p}{\partial n} = -\left[\mu\Delta\vec{v}-\rho\vec{v}_1\cdot\nabla\vec{v}-\vec{f}\right]\cdot\vec{n} \qquad \text{at } \partial G.$$

It follows that the prescription of the pressure at the bounding walls or at the initial time independently of $\vec{v}$, could be incompatible with the initial and boundary conditions of the NSE PDE system, and therefore, could render the problem ill-posed (GaG), (HeJ). A $H_{-1/2}$ based representation with correspondly extended domains of the related operators overcomes this issue.

The related Prandtl operator $\overline{\mathrm{P}}$ is the double layer (hyper-singular integral) potential operator of the Neumann problem. It fulfills the following properties ((LiI) Theorems 4.2.1, 4.2.2, 4.3.2):

i) the Prandtl operator $\overline{P}: H_r \to \hat{H}_{r-1}$ is bounded for $0 \le r \le 1$

ii) the Prandtl operator $\overline{P}: H_r \to \hat{H}_{r-1}$ is Noetherian for $0 < r < 1$

iii) for $1/2 \le r < 1$, the exterior Neumann problem admits one and only one generalized solution.

The related Leray-Hopf projector is the matrix valued Fourier multiplier given by

$$P(\xi) = Id - \frac{\xi\otimes\xi}{|\xi|^2} = (\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2})_{1\le j,k\le n} \quad , \; P = Id - R\otimes R =: Id - Q$$

resp.

$$P = Id - R\otimes R =: Id - Q = Id - \frac{D\otimes D}{D^2}Id - \Delta^{-1}(\nabla\times\nabla).$$

As the operator $Q := R\otimes R = (R_jR_k)_{1\le j,k\le 1} = Q^2$ (R_i denote the Riesz operators [(*)]) is an orthogonal projector, the Leray-Hopf operator is also an orthogonal projection, where the domain can be defined on each Hilbert scale. In (LeN1) an explicit expression for the kernles of the Fourier multipliers of the corresponding Ossen operators are provided, which involves the incomplete gamma function and the confluent hypergeometric function of first kind.

(*) The Riesz transforms (the n-dimensional generalization of the Hilbert transform) are special Calderón-Zygmund (Pseudo Differential, convolution) operators with symbols $m(\omega)\in C^\infty(R^n-\{0\})$, where $m(\mu\omega)=m(\omega)$, $\mu>0$, where the mean of $m(\omega)$ on the unit sphere is zero and where it holds $m(\omega)=\frac{\omega_j}{|\omega|}$. They arise when study the Neumann problem in upper half-plane. The Riesz transforms $R_k u = -ic_n p.v.\int_{-\infty}^{\infty}\frac{x_k-y_k}{|x-y|^{n+1}}u(y)dy$ *(with* $c_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$ *) commutes with translations and homothesis, having nice properties relative to rotation. Especially the latter one play a key role in the concepts of the proposed concept of „rotating differentials" with respect to the rotation group $SO(n)$:*

let $m := m(x) := (m_1(x),\dots m_n(x))$ *be the vector of the Mikhlin multipliers of the Riesz operators and* $\rho = \rho_{ik}\in SO(n)$*, then it holds* $m(\rho(x)) = \rho(m(x))$ *(i.e.* $m_j(\rho(x)) = \sum\rho_{jk}m_k(x)$*), because of*

$$m(\rho(x)) = c_n\int_{S^{n-1}}(\frac{\pi i}{2}sign(x\rho^{-1}(y)) + log\left|\frac{1}{x\rho^{-1}(y)}\right|)\frac{y}{|y|}d\sigma(y) = c_n\int_{S^{n-1}}(\frac{\pi i}{2}sign(xy) + log\left|\frac{1}{xy}\right|)\frac{y}{|y|}d\sigma(y)\ .$$

The Riesz operators are related to the Caldéron- Zygmund operators $T(f) = S*F$ with a distribution S defined by a homogeneous function of degree zero, satisfying a kind of average mean zero condition on the unit sphere with its underlying rotation invariant probability measure (MeY). The search for conditions of minimal regularity in the context of the „pointwise multiplication" operator A is about an analysis of the commutator $[T,A]$. This leads to the „Caldéron operator"

$$(\Lambda u)(x) = (\sum_{k=1}^n R_kD_ku)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\sum_{k=1}^n p.v.\int_{-\infty}^{\infty}\sum_{k=1}^n\frac{x_k-y_k}{|x-y|^{n+1}}\frac{\partial u(y)}{\partial y_k}dy = -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}p.v.\int_{-\infty}^{\infty}\frac{\Delta_y u(y)}{|x-y|^{n-1}}dy = -(\Delta\Lambda^{-1})u(x)$$

with symbol $|\nu|$ and its inverse operator ((EsG) (3.15), (3.17), (3.35))

$$(\Lambda^{-1}u)(x) = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}p.v.\int_{-\infty}^{\infty}\frac{u(y)}{|x-y|^{n-1}}dy\ ,\ n\ge 2\ .$$

In dimension 1, this is about $\Lambda = DH$ where H denotes the Hilbert transform and D the Schrödinger momentum operator in the form $P := D = -i\frac{d}{dx}$ ((MeY) p. 5). The Schrödinger momentum operator in dimension n, and its related Hamiltonin operator is given by $P := -i\hbar\nabla = \frac{\hbar}{i}\nabla$ resp. $H := -\frac{\hbar^2}{2m}\Delta = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2$.

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