A geometric Hilbert scale based gravity and quantum field model

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The modelling landscape

Let the linear equation $Au = f$ be given in a Hilbert space H with inner product $(\cdot\,,\,\cdot)$ and norm $\|\cdot\|$. The operator A is assumed to have the properties, (BrK0)

i) A positive, i.e. $(Au, u) > 0$ for $u \neq 0$

ii) A symmetric, i.e. $(Au, v) = (u, Av)$

for $u, v \in D(A)$. Then $a(u, v) := (Au, v)$ defines an inner product in $D(A)$. The corresponding norm will be denoted by

$$|\|u\||^2 := a(u, u).$$

The domain of definition of $a(u, u)$ can be extended to $H_A x H_A$ with $H_A := \overline{D(A)}^{|\|u\||}$.

Regarding the linear operator A we give the following illustrations:

Example 1. $H = L_2(\Omega)$ with Ω a bounded domain in R^3 and the boundary $\partial\Omega$ sufficiently smooth.

$Au := -\Delta u$ and $D(A) = \dot{W}_2^2(\Omega)$, ($\dot{W}_2^2(\Omega)$ being the intersection of $W_2^2(\Omega)$ and $W_0^{1,2}(\Omega)$).

We introduce a Hilbert scale in the following way: let $\{v_i, \lambda_i\}$ be the orthonormal set of eigen-pairs of A, i.e.

$$-\Delta v_i = \lambda_i v_i \text{ in } \Omega$$

$$v_i = 0 \quad \text{on } \ \partial\Omega\,.$$

The Hilbert spaces $\{H_\beta | \beta \in R\}$ are spanned by the functions with finite β -norm defined by

$$\|z\|_\beta^2 := \sum \lambda_i^\beta z_i^2 \text{ with } z_i := (z, v_i).$$

We have the inclusions

$$D(A) \subset H_A = H_1 = W_0^{1,2}(\Omega) \subset L_2\,(\Omega).$$

The operator $A: W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is called potential operator and $a = \frac{1}{2}\|z\|_1^2$ the potential of a potential function a (ChJ).

Additionally, for $t > 0$ there can be an inner product resp. norm defined for an additional governing Hilbert space with an "*exponential decay*" behavior in the form $e^{-\sqrt{\lambda_i}t}$ given by

$$(x, y)_{\rho.(t)}^2 := \sum_{i=1} \lambda_i^\rho e^{-\sqrt{\lambda_i}t}(x, \phi_i)(y, \phi_i), \ \ \|x\|_{\rho.(t)}^2 := (x, x)_{\rho.(t)}^2.$$

The polynomial and exponential decay norms are governed by the inequality

$$\|x\|_{\rho-\beta}^2 \leq \delta^{2\beta}\|x\|_\rho^2 + e^{t/\delta}\|x\|_{\rho.(t)}^2,$$

being valid for any $t, \delta, \beta > 0$ and $\lambda \geq 1$ (the inequality follows from the inequality $\lambda^{-\beta} \leq \delta^{2\beta} + \mathrm{e}^{\mathrm{t}(\delta^{-1}-\sqrt{\lambda})}$). The special choises $\beta = 1/2$, $\rho = 0$ lead to the $H_{-1/2}$ specific inequality

$$\|x\|_{-1/2}^2 \leq \delta\|x\|_\rho^2 + e^{t/\delta}\|x\|_{(t)}^2.$$

Example 2. $H = L_2^{\#}(\Gamma)$ with $\Gamma = S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Then H is the space of $L_2^{\#}$-integrable periodic functions in R.

$$(Au)(x) := \oint k(x-y)u(y)dy \text{ and } D(A) = H$$

with

$$k(y) := -ln\left|2sin\frac{y}{2}\right|.$$

With the help of the Fourier coefficients v_i of a 2π-perodic function v defined by

$$v_\nu := \frac{1}{2\pi}\oint v(x)e^{-i\nu x}dx$$

we may introduce for real β the norms

$$\|v\|_\beta^2 := \sum|\nu|^{2\beta}v_i^2.$$

The Hilbert spaces $H_\beta = H_\beta((\Gamma)$ are defined similar to example 1. The Fourier coefficients of the convolution Au are

$$(Au)_\nu = \frac{1}{2|\nu|}\cdot u_\nu.$$

This time we have the inclusions $D(A) \subset H_A = H_{-1/2}(\Gamma)$.

Example 3. In the theory of conformal mappings the (circular Hilbert transform) integral operator

$$v = Au$$

$$v(s) := \frac{1}{2\pi}\oint \cot\left(\frac{1}{2}(s-t)u(t)dt\right.$$

plays a central role. It is skew symmetric in $L_2^{\#}(0{,}2\pi)$. Further A maps the space $\dot{L}_2^{\#}(0{,}2\pi) := L_2^{\#}(0{,}2\pi)/R$ i.e. (mean-value zero $L_2^{\#}(0{,}2\pi)$-function on Γ) isometrically onto itself, i.e. $\|Au\| = \|u\|$. The Hilbert inverse relations may be written in the form $A^2 = -I$.

The Hilbert space, (NaS),

$$H := \dot{H}_{1/2}^{\#}(S^1)/R$$

is the subspace of $L_2^{\#}(S^1)/R$ comprising real functions of mean-value zero on S^1 which have a half-order derivative also in $L_2^{\#}(S^1)$. The Hilbert space H is isometric to the sequences space

$$l_2^{1/2} := \left\{\text{complex sequences } \mathrm{u} = u_1, u_2, u_3, \ldots \middle| \sqrt{n}u_n \in l_2 \right\}.$$

The interconnection of the symplectic form

$$S(u,v) := \frac{1}{2\pi}\oint udv$$

between the inner product on H and the Hilbert transform operator A is given by

$$S(u,Av) = (u,v)_{1/2} \text{ for all } u,v \in H.$$

We note the relations to Plemelj's extension of the Green formulae, (PlJ). The extended formulae are accompanied by an extended Dirichlet integral $(\nabla u, \nabla v)_0 \cong (u,v)_1 \rightarrow (u,v)_{1/2}$, and a related new physical notion of a *„mass element*" and a *„flux through a surface*" avoiding the mathematical concept of the normal derivative.

We further mention the relationship to the 2-parameter wavelets [(*)], which can be interpreted as the appropriate extension of Fourier waves (which govern the „Dirichlet" inner product related Hilbert space H_1) governing the $H_1^\perp$ closed sub-space of $H_{1/2}$.

Example 4. Seeking the solution of the Neumann boundary value problem

$$\Delta u = 0 \quad \text{in } R^3 - \Omega$$

$$\frac{\partial u}{\partial n} = f \quad \text{on } S \coloneqq \partial\Omega.$$

leads to the Prandtl operator, (LiI),

$$(\prod v)(x) := \frac{1}{4\pi} \oiint_{\partial\Omega} v(y) \frac{\cos\phi_{xy}}{|x-y|^2} dS_y = f(x).$$

The Prandtl operator $\prod : H_{\mathrm{r}} \to H_{\mathrm{r}-1}$ is

i) is bounded for $0 \leq r \leq 1$

ii) is Noetherian for $0 < r < 1$.

The solution function of the Neumann problem is represented as double layer potential

$$u(x) := \frac{1}{4\pi} \oiint_{\partial\Omega} v(y) \frac{\cos\phi_{xy}}{|x-y|^2} dS_y$$

and the exterior Neumann problem admits one and only one generalized solution for $\frac{1}{2} \leq r < 1$.

Example 5. Let P denotes the orthogonal projection operator of $(L_2(\Omega))^3$ (PDO of degree zero) onto the divergence free vector field H_ω consisting of all solenoidal vector functions u, i.e. P denotes the orthogonal projection operator onto the kernel of the divergence operator.

The Stokes operator A is a selfadjoint (Friedrichs extension) operator in $H_A \coloneqq H_\sigma$ of the non-negative symmetric operator

$$A \coloneqq -P\Delta \text{ in } D(A) \coloneqq H_\omega \coloneqq \left\{u \in C^2 \middle| divu = 0, u_{n|\partial\Omega} = 0\right\},$$

and the corresponding Hilbert scale norms are defined by

$$\|u\|_\beta := \left\|A^{\beta/2}u\right\|.$$

Putting $B(u) := P(u, \nabla u)$ and assuming $Pu_0 = u_0$ the initial-boundary NSE equations can be representated by

$$\frac{du}{dt} + Au + Bu = Pf \text{ , } u(0) = u_0$$

with

$$Au = Pf \quad \text{in } H_0.$$

As u is divergence free and $u \cdot v$ identically vanishes on $\partial\Omega$ one gets

$$b(u,v,w) := ((u,\nabla)v, w) = \iint_\Omega (u,\nabla)v \cdot w dx = -b(u,w,v)$$

and especially $b(u,v,v) = 0$.

(*) **Wavelets in a nutshell** (from „A really friedly guide to wavelets")

„The problem here is that cutting the signal corresponds to a convolution between the signal and the cutting window. Since convolution in the time domain is identical to multiplication in the frequency domain and since the Fourier transform of a Dirac pulse contains all possible frequencies the frequency components of the signal will be smeared out all over the frequency axis. In fact this situation is the opposite of the standard Fourier transform since we now have time resolution but no frequency resolution whatsoever.

In signal processing terms Heisenberg's uncertainty principle, states that it is impossible to know the exact frequency and the exact time of occurrence of this frequency in a signal. In other words, a signal can simply not be represented as a point in the time-frequency space. The uncertainty principle shows that it is very important how one cuts the signal.

In wavelet analysis the use of a fully scalable modulated window solves the signal-cutting problem. The window is shifted along the signal and for every position the spectrum is calculated. Then this process is repeated many times with a slightly shorter (or longer) window for every new cycle. In the end the result will be a collection of time-frequency representations of the signal, all with different resolutions."

We note that wavelets must have a bandpass like spectrum and that the main function of a bandpass filter in a transmitter is to limit the bandwidth of the output signal to the band allocated for the transmission.

Classical mechanics is concerned with two systems,

1. particle systems, which describe the positions and velocities of a finite number of particles
2. field systems, which are described by one or multiple functions defined in the whole space, providing the force of the field at each point of the space.

For systems 1 the equations of motion are Ordinary Differential Equations (ODE), while for system 2 the equations of motion are Partial Differential Equations. We note that ODE and only parabolic / hyperbolic PDE are accompanied with the notion of *time*. In case of ODE and parabolic PDE this notion comes along with the concept of oa „*time arrow*“, while a hyperbolic PDE comes along with the notion of „*time reversibilty*“.

In quantum theory the counterpart of system 1 is the quantum mechanics; the counterpart of system 2 is the quantum field theory. Out of the multiple challenges in this context we recall

> from (FeE):
>
> „*Dirac's theory of radiation is based on a very simple idea; instead of considering an atom and the radiation field with which it interacts as two distinct systems, he treats them as a single system whose energy is the sum of three terms: one representing the energy of the atom, a second representating the electromagnetic energy of the radiation field, and a small term representing the coupling energy of the atom and the radiation field*.“
>
> from (EiA) p. 52 (translation by the author):
>
> *„However, the laws governing the currents and charges (*in the Maxwell equations*), are unknown to us. We know, that electricity exists within elementary particles (electrons, positive kernels), but we don't understand it from a theoretical perspective. We do not know the energetical factors, which determine the electricity in particles with given size and charge; and all attempts failed to complete the theory in this directions. Therefore, if at all we can built on the Maxwell equations, we know the energy tensor of electromagnetic fields only outside of the particles*“.

The essential new element of the proposed quantum field model is the concept of a „*quantum element*“, which carries both, kinematical and potential energy, in the following form

$$|||x|||_E^2 = |||x|||_{E_{kin}}^2 + |||x|||_{E_{pot}}^2 .$$

We mention that in physics the „*potential*“ is defined as the ratio of $\frac{E_{pot}}{c_{coupl}}$, where the coupling constant c_{coupl} depends from the considered physical problem.

In case two quantum elements have the same kinetimatical energy, but different potential energy, there is a potential energy difference, which interacts with the common kinematical part of the two quantum elements. In the proposed model the Hilbert space $H_{EP_{kin}}$ of „kinematical quantum elements“ (~ „fermions“) is compactly embedded into the overall Hilbert space of quantum elements H_{EP}, i.e., mathematically speaking, the cardinality of the space $H_{EP_{kin}}$ is identical with the cardinality of the rational numbers ($\aleph$), while the cardinality of the overall quantum element Hilbert space is identical with the cardinality of the real numbers ($2^{\aleph}$). This properties enable corresponding discrete and continuous spectra. In case of purely potential quantum elements this corresponds to the concept of wave packes governed by wavelets (DeL), (HoM), while in case of purely kinematical quantum elements this corresponds to the concept of Fourier waves.

The model replaces Dirac's model of the „*charge of a point particle*“. Mathematically speaking, it replaces the Dirac „function“, which is an element of the Hilbert space $H_{-n/2-\varepsilon}$ (n denotes the space dimension, and $\varepsilon > 0$), by „*quantum elements*“ of the smaller Hilbert space $H_{-1/2}$ (independently from the space dimension), where the complementary space $H_1^{\perp}$ of the split $H_{1/2} = H_1 \otimes H_1^{\perp}$ is governed by wavelets. For an approximation theory in Hilbert scales we refer to (NiJ), (NiJ1).

Mathematically speaking, the concept of a quantum element in the form $x = x_{Fermi} + x_{Boson}$, accompanied with a total average energy $|||x|||_E^2 = |||x_{Fermi}|||_{E_{kin}}^2 + |||x_{Boson}|||_{E_{pot}}^2$ requires a decomposition of the concerned Hilbert space H into an orthogonal sum of two spaces H^1 and H^2. This leads to the theory of indefinite inner product spaces which provides also appropriate definitions of the notions „*potential*“ and „(*quantum*) *potential operator*“ (appendix). In a Hilbert space $H = H^1 \otimes H^2$ the potential operator $\boldsymbol{W}(x) := \frac{1}{2}\mathrm{grad}\big((x)\big)^2 = P^1 x - P^2 x$ defines the inner product $(x,y)_W := (\boldsymbol{W}(x), y)$, (BoJ) p. 52. It enables e.g. the definition of a Fokker-Planck like operator in the form

$$\dot{u} = div W(u) + \beta^{-1} \Delta u = div[W(u) + \beta^{-1} \nabla u]$$

leading to a problem related inner product given by $(x,y)_W + \beta^{-1}(x,y)_{1/2}$ (see also (ArA)). The construction puts the spot on (ChJ), dealing with a characteristic feature of the Euler-Lagrange equation (as the difference of two potential operators), two concentration-compactness principles and the loss of mass at infinity for the critical one.

An accepted extended energy Hilbert space $H_{1/2}$ enables (weak variational) well-posed 3D non-linear, non-stationary Navier-Stokes equations (NSE) resp. supports the construction of counter examples that the classical NSE problem is not well-posed. As a consequence of the Sobolevskii estimate (SoP), the generalized 3D NSE initial value problem, $(v \in H_{-1/2})$,

$$(\dot{u}, v)_{-1/2} + (Au, v)_{-1/2} + (Bu, v)_{-1/2} = 0$$
$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2} \quad .$$

fulfills the energy inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le \left|(Bu, u)_{-1/2}\right| \le c \cdot \|u\|_{-1/2}\|u\|_1^2.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets

$$y'(t) \le c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$$

resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \le \|u_0\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\left\{\|u_0\|_{-1/2} + \|u_0\|_0^2\right\} \quad .$$

For the wavelet transform of white noise we refer to (HoM). In the context of the role of Fourier waves and wavelets in (Kolmogorow's) turbulence theory (e.g. for an analysis of turbulence flow or energy transfer of turbulence) we refer to (DeL). For the relationship of the Brownian motion and the Fokker-Planck equation we refer to (RiH) [(*)].

An accepted extended energy Hilbert space $H_{1/2}$ ensures a well defined Vlasov equation without the need of the (mathematical) Penrose conditions (i.e. the condition is without any physical meaning) governing the Coulomb (Newton) potential term. The Vlasov equation is the baseline equation for the theory of electromagnetic plasma turbulence, where particle collisions are ignored. Its non-linear term generates interactions between the quasi-linear waves of the quasi-stationary spectrum changing that spectrum (weak turbulance). Then this weak turbulence generates *virtual* waves, which emitter their energy to the electromagnetic particles like $\sim 1/\mathrm{t}$ (strong turbulence: the non-linear Landau damping phenomenon). In other words, the plasma is heated (turbulence heating) in a faster way as being possible possible by only electromagnetic particle interactions (CaF) 4.8. The proposed $H_1^\perp$ provides a model for the turbulence heating energy. Regarding the spectral theory for Fokker-Planck operators including the compactness of the resolvent we refer to (HeB).

The Maxwell Equations are not well posed in a mathematical sense (like all PDE w/o any boundary and/or initial value conditions). The extended energy Hilbert space support the building of a coercive bilinear form (CoM), which is a necessary condition to apply energy methods enabling variational calculus and corresponding approximation theory (VeW). Maxwell's equations hold only in regions with smooth parameter functions. If one considers a situation in which a surface S separates two homogeneous media from each other, the constitutive parameters are no longer continuous but piecewise continuous with finite jumps on S. These jumps imply that the field satisfy certain (transmission boundary) conditions on the surface, (KiA). The solution of the time-harmonic Maxwell equations in a vacuum is related to the fundamental solution of the Helmholtz equation at the origin. Physically the spherical wave fronts solution of the Helmholtz equation can be interpreted as the solution generated by a point charge source at the origin. Therefore, the radiation solutions of the Helmholtz equation can be characterized by the Sommerfeld radiation condition, showing the same (singularity) behavior at the origin as the Coulomb and the Newton potential. The extended energy Hilbert space for the Maxwell equations makes the Yang-Mills equations obsolet, i.e. the related Millennium problem becomes obsolet, as well.

We further mention that the model inherent concept of a „*potential barrior*" can be applied to the role of a chemical potential in the context of the theories of superconductivity, superfluids, and condensates (AnJ). The closed sub-space $H_1^\perp$ is supposed as alternative model of the density of supra-fluid electrons w/o any scattering effects. The BCS theory is about „*The Problem of the Molecular Theory of Superconductivity*". It leads to the London equation modelling the „*Meißner effect*" phenomenon. The concept of a „*potential barrior*" can also be applied in the context of the concept of „*transition temperature*". For the relationship to the Boltzmann collision operator with inverse-power intermolecular potentials we refer to (PaY).

The extended energy Hilbert space framework also allows to revisite the concept of „*reciprocal lattices*" in solid state physics, (KiC), accompanied with the standard energy Hilbert space H_1 governed by Fourier waves. The finer granularity of the Hilbert space $H_{1/2} = H_1 \otimes H_1^\perp$ compared to the standard „energy" Hilbert space H_1 provides the concept of a quantum potential as elements of the sub-space $H_1^\perp$ governed by wavelets.

[(*)] A noise force with a δ-correlation is called white noise, because the spectral density which is given by the Fourier transform is independent of the frequency ω. If the stochastic Langevin forces $\Gamma(t)$ are not δ correlated (i.e. the spectral density depends on the frequency ω) one uses the term colored noise. The δ appears because otherwise the average enrgy of a small particle cannot be finite as it should be according to the equipartition law $\frac{1}{2}m\langle v^2\rangle = \frac{1}{2}kT$. If one multiplies two Langevin forces at different times one assumes that the average value is zero for time differences $t' - t$ which are larger than the duration time τ_0 of a collision.

Referring to example 3 we recall that the interconnection of the symplectic form $S(u,v) := \frac{1}{2\pi}\oint u dv$ between the inner product on $H := \dot{H}^{\#}_{1/2}(S^1)/R$ and the Hilbert transform operator A , (**), given by

$$S(u, Av) = (u,v)_{1/2} \text{ for all } u,v \in H,$$

is linked to Plemelj's concepts of a „*mass element*" on a surface and a „*flux/current*" leading to a corresponding extension of the Green formulae. Therefore, the proposed model allows to revist the „*two containers model*" paradoxes of kinetic theory regarding the Second Law of Thermodynamics (**):

From (KaM) we quote:

„The laws of mechanics are time reversible, i.e. invariant under the change of t into $-t$. The Second Law of Thermodynamics postulates a typical time irreversible behavior. It thus seems impossible to ever derive the Second Law from purely mechanistic considerations. … Zermelo invoked a simple but fundamental theorem of Poincaré to the effect that a conservative dynamical system, satisfying certain mild conditions, has the property that „*almost every*" initial state of the system is bound to recur, to any degree of accurary. This too is in contradiction with irreversible behavior. …

To appreciate these paradoxes consider two containers, one containing a gas and the other completely evacuated. At some time we connect the containers. The second Law predicts then that the gas will flow from the first container into the second and that the amount of gas in the first container will decrease monotonically in time. Such behavior of the gas shows definite arrow of time.

From the kinetic (mechanical) point of view we are dealing with a dynamical system which cannot possibly show the time arrow and which moreover will behave in a quasi-periodic way as implied by Poincaré's theorem. For a conservative dynamical system the Hamiltonian function describes the total energy and the equations of motion with known initial positions and momenta determine a unique motion solution for $t \geq 0$. At the time t the dynamical system is representated by the point

$$P_t = (q_1(t), \dots, q_n(t), p_1(t), \dots, p_n(t))$$

Now, the motion of a dynamical system defines a one-parameter family of transformations T_t by the relation $T_t(P_0) = P_t$.

Suppose now that we have a set A of points P_0, and denote by $T_t(A)$ the set of corresponding points P_t. Then the Liouville theorem states that the transformations are measure preserving, the measure being the ordinary Lebesgue measure in Γ-space.

From the conservation of law it follows that the points representing the dynamical system lie on an „energy surface" Ω

$$H(q_1, \dots, q_n, p_1, \dots, p_n) = c.$$

Let us assume that the energy surface Ω is compact and sufficiently „regular" so that the elementary theory of Lebesgue measure based surface integration is applicable and assume also that on Ω

$$\|\nabla H\|^2 = \sum_{i=1}^{n} (\frac{\partial H}{\partial p_i})^2 + (\frac{\partial H}{\partial q_i})^2 > c > 0.$$

Let B be a subset of Ω on the surface such that

$$\int_B \frac{d\sigma}{\|\nabla H\|} < \infty$$

where $d\sigma$ is the surface element. We define the measure $\mu\{B\}$ of B by the formula

$$(*) \qquad \mu\{B\} = \frac{\int_B \frac{d\sigma}{\|\nabla H\|}}{\int_\Omega \frac{d\sigma}{\|\nabla H\|}}$$

so that $\mu\{\Omega\} = 1$. It now follows from Liouville's theorem, by simple geometric consideration, that

$$\mu\{T_t(B)\} = \mu\{B\}.$$

In other words, T_t preserves the measue μ on the „energy surface" Ω. The formula (*) assigns measures only to certain elementary sets to which the elementary theory of surface integration is applicable (especially it requires the concept of „normal derivative" coming along with the Green formula).

Regarding the above two container model let's asumme we know the precise function form of the Hamiltonian $\mathrm{H}(q_1, \dots, q_n, p_1, \dots, p_n) = c$ and its value C at $t = 0$. There is clearly a set B of points of Ω corresponding to the condition that at $t = 0$ all the particles are in one of the two containers, and we know that our system starts from the set B.

The first assertion of Boltzmann was that the μ-measure $\mu\{B\}$ of B is „*extremely*" small, corresponding to our intuition that we are starting from a highly unusual or rare state. On the other hand the set R of points of Ω, corresponding to states in which the number of particles in the two containers are „very nearly" proportial to the volumes of the two containers, is such that $\mu\{R\}$ is „extremely" close to 1.

Of course this statements depend to a large extent on the meanings of „*extremely*" and „*very nearly*", but suffice it to say that because of the enormity of the number of atoms per cubic centimeter (of the order of 10^{26}) it is quite safe to intepret „*extremely*" as being less than 10^{-10} and „*very nearly*" as being within 10^{-10} of the proper ratio. Boltzmann's second assertion was that the first assertion implies that the relative times which the actual curve describing the motion of the system spends in B and R are respectively „*extremely*" small and „*extremly*" large. … to justify the second assertion he introduced the „*quasi-ergodic hypothesis*", postulating that the curve of motion passes arbitrarily close to every point on the energy surface. This hypothesis came out to be not sufficient to establish a connection between the relative time spent in a sub-set A of Ω and its μ-measure, $\mu\{A\}$.

Let denote the time the curve of motion starting from P_0 spends in A up to time τ. The relative time is then the limit

$$\lim_{\tau \to \infty} \frac{t(\tau, P_0, A)}{\tau}$$

if, of course, it exists. It turns out that the proof of existence of this limit constitutes the real difficulty. Once this is done one needs only an additional assumption of T_t to conclude that the limit is equal to $\mu\{A\}$, which is the famous Birkhoff theorem. A little bit earlier J. v. Neumann proved that the limit

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau g((T_t(P_0))dt$$

exist in the sense of mean square, (HaP)."

For the statistical (time-mean and ergodic hypothesis) and individual (convergence almost everywhere) ergodic theory we refer to (HoE).

(*) We note that the Riesz transforms are the generalization of the Hilbert transform for space dimensions $n > 1$

(**) With respect to part A we note the link of the related Birkhoff ergodic theorem to the theory of continued fractions, (KaM).

The claim is, that an extended special relativity theory (*), based on the *complex* Lorentz group $L(C)$ (the group of complex 4x4 matrices with $\det(\Lambda) = 1$ preserving the metric $g = \Lambda^T g\Lambda$)) in the proposed extended Hilbert space framework enables a geometric (Hilbert space based) gravity theory in line with the proposed quantum field model, whereby the Mach principle governs the (classical) gravitational „*forces*" of the universe down to the kinematical „fermions" level:

The fundamental principle of the SRT is the (Maxwell equations based) invariance principle building on the Lorentz transformation.

> (StR): „*The corresponding Lorentz group L has four disconnected components, where each of which is connected in the sense that any one point can be connected to any other, but no Lorentz transformation in one component can be connected to another in another component. This results to three subgroups of L, which are the orthochronous Lorentz group, the proper Lorentz group, and the orthochorous Lorentz group. Associated with the restricted Lorentz group is the group of 2x2 complex matrices of determinant one ($SL(2, C)$).*"

The alignment of the SRT with the proposed quantum field model is enabled by the *complex* Lorentz group $L(C)$, which is also essential in the proof of the PCT theorem. The central differentiator to the Lorentz group is the fact, that $L(C)$ has the (only two) *connected* components $L_{+/-}(C)$, where $L_{+}(C)$ denotes the proper *complex* Lorentz group (*).

> (StR): *„For a general analysis of relativistic invariance it is reasonable that any relativistically invariant theory in which the states are spanned by the collision states of the elementary particles of the theory has, in a suitable basis, an essentially uniquely determined relativistic transformation law. This transformation law is identical to that of a theory of non-interacting elementary particles of the same masses and spins. Any relativistic theory of particles which does not have this transformation law will, in our opinion, require a novel physical interpretation. (as usual, in making this statement we are ignoring the special difficulties associated with zero mass particles.)"*

The proposed decomposition $H_{-1/2} = H_0 \otimes H_0^{\perp}$ addresses the „zero mass problem" of the above sketched relativistically invariant theory concept, while the statistical Hilbert subspace L_2 supports the corresponding properties of the vacuum expectation values expressed by the quantities

$$(*) \quad (\Psi_0, \varphi_1(x_1)\varphi_2(x_2) \dots \varphi_n(x_n)\Psi_0)$$

where φ_j, $j = 1, \dots n$ is a component of an irreducible tensor, (StR) p. 106. Two of the φ_j might be components of different fields or could be the same component of the same field or could be hermitian conjugate to each other. In other words, all such quantities are considered, for all components of the arbitrary labels, and all permutations of the indices.

Based on the three axioms,

(I) assumptions about the domain & continuity of the field

(II) transformation law of the field

(III) local cummutativity

several laws can be derived as mathematical theorems concerning

- a relativistic transformation law
- spectral, hermiticity, and local commutativity conditions
- positive definiteness conditions
- a cluster decomposition property
- holomorphic functions expressions of (**), also considering permutations in the form

$$\varphi_1(x_1)\varphi_2(x_2) \dots \varphi_n(x_n) \to \varphi_{i_1}(x_{i_1})\varphi_{i_2}(x_2) \dots \varphi_{i_n}(x_{i_n}).$$

We note that the existence of a PCT operator for a set of fields is equivalent to the validity of the identities

$$(\Psi_0, \varphi_1(x_1)\varphi_2(x_2) \dots \varphi_n(x_n)\Psi_0) = (-1)^J i^F (\Psi_0, \varphi_1(-x_n)\varphi_2(-x_{n-1}) \dots \varphi_n(-x_1)\Psi_0)$$

where F is the number of half-odd intger spin fields and J is the total number of undotted indices (the dot over the index simply means that this index transforms according to $\bar{A}$ instead of A, (StR) p. 15).

(*) Einstein's Special Relativity Theory (SRT) requires that the form of every physical law is covariant and that the speed of light in a „vacuum" is a universal constant independent of the motion of the source (of photons).

(**) This puts the spot on the three dimensional unit sphere as a model for the mathematical universe (UnA1) accompanied by Hamilton's quaternions algebra $|\mathbf{H}$ (and its underlying 3-D imaginary sub-space Im($|\mathbf{H}$)), which is isomorphic to a 4-D associative divisonal R-sub-algebra of mat(2,C), (EbH).

In order to motivate the superstring theory in (KaM) it is pointed out, that because general relativity and quantum mechanics can be derived from a small set of postulates, and the theories are incompatible one or more of these postulates must be wrong:

> (KaM): „*Gravity research was totally uncoupled from research in other (weak, strong elementary particles) interactions. Classical relativists continued to find more and more classical solutions in isolation from particle research. Attempts to cannonically quantize the theory were frustrated by the presence of the tremendous redundancy of the theory. There was also the discouraging realization that even if the theory could be successfully quanized, it would still be nonrenormalizable.*
>
> *… General relativity is (also) plagued with similar difficulties when pushed to ist limits:*
>
> *(1) … Einstein's equations necessarily exhitbits pointlike singularities, where we expect the laws of general relativity to collapse. Quantum corrections must dominate over the classical theory in this domain*
> *(2) The action is not bounded from below, it is linear in the curvature tensor. Thus, it may not be stable quantum mechanically*
> *(3) Genereal relativity is not renormalizable. Computer calculations, for example, have now conclusively shown that there is a nonzero counterterm in Einstein's theory at the two-loop level.*
>
> *Physicists have concluded that perhaps one or more of our cherished assumptions about our Universe must be abandoned. Because general relativity and quantum mechanics can be derived from a small set of postulates, one or more of these postulates must be wrong. The key must be to drop one of our commonsense assumptions about Nature on which we have constructed general relativity and quantum mechanics. Over the years, several proposals have been made to drop some of our commonsense notions about the Universe (for more detail we refer to the appendix):*
>
> *(1) Continuity*
> *(2) Causality*
> *(3) Unitary*
> *(4) Locality*
> *(5) Point Particle.*
>
> *The super string theory, because it abandons only the assumption that the fundamental constituents of matter must be point particles, does the least amount of damage to cherished physical principles and continues the tradition of increasing the complexity and sophistication of the gauge group. Superstring theory does not violate any of the laws of quantum mechanics, yet manages to eliminate most, if not all, of divergences of the Feynman diagrams*."

ALL physical laws are built on the mathematical concepts of ODE or PDE, built on the mathematical concept of **point-individuals** (the elements of the straight line) completing in a certain sense the domain of **number-individuals** given by the field of rational numbers ((DeR1) and appendix).

> (DeR1): „*Of the greatest importance, however, is the fact that in the straight line L there are infinitely many points which correspond to no rational number. …. The straight line L is infinitely richer in **point-individuals** than the domain Q of rational numbers in **number-individuals***."

In other words, the „commonsense" assumption (5), i.e. the concept and „mathematical existence" of a point particle, is the fundamental axiom of ALL mathematical arithmetic and analysis [(*)]. This means, the „*most least amount of damage*" from a physical principle (i.e. dropping (5)) finally damages the whole mathematical physics and statistics (including thermostatistics and the statistics of flows (ergodic theory), except the notion „*Point Particle*" is interpreted as a hyper-real number. However, in this case the physical interpretation of „string" interaction (~ „vibration / energy" „infinitesimal contact" between „strings") is already incorporated into that „Ideal Point Particle" definition (or two physical strings become two connecting/interacting straight lines). Hyper-real numbers are extensions of the real numbers, i.e. the field of real numbers is extended to still an ordered, but Non-Archimedian field $^*R$ (with same cardinality building the baseline of Non-Standard Analysis), which could be interpreted as the mathematical model of Leibniz' monads (in line with the proposed $H_1^\perp$ model).

The proposed extended SRT does not require assumption (1), and the tensor calculus with its underlying concept of an „exterior product". The essential new concepts are

1. a coarse-grained standard (physical measurabled) energy Hilbert space H_1 as a sub-space of the extended (not measureable mathematical continuum) „ether" energy Hilbert space $H_{1/2} = H_1 \otimes H_1^\perp$ (with its sub-space $H_{\mathrm{n}/2+\varepsilon}$ being compactly embedded into C^0, (Sobolev embedding theorem), [(**)]
2. Einstein's lost key concerning the concept of *„a variable speed of light",* (UnA)
3. the complex Lorentz group (providing the key tool to prove the PCT theorem, (StR), and „*a larger and more elegant group*" than used in gauge theory, (appendix, (KaM) 1.2 (1)).

[(*)] There is a game theory based approach to build irrational numbers, which is conceptually a generalization of the Dedekind cuts; it is based on the definition of so-called Conway games, (CoJ), (EbH).

[(**)] (DeR1): „*If space has at all a real existence it is not necessary for it to be continuous; many of its properties would remain the same even were discontinuous. And if we knew for certain that space was discontinuous there would be nothing to prevent us, in case we so desire, from filling up ist gaps, in thought, and thus making it continuous; this filling up would consist in a creation of new point-individuals and would have to be effected in accordance with the above principle*."

Appendix

Extract (VaM) chapter IV

A decomposition of a Hilbert space H into an orthonal sum of two spaces H^1 and H^2 with corresponding projection operators P^1 and P^2 enables a definition of a „potential“ and a related „potential operator“:

Let P^1 and P^2 denote the orthogonal projections from a Hilbert space H decomposed into an orthogonal sum of two spaces $H = H^1 \otimes H^2$. Putting

$$\varphi(x) := \big((x)\big)^2 := \|P^1x\|^2 - \|P^2x\|^2$$

then the manifold

$$\big((x)\big) = c > 0$$

represents a hyperboloid in the Hilbert space H. Since $\|x\|^2 = \|P^1x\|^2 + \|P^2x\|^2$ we have

$$\|P^1x\|^2 = \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\big((x)\big),\ \|P^2x\|^2 = \tfrac{1}{2}\|x\|^2 - \tfrac{1}{2}\big((x)\big).$$

The gradient of the potential $\varphi(x) := \big((x)\big)^2$, which is an indefinite metric, is given by

$$grad\varphi(x) = \mathrm{grad}\big((x)\big)^2 = 2P^1x - 2P^2x.$$

The related potential operator is given by

$$\boldsymbol{W}(x) := \tfrac{1}{2}\mathrm{grad}\big((x)\big)^2 = P^1x - P^2x.$$

The fundamental properties of the potential operator $\boldsymbol{W}(x)$ are (1) completely (2) invertible, ($\boldsymbol{W} = \boldsymbol{W}^{-1}$), (3) symmetric, (4) isometric, and (5) the bilinear form $(x,y)_W := (\boldsymbol{W}(x), y)$ defines an inner product, (BoJ) p. 52.

We will consider the hyperbolic region V_c, whose points satisfy the condition

$$\big((x)\big) = \sqrt{\|P^1x\|^2 - \|P^2x\|^2} \geq c > 0,$$

and the conical region V_0:

$$\big((x)\big) = \sqrt{\|P^1x\|^2 - \|P^2x\|^2} \geq 0.$$

In other words, the potential criterion $\varphi(x) = c > 0$ defines a manifold, which represents a hyperboloid in the Hilbert space H with corresponding hyperbolic and conical regions.

Evidently V_c is a subspace of V_0. We remark that if x is an exterior point of the conical region V_0, i.e.

$$\sqrt{\|P^1x\|^2 - \|P^2x\|^2} = \alpha > 0,$$

then those points of the ray $tx, t \in [0,\infty)$ for which $t \geq c/a$ belong to the hyperbolic region V_c, and those for which $0 \leq t < c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t \in [0,\infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $\big((x)\big) = c > 0$ in a single point. We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $\big((x)\big) = 0$. If we look at the unit sphere S^1 ($\|x\|^2 = 1$), then those points of S^1 for which $\|P^1x\| = \|P^2x\|$ belong to K, and those points of S^1 for which $\|P^1x\| > \|P^2x\|$ intersect the hyperboloid $\big((x)\big) = c > 0$ at the point whose distance from θ is given by

$$t = \mathrm{c}\sqrt{\|P^1x\|^2 - \|P^2x\|^2}\,.$$

From this it is seen that $t \to \infty$ if $\|P^1x\|^2 - \|P^2x\|^2 \to 0$, i.e. the manifold K is an asymptotic conical manifold for the hyperboloid $\big((x)\big) = c > 0$.

Extract from

M. Kaku, Introduction to Superstrings and M-Theory, (KaM)

1.2 Historical Review of Gauge Theory

(KaM): „Over the years, several proposals have been made to drop some of our commonsense about the Universe:

(1) Continuity

This approach assumes that space-time must be granular. The size of these grains would provide a natural cutoff for the Feynman integrals, allowing us to have a finite S-matrix. Integrals like

$$\int_{\varepsilon}^{\infty} d^4x$$

would then diverge as ε^{-n}, but we would never take the limit as ε goes to zero. Lattice gravity theories are of this type. In Regge calculus, for example, we latticize Riemannian space with discrete four-simplexes and replace the curvature tensor by the angular deficit calculated when moving in a circle around the simplex:

$$-\frac{1}{2\mu^2}\sqrt{-g}R \rightarrow angular\ deficit.$$

At present, however, there is no experimental evidence to support the idea that space-time is granular. Although we can never rule out this approach, it seems to run counter to the natural progression of particle physics, which has been to postulate larger and more elegant groups“

(2) Causality

This approach allows small violations in causality. Theories that incorporate the Lee-Wick mechanism are actually renormalizable, but permit small deviations from causality. These theories make the Feynman diagrams converge by adding a fictitious Pauli-Villars field of a mass M that changes the ultraviolet behavior of the propagator. … This means that the theory will be riddled with negative probabilities. …. that is, you can meet your parents before you are born

(3) Unitarity

We can replace Einstein's theory, which is based on the curvature tensor, with a conformal theory based on the Weyl tensor by conformal tensor based on the Weyl tensor:

$$\sqrt{-g}R_{ij}g^{ij} \rightarrow \sqrt{-g}C^2_{ij\rho\sigma}.$$

…. the conformal tensor posesses a larger symmetry group than the curvature tensor, that is invariant under local conformal transformations. … The Weyl theory is a higher derivative theory. …. The most optimistic scenario would be to have these unitary ghosts „confined“ by a mechanism similar to quark confinement.

(4) Locality

Over the years, there have also been proposals to abandon some of the important postulates of quantum mechanics, such that locality. After all, there is no guarantee that the laws of quantum mechanics should hold down to distances of 10^{-33} cm. However, there have always been problems whenever physicists tried to deviate from the laws of quantum mechanics, such that causality. At present, there is no successful alternative to quantum mechanics.

(5) Point Particles

Finally, there is the approach of superstrings, which abandons the concept of idealized point particles. … „

Extract from

H. Weyl, Philosophy of Mathematics and Natural Science (WeH3)

The Physical Picture of the World
B. Matter and Fields. Ether

p. 171: „Just as the velocity of a water wave is not a substantial but a phase velocity, so the velocity with which an electron moves is only the velocity of an ideal „center of energy", constructed out of the field distribution. According to this view, there exists but one kind of natural laws, namely, field laws of the same transparent nature as Maxwell had established for the electromagnetic field. The obscure problem of laws of interaction between matter and field does not arise. This conception of the world can hardly be described as dynamical any more, since the field is neither generated nor acting upon an agent separate from the field, but following its own laws is in a quiet continuous flow. It is of the essence of the continuum. Even the atomic nuclei and the electrons are not ultimate unchangeable elements that are pushed back and forth by natural forces acting upon them, but they are themselves spread out continuously and are subject to fine fluent changes.

On the basis of rather convincing general considerations, G. Mie in 1912 pointed out a way of modifying the Maxwell equations in such manner that they might possibly solve the problem of matter, by explaining why the field possesses a „granular" structure and why the knots of energy remain intact in spite of the back and forth flux of energy and momentum. The Maxwell equations will not do because they imply that the negative charges compressed in an electron explode; to guarantee their coherence in spite of Coulomb's repulsive forces was the only service still required of substance by H. A. Lorentz's theory of electrons. The preservation of the energy knots must result from the fact that the modified field laws admit only of one state of field equilibrium – or of a few between which there is no continuous transition (static, spherically symmetry solutions of the field equations). The field laws should thus permit us to compute in advance charge and mass of the electron and the atomic weights of the various chemical elements in existence. And the same fact, rather than the contrast of substance and field, would be the reason why we may decompose the energy or inert mass of a compond body (approximately) into the non-resolvable energy or its last elementary constituents and the resolvable energy of their mutual bond."

Extract from

H. Weyl, Philosophy of Mathematics and Natural Science (WeH3)

Appendix C:
Quantum Physics and Causality

p. 258: „Quantum physics does not force a discontinuous time upon us even if the number of quantum states separable by a grating is universally limited. During the infinitesimal time interval dt the vector space experiences a certain infinitesimal rotation imparting the increment $\overrightarrow{dx} = L\vec{x} \cdot dt$ to the arbitrary vector $\vec{x}$. This dynamical law $\overrightarrow{dx}/dt = L\vec{x}$ (in which the operation L is independent of t and $\vec{x}$) is expressed in terms of Cartesian coordinates x_i by equations of the form

$$\frac{dx_i}{dt} = \sum_J l_{ij}\, x_j(t),\ i,j = 1, \dots, n$$

with given constant antisymmetric coefficients l_{ij} ($l_{ij} = -l_{ij}$). The salient point is that the wave state $\vec{x}$ varies according to a strict causal law; its mathematical simplicity is gratifying. A grating $G = \{E_1, \dots E_r\}$ and the corresponding quantum states $(G;1), \dots (G;r)$, are stationary if the subspaces $\boldsymbol{E}_i$ are invariant in time, i.e. if the linear operators E_i commute with the linear operator L.

*p. 259: „A system is never completely isolated from its surroundings, and its wave state is therefore subject to perpetual disturbances. This is the reason why the **secondary statistics of thermodynamics** is to be superimposed upon the **primary statistics** dealing with a given **wave state and its reaction to a grating**."*

p. 260: „The description here given must be corrected throughout in one point: the coordinates x_j in the underlying n-dimensional vector space are not real but arbitrary complex numbers and as such have an absolute value $|\vec{x}|$ and a phase. The square of the length of the vector is expressed in terms of the absolute values of the coordinates. The simplest of all dynamical laws $\overrightarrow{dx}/dt = L\vec{x}$ in such a complex space is of the form

$$(*)\quad \overrightarrow{dx}/dt = i \cdot \nu \cdot \vec{x}.$$

Here ν is a real constant. The wave state $\vec{x}$ then carries out a simple oscillation of frequency

$$\vec{x} = \overrightarrow{x_0}\{\cos(\nu t) + i sin(\nu t)\}, \quad \overrightarrow{x_0} = const.,$$

and hence the energy has the definite constant value νh (Planck's law). But whatever the dynamical law $\overrightarrow{dx}/dt = L\vec{x}$, the space can always be broken up into a number of mutually orthogonal subspaces $\boldsymbol{E}_j$ ($j = 1, \dots, r$) such that an equation (*) with a definite frequency $\nu = \nu_j$ holds in $\boldsymbol{E}_j$. The grating $G = \{E_1, \dots E_r\}$ thus obtained is stationary and effects a sifting with respect to different frequencies ν_j and corresponding energy levels $U_j = h\,\nu_j$. Thermodynamics is based on this G. Any vector $\vec{x}$ in $\boldsymbol{E}_j$ satisfies the equation $L\vec{x} = i\nu\vec{x}$ ($\nu = \nu_j$), and this fact is expressed in mathematical language by saying that $\vec{x}$ is an eigenvector of the operation L with the eigenvalue $i\nu$. The operator $H = \frac{h}{i}L$, called energy, has the same eigenvectors, but the corresponding eigenvalues are the energy levels $= h\nu$. The general equation $\overrightarrow{dx}/dt = L\vec{x}$ now reads

$$\frac{h}{i}\frac{\overrightarrow{dx}}{dt} = H\vec{x} \quad \text{(Schrödinger's equation)."}$$

Extract from

H. Weyl, Philosophy of Mathematics and Natural Science (WeH3)

Appendix E:
Physics and Biology

topic „virus“

p. 276: „Incidentally, the gap between organic and inorganic matter has been bridged to a certain extent by the discovery of viruses. Viruses are submicroscopic entities that behave like dead inert matter unless placed in certain living cells. As parasites in these cells, however, they show the fundamental chracteristics of life – self-duplication and mutation. On the other hand many viruses have the structure typical of inorganic matter; they are crystals. In size they range from the more complex protein molecules tot he smaller bacteria. Chemically they consist of nucleo-protein, as the genus do. A virus is clearly something like a naked gene. The best studied virus, that of tobacco mosaic disease, is a nucleo-protein of high molecular weight consisting of 95 per cent protein and 5 per cent nucleic acid; it cristallizes in long thin needles.“

p. 277: „The specific properties of living matter will have to be studied within the general laws valid for all matter; the viewpoint of holism that the theory of life comes first and that one descends from there sort of deprivation to inorganic matter must be rejected. It is therefore significant that certein simple and clearcut traits of wholeness, organization, acausality, are ascribed by quantum mechanics to the elementary constituents of all matter.“

p. 277: „The quantum physics of atomic processes will become relevant for biology wherever in the life cycle of an organism a moderate number of atoms exercises a steering effect upon the large scale happenings. …. On a broad empirical foundation, genetics furnishes the most convincing proof that organisms are controlled by processes of atomic range, where the acausality of quantum mechanics may make itself felt. … The mere fact of such X-rays induced mutations proves that the genes are physical structures.“

p. 278: „By ingenious methods H. J. Muller, N. W. Timoféeff-Ressowsky, and others have succeeded in establishing simple quantitive laws concerning the rate of induced mutations. These results indicate that the mutation is brought about by a single hit, not by the concerted action of several hits, and that this hit consists of an ionization, and is not, as one might have thought, a process directly released by the X-ray photon or absorbing the whole energy of the secondary electron.“

These facts suggest the hypothesis that a gene is a (nucleo-protein) molecule of highly complicated structure, that a mutation consists in a chemical change of this molecule brought about by the effect of an ionization on the bonding electrons, and that thus allele genes are essentially isometric molecules.“

The observed absolute rate of mutations would be explained if a specifc mutation requires that a hit occurs within a critical volume (‚target‘) in the gene, the magnitude of which amounts to about 5-10 A cube (5-10 atomic distances cube). The physicist finds it, if not plausible at least acceptable, that a quantum jump at a specific point requiring an activation energy of about 1.5 is relased by a hit of 30 electron volts within a sensitive volume of 5-10 A cube. The observed thermic variation of the spontaneous mutation rate (van't Hoff's factor) is in good quantitative agreement with the picture.“

Extract from

R. Dedekind, Continuity and Irrational Numbers, (DeR1)

§3 Continuity of the straight line

„Of the greatest importance, however, is the fact that in the straight line L there are infinitely many points which correspond ton o rational number. …. The straight line L is infinitely richer in ***point-individuals*** *than the domain Q of rational numbers in* ***number-individuals****.*

If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary that the instrument Q, constructed by the creation of rational numbers be essentially improved by the creation of new numbers such that the domain of numbers shall gain the same completeness, or as we may say at once, the same ***continuity****, as the straight line.*

… For the way in which the irrational numbers are usually introduced is based directly upon the conception of extensive magnitudes – which itself is nowhere carefully defined – and explains number as the result of measuring such magnitude by another of the same kind. Instead of this I demand that arithmetic shall be developed out of itelf. … Just as negative and fractional rational numbers are formed by a new creation, and as the laws of operating with these numbers must and can be reduced tot he laws of operating with positive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone. The question only remains how to do this.

I find the essence of continuity in the following principle:

> *„If all points of the straight line fall into two classes such than every point of the first class lies to the left of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.“*

As already said I think I shall not err in assuming that every one will at once grant the truth of this statement; the majority of my readers will be very much disappointed in learning that this common-place remark the secret of continuity is to be revealed. To this I may say that I am glad if every one finds the above principle so obvious and so harmony with his his own idea of a line; for I utterly unable to adduce any proof of its correctness, nor has anyone the power. The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we find continuity in the line. If space has at all a real existence it is ***not*** *necessary for it to be continuous; many of its properties would remain the same even were discontinuous. And if we knew for certain that space was discontinuous there would be nothing to prevent us, in case we so desire, from filling up ist gaps, in thought, and thus making it continuous; this filling up would consist in a creation of new point-individuals and would have to be effected in accordance with the above principle.“*

Extract from

Ebbinghaus H.-D. et al, Numbers (EbH)

Chapter 11, §2

The dimension of a divisional algebra and the sphere S^{n-1} is only parallelizable for $n = 1, 2, 4, 8$

Theorem:

If the mod-2 invariant of a continuous mapping $f: S^{n-1} \to GL(n)$ (GL(n) the topology group of nxn invertible matrices) is different from zero, than it is $n = 1,2,4,8$.

Theorem:

The sphere S^{n-1} is only parallelizable for dimensions $n = 1,2,4,8$; actually one should exclude the case $n = 1$ case, because it eventually leads to trivial additional considerations.

We note that $U(n)$ (the group of $n \times n$ unitary matrices, which is a sub-group of the general linear group $Gl(n, C)$) is the semi-direct product of $U(1)$ with $SU(n)$, and $U(1) \cong S^1$ and $SU(2) \cong S^3$.

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