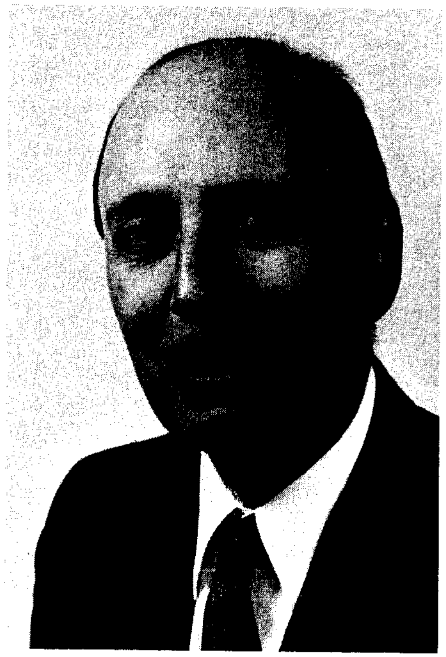

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SYMMETRIES AND REFLECTIONS

Scientific Essays of Eugene P. Wigner

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2

Symmetry and Conservation Laws

Introduction

Symmetry and invariance considerations, and even conservation laws, undoubtedly played an important role in the thinking of the early physicists, such as Galileo and Newton, and probably even before them. However, these considerations were not thought to be particularly important and were articulated only rarely. Newton's equations were not formulated in any special coordinate system and thus left all directions and all points in space equivalent. They were invariant under rotations and displacements, as we now say. The same applies to his gravitational law. There was little point in emphasizing this fact, and in conjuring up the possibility of laws of nature which show a lower symmetry. As to the conservation laws, the energy law was useful and was instinctively recognized in mechanics even before Galileo.[1] The momentum and angular momentum conservation theorems in their full generality were not very useful even though in the special case of central motion they give, of course, one of Kepler's laws. Most books on mechanics, written around the turn of the century and even later, do not mention the general theorem of the conservation of angular momentum.[2] It must have

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[1] G. Hamel, in his *Theoretische Mechanik* (Stuttgart: B. G. Teubner, 1912) mentions (p. 130) Jordanus de Nemore ($\sim$1300) as having recognized essential features of what we now call mechanical energy and Leonardo da Vinci as having postulated the impossibility of the Perpetuum Mobile.

[2] F. Cajori's *History of Physics* (New York: Macmillan Company, 1929) gives exactly half a line to it (p. 108).

been known quite generally because those dealing with the three-body problem, where it is useful, write it down as a matter of course. However, people did not pay very much attention to it.

This situation changed radically, as far as the invariance of the equations is concerned, principally as a result of Einstein's theories. Einstein articulated the postulates about the symmetry of space, that is, the equivalence of directions and of different points of space, eloquently.[3] He also re-established, in a modified form, the equivalence of coordinate systems in motion and at rest. As far as the conservation laws are concerned, their significance became evident when, as a result of the interest in Bohr's atomic model, the angular momentum conservation theorem became all-important. Having lived in those days, I know that there was universal confidence in that law as well as in the other conservation laws. There was much reason for this confidence because Hamel, as early as 1904, established the connection between the conservation laws and the fundamental symmetries of space and time.[4] Although his pioneering work remained practically unknown, at least among physicists, the confidence in the conservation laws was as strong as if it had been known as a matter of course to all. This is yet another example of the greater strength of the physicist's intuition than of his knowledge.

Since the turn of the century, our attitude toward symmetries and conservation laws has turned nearly full circle. Few articles are written nowadays on basic questions of physics which do not refer to invariance postulates, and the connection between conservation laws and invariance principles has been accepted, perhaps too generally.[5] In addition, the concept of symmetry and invariance has been extended into a new area—an area where its roots are much less close to direct experience and observation than in the classical area of space-time symmetry. It may be useful, therefore, to discuss first the relations of phenomena, laws of nature, and invariance principles to each other. This relation is not quite the same for the classical invariance principles, which will be called geometrical, and the new ones, which will be called dynamical.

[3] See, for instance, his semipopular booklet *Relativitätstheorie* (Braunschweig: Friedr. Vieweg und Sohn, various editions, 1916-1956).

[4] G. Hamel, *Z. Math. Phys.*, 50, 1 (1904); F. Engel, *Ges. d. Wiss. Göttingen*, 270 (1916).

[5] See the present writer's article, *Progr. Theoret. Phys.*, 11, 437 (1954); also Y. Murai, *Progr. Theoret. Phys.*, 11, 441 (1954); and more recently D. M. Greenberg, *Ann. Phys.* (N.Y.), 25, 290 (1963).

Finally, I would like to review, from a more elementary point of view than customary, the relation between conservation laws and invariance principles.

Events, Laws of Nature, Invariance Principles

The problem of the relation of these concepts is not new; it has occupied people for a long time, first almost subconsciously. It may be of interest to review it in the light of our greater experience and, we hope, more mature understanding.

From a very abstract point of view, there is a great similarity between the relation of the laws of nature to the events on one hand, and the relation of symmetry principles to the laws of nature on the other. Let me begin with the former relation, that of the laws of nature to the events.

If we knew what the position of a planet will be at any given time, there would remain nothing for the laws of physics to tell us about the motion of that planet. This is true also more generally: if we had a complete knowledge of all events in the world, everywhere and at all times, there would be no use for the laws of physics, or, in fact, of any other science. I am making the rather obvious statement that the laws of the natural sciences are useful because without them we would know even less about the world. If we already knew the position of the planet at all times, the mathematical relations between these positions which the planetary laws furnish would not be useful but might still be interesting. They might give us a certain pleasure and perhaps amazement to contemplate, even if they would not furnish us new information. Perhaps also, if someone came who had some different information about the positions of that planet, we would more effectively contradict him if his statements about the positions did not conform with the planetary laws—assuming that we have confidence in the laws of nature which are embodied in the planetary law.

Let us turn now to the relation of symmetry or invariance principles to the laws of nature. If we know a law of nature, such as the equations of electrodynamics, the knowledge of the subtle properties of these equations does not add anything to the content of these equations. It may be interesting to note that the correlations between events which the equations predict are the same no matter whether the events are viewed by an observer at rest, or an observer in uniform motion. However, all the correlations between events are already given by the equations themselves, and the aforementioned observation of the invariance of the equations does not augment the number or change the character of the correlations.

More generally, if we knew all the laws of nature, or the ultimate law of nature, the invariance properties of these laws would not furnish us new information. They might give us a certain pleasure and perhaps amazement to contemplate, even though they would not furnish new information. Perhaps also, if someone came around to propose a different law of nature, we could more effectively contradict him if his law of nature did not conform with our invariance principle—assuming that we have confidence in the invariance principle.

Evidently, the preceding discussion of the relation of the laws of nature to the events, and of the symmetry or invariance principles to the laws of nature is a very sketchy one. Many, many pages could be written about both. As far as I can see, the new aspects which would be dealt with in these pages would not destroy the similarity of the two relations—that is, the similarity between the relation of the laws of nature to the events, and the relation of the invariance principles to the laws of nature. They would, rather, support it and confirm the function of the invariance principles to provide a structure or coherence to the laws of nature just as the laws of nature provide a structure and coherence to the set of events.

Geometrical and Dynamical Principles of Invariance

What is the difference between the old and well-established geometrical principles of invariance, and the novel, dynamical ones? The geometrical principles of invariance, though they give a structure to the laws of nature, are formulated in terms of the events themselves. Thus, the time-displacement invariance, properly formulated, is: the correlations between events depend only on the time intervals between the events, not on the time at which the first event takes place. If P_1, P_2, P_3 are positions which the aforementioned planet can assume at times t_1, t_2, t_3, it could assume these positions also at times $t_1 + t$, $t_2 + t$, $t_3 + t$, where t is quite arbitrary. On the other hand, the new, dynamical principles of invariance are formulated in terms of the laws of nature. They apply to specific types of interaction, rather than to any correlation between events. Thus, we say that the electromagnetic interaction is gauge invariant, referring to a specific law of nature which regulates the generation of the electromagnetic field by charges, and the influence of the electromagnetic field on the motion of the charges.

It follows that the dynamical types of invariance are based on the existence of specific types of interactions. We all remember having read that, a long time ago, it was hoped that all interactions could be derived from mechanical interactions. Some of us still remember that, early in this century, the electromagnetic interactions were considered to be the source of all others. It was necessary, then, to explain away the gravitational interaction, and in fact this could be done quite successfully. We now recognize four or five distinct types of interactions: the gravitational, the electromagnetic, one or two types of strong (that is, nuclear) interactions, and the weak interaction responsible for beta decay, the decay of the μ meson, and some similar phenomena. Thus, we have given up, at least temporarily, the hope of one single basic interaction. Furthermore, every interaction has a dynamical invariance group, such as the gauge group for the electromagnetic interaction.

This is, however, the extent of our knowledge. Otherwise, let us not forget, the problem of interactions is still a mystery. Utiyama[6] has stimulated a fruitful line of thinking about how the interaction itself may be guessed once its group is known. However, we have no way of telling the group ahead of time; we have no way of telling how many groups and hence how many interactions there are. The groups seem to be quite disjointed, and there seems to be no connection between the various groups which characterize the various interactions or between these groups and the geometrical symmetry group, which is a single, well-defined group with which we have been familiar for many, many years.

Geometrical Principles of Invariance and Conservation Laws

Since it is good to stay on *terra cognita* as long as possible, let us first review the geometrical principles of invariance. These were recognized by Poincaré first, and I like to call the group formed by these invariables the Poincaré group.[7] The true meaning and importance of these principles were brought out only by Einstein, in his special theory of relativity. The group contains, first, displacements in space and time. This means that the correlations between events are the same everywhere and at all times, that the laws of nature—the compendium of the correlations—are the same no matter when and where they are established. If this were not so, it might have been impossible for the human mind to find laws of nature.

[6] R. Utiyama, *Phys. Rev.*, 101, 1597 (1956); also C. N. Yang and R. L. Mills, *Phys. Rev.*, 96, 191 (1954).

[7] H. Poincaré, *Compt. Rend.*, 140, 1504 (1905); *Rend. Circ. Mat. Palermo*, 21, 129 (1906).

It is good to emphasize at this point the fact that the laws of nature, that is, the correlations between events, are the entities to which the symmetry laws apply, not the events themselves. Naturally, the events vary from place to place. However, if one observes the positions of a thrown rock at three different times, one will find a relation between those positions, and this relation will be the same at all points of the Earth.

The second symmetry is not at all as obvious as the first one: it postulates the equivalence of all directions. This principle could be recognized only when the influence of the Earth's attraction was understood to be responsible for the difference between up and down. In other words, contrary to what was just said, the events between which the laws of nature establish correlations are not the three positions of the thrown rock, but the three positions of the rock with respect to the Earth.

The last symmetry—the independence of the laws of nature from the state of motion in which it is observed as long as this is uniform—is not at all obvious to the unpreoccupied mind.[8] One of its consequences is that the laws of nature determine not the velocity but the acceleration of a body: the velocity is different in coordinate systems moving with different speeds; the acceleration is the same as long as the motion of the coordinate systems is uniform with respect to each other. Hence, the principle of the equivalence of uniformly moving coordinate systems, and their equivalence with coordinate systems at rest, could not be established before Newton's second law was understood; it was at once recognized then, by Newton himself. It fell temporarily into dis-

[8] Thus, Aristotle's physics postulated that motion necessarily required the continued operation of a cause. Hence, all bodies would come to an absolute rest if they were removed from the cause which imparts them a velocity. [Cf., e.g., A. C. Crombie's *Augustine to Galileo* (London: Falcon Press, 1952), p. 82 or 244.] This cannot be true for coordinate systems moving with respect to each other. The coordinate systems with respect to which it is true then have a preferred state of motion.

repute as a result of certain electromagnetic phenomena until Einstein re-established it in a somewhat modified form.

It was mentioned already that the conservation laws for energy and for linear and angular momentum are direct consequences of the symmetries just enumerated. This is most evident in quantum-mechanical theory, where they follow directly from the kinematics of the theory, without making use of any dynamical law, such as the Schrödinger equation. This will be demonstrated at once. The situation is much more complex in classical theory, and, in fact, the simplest proof of the conservation laws in classical theory is based on the remark that classical theory is a limiting case of quantum theory. Hence, any equation valid in quantum theory, for any value of Planck's constant h, is valid also in the limit $h = 0$. Traces of this reasoning can be recognized also in the general considerations showing the connection between conservation laws and space-time symmetry in classical theory. The conservation laws can be derived also by elementary means, using the dynamical equation, that is, Newton's second law, and the assumption that the forces can be derived from a potential which depends only on the distances between the particles. Since the notion of a potential is not a very natural one, this is not the usual procedure. Mach, for instance, assumes that the force on any particle is a sum of forces, each due to another particle.[9] Such an assumption is implicit also in Newton's third law, otherwise the notion of counterforce would have no meaning. In addition, Mach assumes that the force depends only on the positions of the interacting pair, not on their velocities. Some such assumption is indeed necessary in classical theory.[10] Under the assumptions just mentioned, the conservation law for linear momentum follows at once from Newton's third law, and, conversely, this third law is also necessary for the conservation of linear momentum. All this was recognized already by Newton. For the conservation law of angular momentum, which was, in its general form, discovered almost 60 years after the *Principia* by Euler, Bernouilli, and d'Arcy, the significance of the isotropy of space is evident. If the direction of the force between a pair of particles were not directed along the line from one particle to the other, it would not be invariant under rotations about that line. Hence, under the assumptions made, only central forces are possible. Since the torque of such forces vanishes if they are oppositely equal, the angular momentum law follows. It would not follow if the forces depended on the positions of three particles or more.

[9] E. Mach, *The Science of Mechanics* (Chicago: Open Court Publ. Co., various editions), Chap. 3, Sec. 3.
[10] See footnote 5.

In quantum mechanics, as was mentioned before, the conservation laws follow already from the basic kinematical concepts. The point is simply that the states in quantum mechanics are vectors in an abstract space, and the physical quantities, such as position, momentum, etc., are operators on these vectors. It then follows, for instance, from the rotational invariance that, given any state ϕ, there is another state ϕ_α which looks just like ϕ in the coordinate system that is obtained by a rotation α about the Z axis. Let us denote the operator which changes ϕ into ϕ_α by Z_α. Let us further denote the state into which ϕ goes over in the time interval τ by $H_\tau\phi$ (for a schematic picture, cf. Fig. 1). Then,

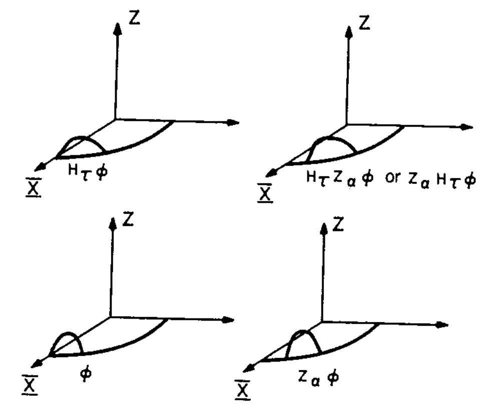

Fig. 1

because of the rotational invariance, ϕ_α will go over, in the same time interval, into the state $H_\tau\phi_\alpha$, which looks, in the second coordinate system, just like $H_\tau\phi$. Hence, it can be obtained from $H_\tau\phi$ by the operation Z_α. It follows that

$$H_\tau Z_\alpha \phi = Z_\alpha H_\tau \phi, \tag{1}$$

and since this is valid for any ϕ,

$$H_\tau Z_\alpha = Z_\alpha H_\tau. \tag{2}$$

Thus the operator Z_α commutes with H_τ, and this is the condition for its being conserved. Actually, the angular momentum about the Z axis is the limit of $(1/\alpha)(Z_\alpha - 1)$ for infinitely small α. The other conservation laws are derived in the same way. The point is that *the transformation operators, or at least the infinitesimal ones among them, play a double role and are themselves the conserved quantities.*

This will conclude the discussion of the geometrical principles of invariance. You will note that reflections which give rise *inter alia* to the concept of parity were not mentioned, nor did I speak about the apparently much more general geometric principle of invariance which forms the foundation of the general theory of relativity. The reason for the former omission is that I will have to consider the reflection operators at the end of this discussion. The reason that I did not speak about the invariance with respect to the general coordinate transformations of the general theory of relativity is that I believe that the underlying invariance is not geometric but dynamic. Let us consider, hence, the dynamic principles of invariance.

Dynamic Principles of Invariance

When we deal with the dynamic principles of invariance, we are largely on *terra incognita.* Nevertheless, since some of the attempts to develop these principles are both ingenious and successful, and since the subject is at the center of interest, I would like to make a few comments. Let us begin with the case that is best understood, the electromagnetic interaction.

In order to describe the interaction of charges with the electromagnetic field, one first introduces new quantities to describe the electromagnetic field, the so-called electromagnetic potentials. From these, the components of the electromagnetic field can be easily calculated, but not conversely. Furthermore, the potentials are not uniquely determined by the field; several potentials (those differing by a gradient) give the same field. It follows that the potentials cannot be measurable, and, in fact, only such quantities can be measurable which are invariant under the transformations which are arbitrary in the potential. This invariance is, of course, an artificial one, similar to that which we could obtain by introducing into our equations the location of a ghost. The equations then must be invariant with respect to changes of the coordinate of that ghost. One does not see, in fact, what good the introduction of the coordinate of the ghost does.

So it is with the replacement of the fields by the potentials, as long as one leaves everything else unchanged. One postulates, however, and this is the decisive step, that in order to maintain the same situation, one has to couple a transformation of the matter field with every transition from a set of potentials to another one which gives the same electromagnetic field. The combination of these two transformations, one on the electromagnetic potentials, the other on the matter field, is called a gauge transformation. Since it leaves the physical situation unchanged, every equation must be invariant thereunder. This is not true, for instance, of the unchanged equations of motion, and they would have, if left unchanged, the absurd property that two situations which are completely equivalent at one time would develop, in the course of time, into two distinguishable situations. Hence, the equations of motion have to be modified, and this can be done most easily by a mathematical device called the modification of the Lagrangian. The simplest modification that restores the invariance gives the accepted equations of electrodynamics which are well in accord with all experience.

Let me state next, without giving all the details, that a similar procedure is possible with respect to the gravitational interaction. Actually, this has been hinted at already by Utiyama.[11] The unnecessary complication that one has to introduce in this case is, instead of potentials, generalized coordinates. The equations then have to be invariant with respect to all the coordinate transformations of the general theory of relativity. This would not change the content of the theory but would only amount to the introduction of a more flexible language in which there are several equivalent descriptions of the same physical situation. Next, however, one postulates that the matter field also transforms as the metric field so that one has to modify the equations in order to preserve their invariance. The simplest modification, or one of the simplest ones, leads to Einstein's equations.

The preceding interpretation of the invariance of the general theory of relativity does not interpret it as a geometrical invariance. That this should not be done had already been pointed out by the Russian physicist Fock.[12] With a slight oversimplification, one can say that a geometrical invariance postulates that two physically different situations, such as

[11] See footnote 6.

[12] V. Fock, *The Theory of Space, Time and Gravitation* (New York: Pergamon Press, 1959). See also A. Kretschman, *Ann. Phys.*, 53, 575 (1917).

those in Figure 1, should develop, in the course of time, into situations which differ in the same way. This is not the case here: the postulate is merely that two different descriptions of the same situation should develop, in the course of time, into two descriptions which also describe the same physical situation. The similarity with the case of the electromagnetic potentials is obvious.

Unfortunately, the situation is by no means the same in the case of the other interactions. One knows very little about the weaker one of the strong interactions. The strong one, as well as the weak interaction, has a group which is, first of all, very much smaller than the gauge group or the group of general coordinate transformations.[13] Instead of the infinity of generators of the gauge and general transformation groups, they have only a finite number, that is, eight, generators. They do suffice, nevertheless, to a large extent to determine the form of the interaction, as well as to derive some theorems, similar to those of spectroscopy, which give approximate relations between reaction rates and between energies, that is, masses. Figure 2 shows the octuplet of heavy masses—its members are joined to each other by the simplest nontrivial representation of the underlying group which is equivalent to its conjugate complex.

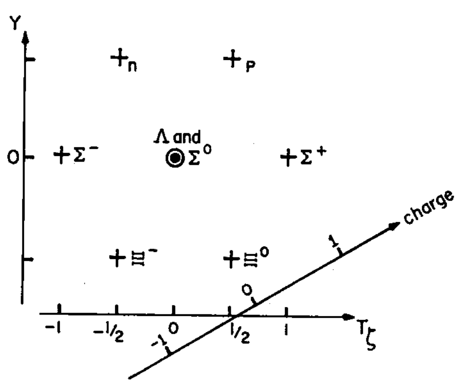

Fig. 2

[13] For the strong interaction, cf. Y. Ne'eman, *Nucl. Phys.*, 26, 222 (1961), and M. Gell-Mann, *Phys. Rev.*, 125, 1067 (1962). For the weak interaction, R. P. Feynman and M. Gell-Mann, *Phys. Rev.*, 109, 193 (1958), and E. C. G. Sudarshan and R. E. Marshak, *Phys. Rev.*, 109, 1960 (1958); also J. J. Sakurai, *Nuovo Cimento*, 7, 649 (1958), and G. S. Gershtin and A. B. Zeldovitch, *J. Exptl. Theoret. Phys. USSR*, 29, 698 (1955).

Another difference between the invariance groups of electromagnetism and gravitation on one hand, and at least the invariance group of the strong interaction on the other hand, is that the operations of the former remain valid symmetry operations even if the existence of the other types of interactions is taken into account. The symmetry of the strong interaction, on the other hand, is "broken" by the other interactions, i.e., the operations of the group of the strong interaction are valid symmetry operations only if the other types of interactions can be disregarded. The symmetry group helps to determine the interaction operator in every case. However, whereas all interactions are invariant under the groups of the electromagnetic and gravitational interactions, only the strong interaction is invariant under the group of that interaction.

We have seen before that the operations of the geometric symmetry group entail conservation laws. The question naturally arises whether this is true also for the operations of the dynamic symmetry groups. Again, there seems to be a difference between the different dynamic invariance groups. It is common opinion that the conservation law for electric charge can be regarded as a consequence of gauge invariance, i.e., of the group of the electromagnetic interaction. On the other hand, one can only speculate about conservation laws which could be attributed to the dynamic group of general relativity. Again, it appears reasonable to assume that the conservation laws for baryons and leptons can be deduced by means of the groups of the strong and of the weak interaction.[14] If true, this would imply that the proper groups of these interactions have not yet been recognized. One can adduce two pieces of evidence for the last statement. First, so far, the conservation laws in question[15] could not be deduced from the symmetry properties of these interactions, and it is unlikely that they can be deduced from

[14] For the baryon conservation law and the strong interaction, this was suggested by the present writer, *Proc. Am. Phil. Soc.*, 93, 521 (1949), and these *Proceedings*, 38, 449 (1952). The baryon conservation law was first postulated by E. C. G. Stueckelberg, *Helv. Phys. Acta*, 11, 299 (1938).

[15] For the experimental verification of these and the other conservation laws, see G. Feinberg and M. Goldhaber, these *Proceedings*, 45, 1301 (1959). The conservation law for leptons was proposed by G. Marx in *Acta Phys. Hung.*, 3, 55 (1953); also A. B. Zeldovitch, *Dokl. Akad. Nauk USSR*, 91, 1317 (1953), and E. J. Konopinski and H. M. Mahmoud, *Phys. Rev.*, 92, 1045 (1953). It seemed to be definitely established by T. D. Lee and C. N. Yang, *Phys. Rev.*, 105, 1671 (1957). See also Fermi's observation mentioned by C. N. Yang and J. Tiomno, *Phys. Rev.*, 79, 497 (1950).

them.[16] Second, the symmetry properties in question are not rigorous but are broken by the other interactions. It is not clear how rigorous conservation laws could follow from approximate symmetries—and all evidence indicates that the baryon and lepton conservation laws are rigorous.[17] Again, we are reminded that our ideas on the dynamical principles of invariance are not nearly as firmly established as those on the geometrical ones.

Let me make a last remark on a principle which I would not hesitate to call a symmetry principle and which forms a transition between the geometrical and dynamical principles. This is given by the crossing relations.[18] Let us consider the amplitude for the probability of some collision, such as

$$A+B+\ldots \rightarrow X+Y+\ldots. \tag{3}$$

This will be a function of the invariants which can be formed from the momenta four-vectors of the incident and emitted particles. It then follows from one of the reflection principles which I did not discuss, the "time reversal invariance," that the amplitude of (3) determines also the amplitude of the inverse reaction

$$X+Y+\ldots \rightarrow A+B+\ldots \tag{4}$$

in a very simple fashion. If one reverses all the velocities and also interchanges past and future (which is the definition of "time reversal"), (4) goes over into (3) so that the amplitudes for both are essentially equal. Similarly, if we denote the antiparticle of A by $\overline{A}$, that of B by $\overline{B}$, and so on, and consider the reaction

$$\overline{A}+\overline{B}+\ldots \rightarrow \overline{X}+\overline{Y}+\ldots, \tag{5}$$

its amplitude is immediately given by that of (3) because (according to the interpretation of Lee and Yang), the reaction (5) is obtained from (3) by space inversion. The amplitudes for

$$\overline{X}+\overline{Y}+\ldots \rightarrow \overline{A}+\overline{B}+\ldots \tag{6}$$

[16] For the baryon conservation and strong interaction, this was emphatically pointed out in a very interesting article by J. J. Sakurai, *Ann. Phys.* (N.Y.), 11, 1 (1960). Concerning the conservation of lepton members, see G. Marx, *Z. Naturforsch.*, 9a, 1051 (1954).

[17] See footnote 15.

[18] M. L. Goldberger, *Phys. Rev.*, 99, 979 (1955); M. Gell-Mann, and M. L. Goldberger, *Phys. Rev.*, 96, 1433 (1954). See M. L. Goldberger and K. M. Watson, *Collision Theory* (New York: John Wiley and Sons, 1964), chap. 10.

can be obtained in a similar way. The relations between the amplitudes of reactions (3), (4), (5), and (6) are consequences of geometrical principles of invariance.

However, one can go further. The crossing relations tell us how to calculate, for instance, the amplitude of

$$\overline{X}+B+\ldots \rightarrow \overline{A}+Y+\ldots \tag{7}$$

from the amplitude system of (3). To be sure, the calculation, or its result, is not simple any more. One has to consider the dependence of the reaction amplitude for (3) as an analytic function of the invariants formed from the momenta of the particles in (3), and extend this analytic function to such values of the variables which have no physical significance for the reaction (3) but which give the amplitude for (7). Evidently there are several other reactions the amplitudes of which can be obtained in a similar way; they are all obtained by the analytic continuation of the amplitude for (3), or any of the other reactions. Thus, rather than exchanging A and X to obtain (7), A and Y could be exchanged, and so on.

The crossing relations share two properties of the geometrical principles of invariance: they do not refer to any particular type of interaction and most of us believe that they have unlimited validity. On the other hand, though they can be formulated in terms of events, their formulation presupposes the establishment of a law of nature, namely, the mathematical, in fact analytic, expression for the collision amplitude for one of the aforementioned reactions. One may hope that they will help to establish a link between the now disjoint geometrical and dynamical principles of invariance.