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On a question of Brézis and Nirenberg concerning the degree of circle maps

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Abstract. One considers maps F of the unit circle into itself. For maps of class C^1 or in the Sobolev class $H^{\frac{1}{2}}$, the degree or winding number is equal to the sum of the series $\sum n|c_n|^2$ in terms of the Fourier coefficients c_n of $F(e^{it}) = f(t)$. Here one studies the possible relation between this series (in symmetric arrangement) and the degree for arbitrary continuous maps F. It is shown that for such maps, the series may fail to be convergent or Abel summable. If the series does converge, the sum may have any value different from the degree of F.

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1. Introduction

Let F be any measurable map of the positively oriented unit circle S^1 into itself. We will work with the periodic unimodular *defining function,* $F(e^{it}) = f(t)$. It gives the Fourier series associated with the map

$$F(e^{it}) = f(t) \sim \sum_{-\infty}^{\infty} c_n e^{int}, \quad c_n = c_n[f] = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt. \tag{1.1}$$

The condition $|f(t)| \equiv 1$ has an equivalent formulation in terms of the Fourier coefficients

$$\sum_{n=-\infty}^{\infty} c_n \overline{c}_{n-k} = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{f(t)}e^{-ikt}dt = \begin{cases} 0 & \text{if } k \neq 0, \\ 1 & \text{if } k = 0. \end{cases}. \tag{1.2}$$

Following a question of Gelfand [5], H. Brézis and L. Nirenberg discovered that for circle maps F of class C^1, the degree, index or winding number can be expressed in terms of the Fourier coefficients as follows, cf. [4]:

$$\deg F = \frac{1}{2\pi i}\Delta \log F|_{S^1} \tag{1.3}$$

$$= \frac{1}{2\pi i}\int_{S^1} \frac{1}{F} dF = \frac{1}{2\pi i}\int_{S^1} \overline{F} dF \tag{1.4}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \overline{f(t)} f'(t)dt = \sum_{-\infty}^{\infty} n|c_n|^2. \tag{1.5}$$

A degree can also be defined for more general maps $F : S^1 \to S^1$. The continuous case is classical, cf. (1.3). Recent extensions have been motivated by the Ginzburg-Landau equations, see L. Boutet de Monvel and O. Gabber in [1, Appendix] and H. Brézis and L. Nirenberg [2], [3]. For the case of S^1, these authors considered maps in the Sobolev class $H^{\frac{1}{2}}$, using duality in (1.4), and in the larger class VMO, using approximation by continuous functions. In their papers, Brézis and Nirenberg made a thorough study of much more general maps of class VMO, also for higher dimensions, cf. the surveys in Brézis [4] and Nirenberg [9].

For circle maps F, Brézis and Nirenberg [2], [4] have asked if one could give a meaning to the formula

$$\deg F = \sum_{-\infty}^{\infty} n|c_n|^2 \tag{1.6}$$

in all cases where $\deg F$ is well-defined. Their answer was yes if F is in $H^{\frac{1}{2}}(S^1, S^1)$, a condition which is equivalent to the absolute convergence of the series in (1.6). Since such an F can be written as $H^{\frac{1}{2}}$-limit of functions in $C^1(S^1, S^1)$ it follows that $\deg F$ as defined by duality in (1.4) is an integer (also observed in [1, Appendix]) and that (1.6) is satisfied. Thus Brézis and Nirenberg obtained the surprising result that the orthogonality relations (1.2) for a sequence $\{c_n\}$ in l^2, together with the absolute convergence of the series $\sum n|c_n|^2$, imply that *the sum of the series is an integer*! It would be interesting to have a direct proof of this fact.

Remarks 1.1. Without the absolute convergence of (1.6), the relations (1.2) do not imply that the symmetric sum $S = \lim_{N\to\infty} \sum_{-N}^{N} n|c_n|^2$ is an integer whenever it exists. In fact, as observed by Jan Wiegerinck [10], for the discontinuous circle map defined by $g(t) = e^{\frac{1}{2}it}$ on $[0, 2\pi)$, one has $2\pi i c_n[g] = 4/(2n-1)$, so that

$$S = \frac{4}{\pi^2}\sum \frac{n}{(2n-1)^2} = \frac{1}{2}!$$

For the more general map defined by $g(t) = e^{i(\alpha/2\pi)t}$ on $[0, 2\pi)$, one finds that

$$S = \frac{\sin^2 \alpha/2}{\pi^2}\sum \frac{n}{(n-\alpha/2\pi)^2} = \frac{1}{2\pi}(\alpha - \sin\alpha). \tag{1.7}$$

This result has an interesting geometric meaning, cf. Remark 4.1 below.

We do not use degrees or indices for discontinuous maps, as one does for example in the theory of integral equations, cf. the books by I.C. Gohberg and coauthors [6], [7], [8].

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2. The case of continuous maps

In this note we address the following question.

Question 2.1 (cf. Brézis [4])**.** Let $F : S^1 \to S^1$ be continuous, $F(e^{it}) = f(t)$ and define

$$A_N = A_N[f] = \sum_{-N}^{N} n|c_n|^2, \quad A_r = A_r[f] = \sum_{-\infty}^{\infty} n|c_n|^2 r^{|n|}, \tag{2.1}$$

where $c_n = c_n[f]$ and $0 \le r < 1$. Can one then compute $\deg F$ as

$$S = S[f] \stackrel{\text{def}}{=} \lim_{N\to\infty} A_N \quad \text{or as} \quad A = A[f] \stackrel{\text{def}}{=} \lim_{r\to 1} A_r? \tag{2.2}$$

If the sum $S[f]$ exists, then so will the Abel sum $A[f]$, and the two will be the same. We can therefore use the notation $A[f]$ also for $S[f]$.

Observation 2.2. There is one simple case where the answer to Question 2.1 is yes. If f is (continuous and) *of bounded variation*, the Fourier series (1.1) is uniformly convergent to f while *df* is a complex measure; hence by (1.3), (1.4), integration by parts shows that

$$\begin{aligned}\deg F &= \frac{1}{2\pi i}\int_0^{2\pi} \overline{f(t)}df(t) = \lim_{N\to\infty} \frac{1}{2\pi i}\int_0^{2\pi} \sum_{-N}^{N} \overline{c}_n e^{-int} df(t) \\ &= \lim_{N\to\infty} \sum_{-N}^{N} n|c_n|^2.\end{aligned} \tag{2.3}$$

However, in general the answer to Question 2.1 is no.

Theorem 2.3. *For continuous maps $F : S^1 \to S^1$ of degree zero (or any other degree), the limits* (2.2) *for $f(t) = F(e^{it})$ may fail to exist, or they may have any value different from the degree, including $\pm\infty$.*

We know that for discontinuous maps the limit S can have any nonintegral value, cf. (1.7). The proof of the theorem uses the fact that for our problem, certain continuous maps can be made to behave like discontinuous ones.

In the following, it is best to *think of* S *or* A, A_r *and* A_N *as areas.* It is well-known that for smooth F on the positively oriented unit circle S^1, the signed area enclosed by the image curve $\Gamma = F(S^1)$ in the complex plane is equal to

$$\frac{1}{2i}\int_\Gamma \overline{z}dz = \frac{1}{2i}\int_{S^1} \overline{F}dF = \pi \sum_{-\infty}^{\infty} n|c_n|^2. \tag{2.4}$$

For integrable $F(e^{it}) = f(t)$, let f_r, with $0 \le r < 1$, be the *Poisson transform*

$$f_r(t) = \int_0^{2\pi} f(s)P_r(t-s)ds, \quad P_r(x) = \frac{1-r^2}{2\pi(1-2r\cos x + r^2)}. \tag{2.5a}$$

Similarly, the symmetric partial sum $f_N = s_N f$ of the Fourier series for f is given by its *Dirichlet transform*

$$f_N(t) = \int_0^{2\pi} f(s)D_N(t-s)ds, \quad D_N(x) = \frac{\sin(N+\frac{1}{2})x}{2\pi\sin\frac{1}{2}x}. \tag{2.5b}$$

In the following, the transforms of f will be treated together under the name f_ρ, where $\rho = r$ or N. The corresponding kernels will both be denoted by Q_ρ. The proof of the theorem will show that one can also use other transforms.

If F is continuous, $\pi A_r[f]$ is equal to the signed area enclosed by the curve given by $f_{\sqrt{r}}$, so that we get the following useful representation for A_r of (2.1):

$$A_r[f] = \sum_{-\infty}^{\infty} n|c_n|^2 r^{|n|} = \frac{1}{2\pi i}\int_0^{2\pi} \overline{f}_{\sqrt{r}} f'_{\sqrt{r}} = \frac{1}{2\pi i}\int_0^{2\pi} \overline{f} f'_r. \tag{2.6a}$$

The analog for $A_N[f]$ is

$$A_N[f] = A[f_N] = \frac{1}{2\pi i}\int_0^{2\pi} \overline{f} f'_N, \quad \text{where} \quad f_N = s_N f. \tag{2.6b}$$

We will use these formulas to define A_ρ with $\rho = r$ or N also for discontinuous maps.

3. A basic proposition

In this section we carefully discuss f_ρ and A_ρ for a special kind of defining function f. The proof of Theorem 2.3 will be based on infinite products of closely related functions.

For given $\alpha \in (0, \pi]$ and positive numbers a and b such that $a + b \leq 2\pi$, set

$$\frac{\alpha}{a} = v, \quad \frac{\alpha}{b} = w, \quad \frac{1}{2\pi}(\alpha - \sin\alpha) = \beta. \tag{3.1}$$

We now define

$$f = e^{i\phi}, \tag{3.2}$$

with continuous real periodic ϕ given by

$$\phi(t) = \begin{cases} vt & \text{for } 0 \leq t \leq a, \\ -w(t - a - b) & \text{for } a \leq t \leq a + b, \\ 0 & \text{for } a + b \leq t \leq 2\pi. \end{cases} \tag{3.3}$$

Applying integration by parts in (2.6) and using the periodicity of $\overline{f} f_\rho$, one obtains the formula

$$2\pi A_\rho[f] = i \int_0^{2\pi} \overline{f}' f_\rho = \int_0^a v e^{-ivt} f_\rho(t)dt - \int_a^{a+b} w e^{iw(t-a-b)} f_\rho(t)dt. \tag{3.4}$$

Since f_ρ converges uniformly to f as $\rho = r \to 1$ or $\rho = N \to \infty$, it follows that

$$2\pi A_\rho[f] \to \int_0^a v\overline{f} f - \int_a^{a+b} w\overline{f} f = av - bw = 0; \quad 2\pi A[f] = 0. \tag{3.5}$$

At this stage, it is crucial to remark that for $b \ll a$ (meaning that b/a is very small), the final integral in (3.4) will approach its limit α much more slowly than the integral over $(0, a)$. Indeed, as will be shown below, for

$$b \ll \rho = 1 - r \ll a, \quad \text{or} \quad v \ll \rho = N \ll w, \tag{3.6}$$

the integral over $(0, a)$ is close to α, whereas the integral over $(a, a + b)$ is close to $\sin\alpha$. For our ρ's, the transforms f_ρ do not "see" that f changes continuously on the short arc corresponding to $a \leq t \leq a + b$. The transforms behave as if f has a jump discontinuity at a, changing abruptly from the value $f(a) = e^{i\alpha}$ to the value $f(a + b) = 1$.

The discontinuous function obtained from f by letting $b \to 0$ will be called g. Normalizing it, we have

$$g(t) = \begin{cases} e^{ivt} & \text{for } 0 \leq t < a, \\ 1 & \text{for } a < t \leq 2\pi, \\ \frac{1}{2}(e^{i\alpha} + 1) & \text{for } t = a. \end{cases} \tag{3.7}$$

Proposition 3.1. *Let f, g and β be as in* (3.1)–(3.3) *and* (3.7).

(1) *For $b \ll 1-r$ or $N \ll w$, the transform f_ρ with $\rho = r$ or N is close to g_ρ everywhere.*

(2) *For $1-r \ll a$ or $v \ll N$, the function f_ρ is close to f outside the subinterval $J = [a-K(1-r), a+b+K(1-r)]$ or $[a-K/N, a+b+K/N]$ of $[0, 2\pi]$ for every constant K such that $1 \ll K \ll \{a/(1-r)\}^{\frac{1}{2}}$ or $1 \ll K \ll (N/v)^{\frac{1}{2}}$.*

(3) *Suppose now that $b \ll a$ and $b \ll 1-r \ll a$ or $v \ll N \ll w$. Then for $a \le t \le a+b$, the values of $f_\rho(t)$ and $g_\rho(t)$ are close to $g(a)$. Furthermore,*

$$A_\rho[f] \text{ is close to } A[g] = \beta. \tag{3.8}$$

(4) *For $\max(a,b) \ll 1-r$ or $N \ll \min(v,w)$, and for $1-r \ll \min(a,b)$ or $\max(v,w) \ll N$, the transforms f_ρ are everywhere close to 1, and close to f, respectively. In both cases, $A_\rho[f]$ is close to 0.*

All these uniform approximations can be made arbitrarily good by making the ratios p/q in the inequalities $p \ll q$ sufficiently small.

There are corresponding results with a and b interchanged (or v and w). In (3) *one then has to use the comparison function h obtained from f by letting $a \to 0$; $A[h] = -\beta$.*

If $0 < \alpha \le \pi/2$, then $|A_\rho[f]| < \alpha^2$ for every ρ.

4. Proof of Proposition 3.1

As far as possible, the Poisson and Dirichlet transform of f will be treated together as f_ρ.

(i) *Comparison of f_ρ and g_ρ.* The values of f and g belong to the circular arc $\Gamma_\alpha : \{z = e^{i\theta},\ 0 \le \theta \le \alpha\}$, so that $|f-g| \le \alpha$. Hence

$$\begin{aligned}|f_\rho(t) - g_\rho(t)| &= \left|\int_0^{2\pi} \{f(s)-g(s)\} Q_\rho(t-s)\,ds\right| = \left|\int_a^{a+b}\right| \\ &< b\alpha/(1-r), \quad \text{or} \quad < b\alpha N, \quad \text{respectively.}\end{aligned} \tag{4.1}$$

These bounds are very small provided $1-r \gg b$, or $N \ll w$.

One can also obtain bounds which are valid for arbitrary $\rho = r$ or N. Since $P_r > 0$, the values of f_r and g_r belong to the convex hull of Γ_α, that is, the closed segment Δ_α of the unit disc Δ bounded by the arc and its chord. (For example, since $\frac{1}{2}\pi \le \arg(f-1) \le \frac{1}{2}(\pi+\alpha)$ or $\mathrm{Re}\, f \le 1$ and $\mathrm{Im}\, e^{-\frac{1}{2}i(\pi+\alpha)}(f-1) \le 0$, the same inequalities hold for f_r, etc.) In particular $|f_r - g_r| \le \alpha$.

In the case of the Dirichlet transform, one has to reckon with the Gibbs phenomenon: $f-g$ jumps from 0 to $e^{i\alpha}-1$ at a. Separately considering the real and imaginary part of $f_N - g_N$, one finds that $|f_N - g_N - \frac{1}{2}(e^{i\alpha} - 1)| < 0.6\alpha$, hence $|f_N - g_N| < 1.1\alpha$ for all N. Likewise $|g_N - \frac{1}{2}(e^{i\alpha} + 1)| < 0.6\alpha$.

(ii) *Estimates for $f - f_\rho$ and f_ρ.* It follows from part (i) that $|f - f_r| < \alpha$ and $|f_N - e^{i\alpha}| < 1.2\alpha$, hence

$$|f(t) - f_\rho(t)| < 2.2\alpha \quad \text{for every } \rho \text{ and } t. \tag{4.2a}$$

For part (iv) and for later, we need rather precise estimates under the assumption that $1 - r \ll a$ or $N \gg v$. We treat the Poisson transform in detail; working modulo 2π, we take t outside an interval of the form $J = [a - K(1-r), a + b + K(1-r)]$. One has

$$|f(t) - f_r(t)| \le \int_{-\pi}^{\pi} |f(t) - f(t+x)| P_r(x) dx = I_1 + I_2,$$

where in I_1 we integrate over $|x| \le K(1-r)$ and in I_2 over $|x| > K(1-r)$. In I_1 we use the inequality $P_r(x) < 1/(1-r)$ and in I_2, taking r close to 1, we use $P_r(x) < (1-r)/x^2$. Observe that in I_1, the points t and $t+x$ lie on the same side of $[a, a+b]$, so that $|f(t) - f(t+x)| \le v|x| = (\alpha/a)|x|$. Thus

$$|f(t) - f_r(t)| \le |I_1| + |I_2| < \alpha K^2 \frac{1-r}{a} + 2\alpha \frac{1}{K}, \quad t \notin J. \tag{4.2b}$$

This bound is small whenever $1 \ll K \ll \{a/(1-r)\}^{\frac{1}{2}}$.

Another calculation shows that $|f(t) - f_N(t)|$ is small for t outside any interval $[a - K/N, a + b + K/N]$ with $1 \ll K \ll (N/v)^{\frac{1}{2}}$. Here one has to integrate by parts in an integral I_2 where $|x| > K/N$.

In this paragraph one assumes $b \le a$. That f_ρ is then close to 1 for $1 - r \gg a$ or $N \ll v$ follows immediately from the integral formulas (2.5) applied to $f - 1$. For $1 - r \ll b$ or $N \gg w$ one may use the coefficient estimates

$$2\pi|nc_n| = \left|\int_0^{2\pi} e^{-int} df(t)\right| = \left|\int_0^a + \int_a^{a+b}\right| \le av + bw = 2\alpha,$$

$$2\pi n^2 |c_n| \le \int_0^{2\pi} |df'| = (2+\alpha)(v+w) < 4\pi w.$$

They imply that f_ρ is close to f everywhere:

$$|f - f_N| \le \sum_{|n|>N} |c_n| < 4w/N, \quad |f - f_r| < 12\sqrt{(1-r)/b}. \tag{4.3}$$

(iii) *Estimates for* g_ρ. There are inequalities (4.2) also for $g - g_\rho$. Let us now assume $b \ll 1 - r \ll a$ or $v \ll N \ll w$. Taking $a \le t \le a + b$, we show that *both* $g_\rho(t)$ *and* $f_\rho(t)$ *are close to*

$$g(a) = \{g(a-) + g(a+)\}/2 = (e^{i\alpha} + 1)/2.$$

Because of (4.1), it is enough to consider $g_\rho(t)$ and we may actually limit ourselves to $t = a$. Indeed, $g(t+x) - g(a+x) \ne 0$ only on the interval $(-t, -t+a)$ of length a where one has a bound $\le v(t-a) \le vb$ and on the interval $[-t+a, 0]$ of length $t - a \le b$ where one has bound α. The usual inequalities for the kernels Q_ρ then show that the integrals for $g_\rho(t) - g_\rho(a)$ are uniformly small for our ρ's.

Finally, by the symmetry of Q_ρ, assuming that $a \le \pi$ in order to obtain a simple formula, one has

$$\begin{aligned} g_\rho(a) - g(a) &= \int_0^\pi \{g(a+x) + g(a-x) - 2g(a)\} Q_\rho(x) dx \\ &= \int_0^a e^{iva}(e^{-ivx} - 1) Q_\rho(x) dx + \int_a^\pi (1 - e^{iva}) Q_\rho(x) dx. \end{aligned}$$

One may split the integral over $(0, a)$ at $x = 1 - r$ or $1/N$ to show that $g_\rho(a) - g(a)$ is small for $1 - r \ll a$ or $N \gg v$, whichever applies.

(iv) *The values of* $A_\rho[f]$. We will compare the integrals in (3.4) and (3.5), assuming initially that $b \ll 1 - r \ll a$ or $v \ll N \ll w$. It will be enough to act as if $\rho = r$. Splitting the integral over $(0, a)$ and using (4.2), one obtains

$$\begin{aligned} \left| \int_0^a v e^{-ivt}(f - f_r) \right| &= \left| \int_{(0,a)\setminus J} + \int_{(0,a)\cap J} \right| \\ &< av \left(\alpha K^2 \frac{1-r}{a} + 2\alpha \frac{1}{K} \right) + \{2K(1-r) + b\} v\alpha. \end{aligned} \tag{4.4}$$

Taking K as before one concludes that $\int_0^a v e^{-ivt} f_\rho$ is close to $\int_0^a v \overline{f} f = av = \alpha$.

We next consider the integral over $(a, a+b)$ in (3.4). By part (iii) above,

$$\begin{aligned} \int_a^{a+b} w e^{iw(t-a-b)} f_\rho &\approx \frac{1}{2}(e^{i\alpha} + 1) \int_a^{a+b} w e^{iw(t-a-b)} dt \\ &= \frac{1}{2}(e^{i\alpha} + 1) \frac{1}{i}(1 - e^{-i\alpha}) = \sin\alpha. \end{aligned} \tag{4.5}$$

Hence by (3.4), for our present values of ρ,

$$A_\rho[f] \approx \frac{1}{2\pi}(\alpha - \sin\alpha) = \beta. \tag{4.6}$$

For $1 - r \gg a \geq b$ or $N \ll v \leq w$, f_ρ is close to 1, cf. part (ii), so that $A_\rho[f]$ is close to 0. For $1 - r \ll b \leq a$ or $N \gg w \geq v$ it follows from (4.4) that $A_\rho[f]$ is again close to 0.

We also have to know what happens for other values of r or N, where we may assume for later that $0 < \alpha \leq \pi/2$. Since the values of f_r belong to the disc segment Δ_α, the real parts of the last two integrals in (3.4) with $\rho = r$ lie between $\sin\alpha\cos\alpha$ and $\sin\alpha\,(2 - \cos\alpha)$. It follows that for every r,

$$2\pi|A_r[f]| \leq 2\sin\alpha\,(1 - \cos\alpha) < \alpha^3. \tag{4.7a}$$

In the case of f_N the inequality $|f_N - e^{i\alpha}| < 1.2\alpha$ of part (ii) implies that for every N,

$$2\pi|A_N[f]| < 5\alpha^2. \tag{4.7b}$$

(v) *The value of* $A[g]$. In the case of g, we treat the integral (2.6) for A_ρ as a Stieltjes integral, integrate by parts and finally let $r \to 1$ or $N \to \infty$. Since g is a normalized function of bounded variation, $g_\rho \to g$ pointwise and boundedly. One thus finds that $\lim A_\rho[g] = A[g]$ exists, cf. also part (iii):

$$\begin{aligned} 2\pi A_\rho[g] &= \frac{1}{i}\int_0^{2\pi} \overline{g}\,dg_\rho = i\int_0^{2\pi} g_\rho\,d\overline{g} \\ &= i\int_0^{a-} g_\rho\overline{g}'\,dt + i\int_{a-}^{a+} g_\rho\,d\overline{g} = \int_0^a g_\rho v\overline{g}\,dt + ig_\rho(a)\{\overline{g}(a+) - \overline{g}(a-)\} \\ &\to \int_0^a gv\overline{g}\,dt + ig(a)\{\overline{g}(a+) - \overline{g}(a-)\} = av + i\frac{1}{2}(e^{i\alpha} + 1)(1 - e^{-i\alpha}) \\ &= \alpha - \sin\alpha = 2\pi A[g]. \end{aligned} \tag{4.8}$$

One may derive from part (iii) that the difference between $A[g]$ and $A_\rho[g]$ is small when $1 - r \ll a$ or $N \gg v$.

This concludes the proof of the proposition.

Remark 4.1. Observe that $\pi A[g] = \frac{1}{2}(\alpha - \sin\alpha)$ is the area of the disc segment Δ_α bounded by the arc Γ_α and its chord, cf. part (i). Thinking of $\pi A[g]$ as the area bounded by a "closed curve" $g([0, 2\pi])$, one has to make the "vertical" transition of g from the value $e^{i\alpha}$ to the value 1 at $t = a$ correspond to the chord! However, this is perfectly reasonable if one considers the curve $g_r([0, 2\pi])$. Taking $\alpha = \pi$ as in (1.7), one can compute g_r exactly and it is then seen that the curve $g_r([0, 2\pi])$ approximates the boundary of Δ_α.

5. Towards the proof of Theorem 2.3

Our defining functions f are given by infinite products

$$f = f_1 f_2 f_3 \cdots, \quad f_k = e^{i\phi_k}, \tag{5.1}$$

where the ϕ_k's are continuous real periodic functions like ϕ in formula (3.3), but with constants indexed by k, and with shifts so that the peaks of different ϕ_k's do not overlap. The constants $\alpha_k \in (0, \pi]$ are taken such that $\alpha_k \downarrow 0$, but

$$\sum_{k=1}^{\infty} \beta_k = \infty, \quad \text{where} \quad \beta_k \stackrel{\text{def}}{=} (\alpha_k - \sin \alpha_k)/2\pi. \tag{5.2}$$

The positive sequences $\{a_k\}$ and $\{b_k\}$ must be nonincreasing and satisfy the condition $\sum_{k=1}^{\infty}(a_k + b_k) \le 2\pi$; we define

$$v_k = \alpha_k / a_k, \quad w_k = \alpha_k / b_k, \quad \sum_{j \le k}(a_j + b_j) = \gamma_k, \quad \sum_{k > p}(a_k + b_k) = \delta_p. \tag{5.3}$$

Other conditions on the constants will be imposed later.

The peak of $\phi_k(t)$ on $[0, 2\pi]$ is to start at $t = \gamma_{k-1}$ (with $\gamma_0 = 0$); it has width $a_k + b_k$ and height α_k. Thus

$$f_k(t) = \begin{cases} e^{iv_k t} & \text{for } \gamma_{k-1} \le t \le \gamma_{k-1} + a_k, \\ e^{-iw_k(t-\gamma_k)} & \text{for } \gamma_k - b_k \le t \le \gamma_k, \\ 1 & \text{on } [0, 2\pi] \text{ outside } (\gamma_{k-1}, \gamma_k). \end{cases} \tag{5.4}$$

We also need the functions g_k and h_k obtained from f_k by letting b_k or a_k go to 0; at the points of discontinuity we use the average of the left-hand and right-hand limit.

Since $\alpha_k \to 0$, our functions f will define continuous maps F of degree 0.

For the subsequent computations we observe that the results of Proposition 3.1 apply *mutatis mutandis* to the shifted functions f_k. We now derive some auxiliary results. One may expect that for our product maps, the "areas" A and A_ρ are in some sense equal to the sum of the areas of the factors. Let us set

$$F_p = f_1 \cdots f_p, \quad T_p = f_{p+1} \cdots, \quad G_p = g_1 \cdots g_p, \quad H_p = h_1 \cdots h_p. \tag{5.5}$$

For given r, the value of $A_r[F_p]$ will depend on how much of the curve $F_p([0, 2\pi])$ is "visible" from the circle $S^1(0, r)$ of radius r.

Proposition 5.1.

(1) *With* β_k *as in* (5.2),

$$A[g_1 \cdots g_p] = \sum_{k=1}^{p} A[g_k] = \sum_{k=1}^{p} \beta_k. \tag{5.6}$$

(2) *For* $1-r \ll a_p$ *or* $v_p \ll N$,

$$A_\rho[F_p] = A_\rho[f_1 \cdots f_p] \approx \sum_{k=1}^{p} A_\rho[f_k]. \tag{5.7}$$

(3) *Suppose for the time being that* $b_1 \ll a_p$. *Then for* $b_1 \ll 1-r \ll a_p$ *or* $v_p \ll N \ll w_1$, $A_\rho[f_1 \cdots f_p]$ *is close to* $A[g_1 \cdots g_p]$. *If moreover* $b_2 \ll b_1$, *then for* $b_2 \ll 1-r \ll b_1$ *or* $w_1 \ll N \ll w_2$, $A_\rho[f_1 \cdots f_p]$ *is close to* $A[g_2 \cdots g_p]$. *During the transition from* $1-r \gg b_1$ *to* $1-r \ll b_1$ *(or from* $N \ll w_1$ *to* $N \gg w_1$*),* $2\pi A_\rho[f_1]$ *goes from about* $\alpha_1 - \sin\alpha_1$ *to about* 0. *While this takes place,* $|A_\rho[f_1]|$ *never exceeds* α_1^2 *(assuming that* $\alpha_1 \le \pi/2$*).*

The role of a's and b's may be interchanged. If $a_1 \ll b_p$, $A_\rho[F_p]$ *is close to* $A[H_p] = -\sum_1^p \beta_k$ *for* $a_1 \ll 1-r \ll b_p$ *or* $w_p \ll N \ll v_1$.

One can make "close" as close as one wishes by appropriate choice of the relative gaps indicated by the symbol $\ll$.

Proof. For definiteness we take $\rho = r$. Since $g_1 \cdots g_p = \sum_{k=1}^p g_k - (p-1)$, one has $(g_1 \cdots g_p)_r = g_{1r} + \cdots + g_{pr} - (p-1)$ and similarly for products of f_k's. Formula (5.6) for $A[g_1 \cdots g_p]$ then follows by a computation similar to (4.8).

We turn to part (2). By (3.4), since $f_k = 1$ outside $I_k = (\gamma_{k-1}, \gamma_k)$,

$$\begin{aligned} 2\pi A_r[f_1 \cdots f_p] = i \sum_{k=1}^{p} \int_{I_k} \overline{f}'_k \{ &(f_{1r} - 1) + \cdots + (f_{k-1,r} - 1) \\ &+ f_{kr} + (f_{k+1,r} - 1) + \cdots + (f_{pr} - 1)\}. \end{aligned} \tag{5.8}$$

Suppose now that $1 - r \ll a_p$. To establish (5.7) we show that the integrals

$$\int_{I_k} \overline{f}'_k (f_{qr} - 1) \quad \text{with} \quad q \ne k \tag{5.9}$$

are small. We know that $1-r \ll a_q$; hence if also $1-r \ll b_q$, then by Proposition 3.1(4), f_{qr} is close to f_q everywhere. Thus f_{qr} is close to 1 on I_k and the smallness of (5.9) follows. We may therefore assume that $b_q \le K(1-r)$ for some constant K. Proposition 3.1(2) then implies that f_{qr} is close to f_q outside any interval $J_q = [\gamma_q - M_q(1-r), \gamma_q + M_q(1-r)]$ with $1 \ll M_q \ll \{a_q/(1-r)\}^{\frac{1}{2}}$. If $q > k$, the exceptional interval starts to the right of I_k; hence f_{qr} is close to f_q and thus again close to 1 on I_k. We finally take $q < k$; it will suffice to consider $q = k-1$. Then J_{k-1} overlaps I_k, but one may take $M_{k-1} \ll \{a_k/(1-r)\}^{\frac{1}{2}}$ and then J_{k-1} lies to the left of $\gamma_{k-1} + a_k$. One thus finds that the contribution to (5.9) due to the overlap is bounded by

$$\int_{I_k \cap J_{k-1}} |\overline{f}'_k (f_{k-1,r} - 1)| \le M_{k-1}(1-r) v_k \alpha_{k-1}$$

and therefore small. The conclusion is that all the integrals (5.9) are small; hence the expression in (5.8) is indeed close to $\sum_{k=1}^{p} A_r[f_k]$.

Part (3). Suppose now that $b_1 \ll a_p$. Then for $b_1 \ll 1-r \ll a_p$,

$$2\pi A_r[f_k] \approx 2\pi A[g_k] = \alpha_k - \sin\alpha_k \tag{5.10}$$

for every k, cf. (3.8), so that $A_r[f_1 \cdots f_p] \approx A[g_1 \cdots g_p]$.

If $b_2 \ll b_1 \ll a_p$ and one takes $b_2 \ll 1-r \ll b_1$, then the approximation (5.10) remains valid for $k \geq 2$, but now $A_r[f_1]$ is close to $A[f_1] = 0$, cf. Proposition 3.1(4). Thus $A_r[F_p]$ is reduced by $A[g_1] = \beta_1$ from its earlier value. The transition result for $1-r$ around b_1 follows from (4.7).

The remaining statements in the proposition require no proof, cf. Proposition 3.1.

We finally need a comparison between $A_\rho[f]$ and $A_\rho[F_p]$. Can one neglect the tail T_p in the product for f when ρ is in a certain range? When is the tail $T_p([0,2\pi])$ "nearly invisible" from $S^1(0,r)$? Here one has to impose a "tail condition" on the numbers a_k, b_k with $k > p$.

Proposition 5.2. *For given ρ and small $\varepsilon > 0$, one can ensure that*

$$|A_\rho[f] - A_\rho[F_p]| < \varepsilon \tag{5.11}$$

by requiring that the remainder δ_p in (5.3) *be small relative to* $(1-r)/p$ *or* $1/pN$.

Proof. It is efficient to use Fourier series here. Observe that by (5.5),

$$f = F_p T_p = F_p + (T_p - 1). \tag{5.12}$$

For convenience, set

$$c_n[F_p] = d_n, \quad c_n[T_p - 1] = e_n, \quad c_n[f] = d_n + e_n = c_n. \tag{5.13}$$

The d_n's may be estimated as follows:

$$\begin{aligned} 2\pi|nd_n| &= \left|\int_0^{2\pi} F_p'(t)e^{-int}dt\right| \leq \int_0^{2\pi} (|\phi_1'| + \cdots + |\phi_p'|) \\ &= \sum_{k\leq p}(a_k v_k + b_k w_k) = 2\sum_{k\leq p}\alpha_k \leq 2\pi p. \end{aligned}$$

For the e_n's one observes that $|T_p| = 1$, while $T_p = 1$ on the part of $[0,2\pi)$ outside $(\gamma_p, \gamma_p + \delta_p)$, so that

$$2\pi|e_n| = \left|\int_{\gamma_p}^{\gamma_p+\delta_p} \{T_p(t) - 1\}e^{-int}dt\right| < 2\delta_p.$$

Hence by (5.13),

$$|A_r[f] - A_r[F_p]| = \left|\sum (n|c_n|^2 - n|d_n|^2) r^{|n|}\right| \le \sum (|n||e_n|^2 + 2|nd_n e_n|) r^{|n|}$$
$$< \left(\frac{\delta_p}{1-r}\right)^2 + 3p\frac{\delta_p}{1-r} < \varepsilon, \tag{5.14}$$

provided $p\delta_p \ll 1 - r$. For (5.11) with $\rho = N$ one sums over $|n| \le N$ in (5.14) (omitting $r^{|n|}$) and replaces $1/(1-r)$ by N.

6. Completion of the proof of Theorem 2.3

The following two constructions and some concluding remarks will establish Theorem 2.3. In both constructions f will have the form $f_1 f_2 \cdots$ with factors f_k as in (5.4). To determine f_k one need only specify α_k and a_j, b_j with $j \le k$ and that is what will be done below.

(i) *Oscillation result for* A_r. For given $\lambda > 0$ and $\mu < 0$, we obtain f and a sequence $r_k \uparrow 1$ such that $A_{r_k}[f]$ is alternately close to λ and to μ.

In the first step, choose $p_1 = q_1$ and $\alpha_1 = \cdots = \alpha_{p_1} \le \pi$ such that $q_1\beta_{p_1} = \lambda$. Then choose $a_1 = \cdots = a_{p_1}$ as well as $b_1 = \cdots = b_{p_1} \ll a_{p_1}$ in such a way that $q_1(a_{p_1} + b_{p_1}) < 2\pi$. For $b_{p_1} \ll 1 - r \ll a_{p_1}$, Proposition 5.1 now shows that $A_r[f_1 \cdots f_{p_1}]$ is close to $A[g_1 \cdots g_{p_1}] = q_1\beta_{p_1} = \lambda$. This will also be true for $A_r[f]$, provided the tail T_{p_1} in the ultimate product for f can be neglected. For this we will require that later a's and b's satisfy the tail condition $4p_1\delta_{p_1} \le b_{p_1}$, cf. (5.14).

In the second step one takes $p_2 = p_1 + q_2$, choosing q_2 and $\alpha_{p_1+1} = \cdots = \alpha_{p_2} \le \alpha_{p_1}$ such that $-q_2\beta_{p_2} = \mu$. One next takes $a_{p_1+1} = \cdots = a_{p_2} \ll b_{p_1+1} = \cdots = b_{p_2} \le b_{p_1}$, making sure that $q_2(a_{p_2} + b_{p_2})$ is small enough. Then the following approximations hold for $1 - r \ll b_{p_2}$, and for $a_{p_2} \ll 1 - r \ll b_{p_2}$, respectively:

$$A_r[f_1 \cdots f_{p_2}] \approx \sum_{k=1}^{p_2} A_r[f_k] \approx \sum_{1}^{p_1} A[f_k] + \sum_{p_1+1}^{p_2} A[h_k] = \mu,$$

cf. Propositions 5.1 and 3.1. Thus $A_r[f] \approx \mu$ under a suitable tail condition on the numbers a_k, b_k with $k > p_2$.

Next, one takes $p_3 = p_2 + q_3$, $\alpha_{p_2+1} = \cdots = \alpha_{p_3} \le \alpha_{p_2}$ such that $q_3\beta_{p_3} = \lambda$. One now obtains a range for r given by $a_{p_2+1} = \cdots = a_{p_3} \ll 1 - r \ll b_{p_2+1} = \cdots = b_{p_3} \le a_{p_2}$ in which $A_r[f]$ is again close to λ. Etc. In all this one makes sure that $\alpha_k \to 0$.

One thus obtains f and ranges in which to choose numbers r_k for which the sequence $\{A_{r_k}[f]\}$ exhibits the desired oscillatory behavior.

(ii) *Convergence result for A_N.* We finally show how to construct a defining function f for which $A_N[f]$ converges to a prescribed real number λ as $N \to \infty$. For definiteness one supposes $\lambda > 0$.

The plan for the construction is as follows. With $A[g_j] = \beta_j$ as in (5.2), (5.6), one determines numbers $p_k \geq k+1$ and $\alpha_k \leq \pi/2$ such that for $k = 1, 2, \ldots,$

$$A[g_k \cdots g_{p_k}] = \sum_{j=k}^{p_k} \beta_j = \lambda, \quad \text{hence} \quad A[g_{k+1} \cdots g_{p_k}] = \lambda - \beta_k. \tag{6.1}$$

One then seeks numbers a_j and b_j or v_j and w_j such that

$$v_{p_1} \ll w_1 \ll v_{p_2} \ll \cdots \ll v_{p_k} \ll w_k \ll v_{p_{k+1}} \ll \cdots,$$

so that for the corresponding functions f_j, cf. Propositions 5.1 and 3.1,

$$\begin{aligned} A_N[f_1 \cdots f_{p_k}] &\approx \sum_{j=1}^{p_k} A_N[f_j] \qquad \text{for } N \gg v_{p_k}, \\ &\approx \begin{cases} A[g_k \cdots g_{p_k}] & \text{for } v_{p_k} \ll N \ll w_k, \\ A[g_{k+1} \cdots g_{p_k}] & \text{for } w_k \ll N \ll v_{p_{k+1}}. \end{cases} \end{aligned} \tag{6.2}$$

It turns out that it is convenient to choose p_k, α_k or β_k and v_k such that

$$p_1 = q \quad \text{with } q \geq 2 \text{ and } \lambda/q \leq (\pi - 2)/4\pi, \quad p_{k+1} = p_k + 2,$$

$$\beta_1 = \cdots = \beta_{p_1} = \lambda/q, \quad \beta_{p_k} = \beta_{p_k+1} = \frac{1}{2}\beta_{p_{k-1}},$$
$$v_1 = \cdots = v_{p_1}, \quad v_{p_k+1} = v_{p_{k+1}}.$$

Then condition (6.1) is satisfied and $\pi/2 \geq \alpha_k \to 0$. Also, to deal with transition values of N around w_k and $v_{p_{k+1}}$ below, one can use the following inequalities:

$$|A_N[f_k]| < \alpha_k^2, \quad |A_N[f_{p_k+1} f_{p_{k+1}}]| \approx 2|A_N[f_{p_{k+1}}]| < 2\alpha_{p_{k+1}}^2. \tag{6.3}$$

Starting with $k = 1$, one now successively chooses constants a_k and b_k or v_k and w_k subject to appropriate conditions to ensure that

$$\begin{aligned} &|A_N[f_1 \cdots f_{p_k}] - \lambda| < \beta_k, \quad |A_N[f] - \lambda| < 2\beta_k && \text{for } v_{p_k} \ll N \ll w_k, \\ &|A_N[f] - (\lambda - \beta_k)| < 2\beta_k && \text{for } w_k \ll N \ll v_{p_{k+1}}, \\ &|A_N[f] - \lambda| < 2\beta_k + \alpha_k^2 && \text{for } N \text{ around } w_k, \\ &|A_N[f] - (\lambda - \beta_k)| < 2\beta_k + 2\alpha_{p_{k+1}}^2 && \text{for } N \text{ around } v_{p_{k+1}}, \end{aligned}$$

cf. (6.1)–(6.3) and Propositions 5.1, 5.2.

The step by step construction leads to a function f which defines a continuous map F of degree zero such that $A_N[f]$ converges to the limit $S[f] = \lambda$.

Obvious changes would produce a function f for which $A_N \to \infty$.

(iii) *Concluding remarks.* By well-known results for infinite series, the sums A_N must oscillate at least as much as the values A_r. Indeed,

$$\begin{aligned} A_r &= \sum_1^\infty n(|c_n|^2 - |c_{-n}|^2)r^n = \\ &= \sum_1^\infty (A_n - A_{n-1})r^n = (1-r)\sum_1^\infty A_n r^n, \end{aligned} \tag{6.4}$$

hence if $\limsup A_N \le \omega$ as $N \to \infty$, then also $\limsup A_r \le \omega$ for $r \to 1$. Thus in part (i), A_N cannot remain well below λ as $N \to \infty$. Similarly, it cannot remain well above μ.

For a function f as in part (ii), where $A_N \to \lambda$, also $A_r \to \lambda$; convergence implies Abel summability to the same sum.

To obtain continuous maps of prescribed degree $m \ne 0$ with properties as in parts (i) and (ii), one may multiply the above functions $f(t)$ by $e^{i(m/2\pi)t}$ and then use suitably adjusted values of λ and μ.

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