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A Coercive Bilinear Form for Maxwell's Equations *

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Abstract

When one wants to treat the time-harmonic Maxwell equations with variational methods, one has to face the problem that the natural bilinear form is not coercive on the whole Sobolev space H^1. On can, however, make it coercive by adding a certain bilinear form on the boundary of the domain. This addition causes a change in the natural boundary conditions. The additional bilinear form (see (2.7), (2.21), (3.3)) contains tangential derivatives of the normal and tangential components of the field on the boundary, and it vanishes on the subspaces of H^1 that consist of fields with either vanishing tangential components or vanishing normal components on the boundary. Thus the variational formulations of the "electric" or "magnetic" boundary value problems with homogeneous boundary conditions are not changed. A useful change is caused in the method of boundary integral equations for the boundary value problems and for transmission problems where one has to use nonzero boundary data. The idea of this change emerged from the desire to have strongly elliptic boundary integral equations for the "electric" boundary value problem that are suitable for numerical approximation [12], [13]. Subsequently, it was shown how to incorporate the "magnetic" boundary data and to apply the idea to transmission problems [3], [7], [5]. In the present note we present this idea in full generality, also for the anisotropic case, and prove coercivity without using symbols of pseudodifferential operators on the boundary.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\Gamma \in C^{1,1}$. This means that the exterior normal $\vec{n}$ can be extended to a Lipschitz continuous vector field of unit length on a neighborhood of Γ.

Consider the time-harmonic Maxwell equations

$$\operatorname{curl} \vec{E} = i\omega\mu\vec{H}\,; \qquad \operatorname{curl} \vec{H} = -i\omega\varepsilon\vec{E} \qquad \text{in } \Omega\,. \tag{1.1}$$

Here ω is a constant, and ε and μ are in general (3×3)-matrix valued functions which we assume to be in $C^1(\overline{\Omega})$. Further assumptions on ε and μ will be made later on. All

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functions are complex-valued. The electric field $\vec{E}$ satisfies the second order equation

$$\text{curl}\,\alpha\,\text{curl}\,\vec{E} - \omega^2 \varepsilon \vec{E} = 0 \qquad \text{in } \Omega\,, \tag{1.2}$$

where $\alpha = \mu^{-1}$. Since also

$$\text{div}\,\varepsilon\vec{E} = 0 \qquad \text{in } \Omega \tag{1.3}$$

holds, $\vec{E}$ satisfies

$$P\vec{E} - \omega^2\varepsilon\vec{E} \;:=\; \text{curl}\,\alpha\,\text{curl}\,\vec{E} - \varepsilon^*\,\text{grad}(s\,\text{div}\,\varepsilon\vec{E}) - \omega^2\varepsilon\vec{E} = 0 \qquad \text{in } \Omega\,, \tag{1.4}$$

where ε^* is the adjoint of ε, and $s \in C^1(\overline{\Omega})$ is an arbitrary function.

The natural bilinear form associated with the second order elliptic system (1.4) is

$$a_0(\vec{E},\vec{F}) \;:=\; \int_\Omega \left\{ (\alpha\,\text{curl}\,\vec{E})\cdot\overline{\text{curl}\,\vec{F}} + s(\text{div}\,\varepsilon\vec{E})(\overline{\text{div}\,\varepsilon\vec{F}}) \right\} dx. \tag{1.5}$$

Let us denote the $L^2(\Omega)$ inner product by $(\cdot,\cdot)$, for scalar as well as for vector functions:

$$(\vec{E},\,\vec{F}) \;:=\; \int_\Omega \vec{E}(x)\cdot\overline{\vec{F}(x)}\,dx.$$

By $<\cdot,\,\cdot>$ we denote the $L^2(\Gamma)$ inner product

$$<u,\,v> \;:=\; \int_\Gamma u\overline{v}\,ds\,,$$

where ds is the surface measure on Γ.

Green's formulas are

$$(\text{curl}\,\vec{u},\,\vec{v}) \;-\; (\vec{u},\,\text{curl}\,\vec{v}) \;=\; <\vec{n}\times\vec{u},\,\vec{v}> \tag{1.6}$$

$$(\text{div}\,\vec{u},\,\varphi) \;+\; (\vec{u},\,\text{grad}\,\varphi) \;=\; <\vec{n}\cdot\vec{u},\,\varphi> \tag{1.7}$$

Thus the bilinear form a_0 is related to the differential operator P by

$$a_0(\vec{E},\vec{F}) \;=\; (P\vec{E},\,\vec{F}) - <\vec{n}\times(\alpha\,\text{curl}\,\vec{E}),\,\vec{F}> \;+\; <s\,\text{div}\,\varepsilon\vec{E},\,\vec{n}\cdot\varepsilon\vec{F}> \;. \tag{1.8}$$

This leads to the well-known (see e.g., [9], [14]) weak formulations of the standard boundary value problems for the operator P:
Let

$$X \;:=\; \left\{\vec{E}\in H^1(\Omega)\mid \vec{n}\times\vec{E} = 0 \text{ on } \Gamma\right\}\,, \tag{1.9}$$

$$Y \;:=\; \left\{\vec{E}\in H^1(\Omega)\mid \vec{n}\cdot\varepsilon\vec{E} = 0 \text{ on } \Gamma\right\}\,. \tag{1.10}$$

Then for $\vec{f}\in L^2(\Omega)$, the weak form of the "electric" boundary value problem

$$P\vec{E} = \vec{f} \quad \text{in } \Omega\,; \qquad \vec{n}\times\vec{E} = 0 \quad \text{on } \Gamma \tag{1.11}$$

is: Find $\vec{E}\in X$ such that

$$a_0(\vec{E},\vec{F}) \;=\; (\vec{f},\,\vec{F}) \qquad \text{for all } \vec{F}\in X\,. \tag{1.12}$$

From (1.8), we see that then $\vec{E}$ satisfies in the weak sense the natural boundary condition

$$\operatorname{div} \varepsilon \vec{E} = 0 \qquad \text{on } \Gamma . \tag{1.13}$$

Similarly, the "magnetic" boundary value problem

$$P\vec{E} = \vec{f} \quad \text{in } \Omega ; \qquad \vec{n} \cdot \varepsilon \vec{E} = 0 \quad \text{on } \Gamma \tag{1.14}$$

has the weak formulation: Find $\vec{E} \in Y$ such that

$$a_0(\vec{E}, \vec{F}) \; = \; (\vec{f}, \vec{F}) \qquad \text{for all } \vec{F} \in Y . \tag{1.15}$$

The natural boundary condition is

$$\vec{n} \times (\alpha \operatorname{curl} \vec{E}) = 0 \qquad \text{on } \Gamma . \tag{1.16}$$

It is well known (see [9], [11]) that, under suitable hypotheses on ε, μ and s, the bilinear form a_0 is coercive on both subspaces X and Y of $H^1(\Omega)$. Thus both boundary value problems can be numerically approximated using finite element methods. Also the spectral theory for strongly elliptic boundary value problems is available and can be used for the analysis of the corresponding time-dependent problems.

The bilinear form a_0 is, however, not coercive on the whole space $H^1(\Omega)$. This causes problems, e.g., if the boundary value problems are to be solved by boundary element methods (see [12], [13], [1], [2]), or if corresponding transmission problems are studied [7].

The boundary integral equations of the first kind studied in [12], [13], [1], [2] are an elliptic system of pseudodifferential equations which, due to the non-coercivity of a_0, is not strongly elliptic. In [1], [2], the problem was therefore treated as a saddle-point problem and a mixed finite element method for its solution was devised. In [12], [13], it was found that the system can be transformed into a strongly elliptic system which is then treatable by ordinary finite element methods. This transformation corresponds to a change in the natural boundary condition (1.13). This together with an analogous change in the other natural boundary condition (1.16) was shown in [3] for the case $\varepsilon = \mu = s = 1$ to correspond to a change in the bilinear form a_0 which makes it coercive over all of $H^1(\Omega)$.

Transmission problems in a more general, but isotropic case, are studied in [7] by boundary integral equation methods and in [5] for inhomogeneous problems by a coupling of boundary integral equation and finite element methods. In [7], the strong ellipticity of the system of pseudodifferential operators is proved by computing their principal symbols.

In this paper, we prove the coercivity (strong ellipticity, Gårding's inequality) for the modified bilinear form in the general case just by using Green's formula. Thus we need less regularity for the boundary Γ than in [7]. There is even a result for polyhedra and piecewise $C^{1,1}$ boundaries.

In section 2, we begin with the simplest case $\varepsilon = \mu = s = 1$ and generalize this then to physically more meaningful isotropic homogeneous cases.

In section 3, we treat the anisotropic inhomogeneous case.

In section 4, we show corresponding results for piecewise smooth boundaries.

2 The isotropic case

We will need the following notations:

For a vector field $\vec{u}$ defined on Γ or on a neighborhood of Γ, the tangential and normal components are

$$\vec{u}_\top := -\vec{n} \times (\vec{n} \times \vec{u}) = \vec{u} - u_n \vec{n}\,; \qquad u_n := \vec{n} \cdot \vec{u}\,. \tag{2.1}$$

Here $\vec{n}$ is the unique extension of the exterior normal vector field on Γ to a neighborhood of Γ as a Lipschitz continuous vector field of unit length (Recall that we assume $\Gamma \in C^{1,1}$ unless stated otherwise). It follows that

$$\operatorname{curl} \vec{n} \equiv \partial_n \vec{n} \equiv 0\,. \tag{2.2}$$

We need the surface divergence $\operatorname{div}_\top \vec{u}_\top$ on Γ which we define as follows

$$\operatorname{div}_\top \vec{u}_\top := \operatorname{div} \vec{u}_\top = \operatorname{div} \vec{u} - u_n \operatorname{div} \vec{n} - \partial_n u_n = \operatorname{div} \vec{u} - \vec{n} \cdot (\partial_n \vec{u}) - u_n \operatorname{div} \vec{n}. \tag{2.3}$$

A little vector analysis together with (2.2) shows that

$$\operatorname{div}_\top \vec{u}_\top = \vec{n} \cdot \operatorname{curl}(\vec{n} \times \vec{u})\,. \tag{2.4}$$

From Green's formulas (1.6), (1.7) we obtain for φ supported in a small neighborhood of Γ

$$\begin{aligned} <\operatorname{div}_\top \vec{u}_\top,\, \varphi> &= <\vec{n} \cdot \operatorname{curl}(\vec{n} \times \vec{u}),\, \varphi> \\ &= (\operatorname{curl}(\vec{n} \times \vec{u}),\, \operatorname{grad} \varphi) = -<\vec{u}_\top,\, \operatorname{grad}_\top \varphi>\,. \end{aligned} \tag{2.5}$$

This formula shows that the mapping $\vec{u} \mapsto \operatorname{div}_\top \vec{u}_\top \big|_\Gamma$ can be extended from smooth functions $\vec{u}$ to $\vec{u} \in H^1(\Omega)$, defining a continuous mapping : $H^1(\Omega) \to H^{-1/2}(\Gamma)$.

Similarly, the mapping $\varphi \mapsto \operatorname{grad}_\top \varphi \big|_\Gamma$ is continuous from $H^1(\Omega)$ to $H^{-1/2}(\Gamma)$. The brackets $<\cdot,\, \cdot>$ then denote the natural duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. We use the usual Sobolev spaces on Ω and Γ (see e.g. [9]), and we use the same notation for spaces of vector-valued functions. Thus, e.g.,

$$\|\vec{u}\|^2_{H^1(\Omega)} = (\operatorname{grad} \vec{u},\, \operatorname{grad} \vec{u}) + (\vec{u},\, \vec{u}) = \int_\Omega \left\{ \sum_{j,k=1}^{3} |\partial_j u_k|^2 + \sum_{j=1}^{3} |u_j|^2 \right\} dx. \tag{2.6}$$

Theorem 2.1 *Define $a_1(\vec{u}, \vec{v})$ by*

$$\begin{aligned} a_1(\vec{u}, \vec{v}) := \; & (\operatorname{curl} \vec{u},\, \operatorname{curl} \vec{v}) + (\operatorname{div} \vec{u},\, \operatorname{div} \vec{v}) \\ & + <\operatorname{grad}_\top u_n,\, \vec{v}_\top> - <\operatorname{div}_\top \vec{u}_\top,\, v_n>\,. \end{aligned} \tag{2.7}$$

Then a_1 is coercive over $H^1(\Omega)$, i.e., there exist constants $\gamma > 0$ and c such that

$$\operatorname{Re} a_1(\vec{u}, \vec{u}) \geq \gamma \|\vec{u}\|^2_{H^1(\Omega)} - c\|\vec{u}\|^2_{L^2(\Omega)} \qquad \textit{for all } \vec{u} \in H^1(\Omega)\,. \tag{2.8}$$

Proof. Since a_1 is continuous on $H^1(\Omega)$ and $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, we need to show (2.8) only for smooth $\vec{u}$.
From the formula

$$\operatorname{grad}(\vec{a}\cdot\vec{b}) = \vec{a}\times\operatorname{curl}\vec{b} + \vec{b}\times\operatorname{curl}\vec{a} + (\vec{a}\cdot\operatorname{grad})\vec{b} + (\vec{b}\cdot\operatorname{grad})\vec{a}$$

together with (2.2) it follows that

$$\operatorname{grad} u_n = \vec{n}\times\operatorname{curl}\vec{u} + \partial_n\vec{u} + (\vec{u}\cdot\operatorname{grad})\vec{n}\,. \tag{2.9}$$

Now we apply Green's formula (1.8)

$$(\operatorname{curl}\vec{u},\,\operatorname{curl}\vec{v}) \;+\; (\operatorname{div}\vec{u},\,\operatorname{div}\vec{v}) \;=\; (-\Delta\vec{u},\,\vec{v}) - \;<\vec{n}\times\operatorname{curl}\vec{u},\,\vec{v}_\top> \;+\; <\operatorname{div}\vec{u},\,v_n>$$

to a_1 and obtain

$$a_1(\vec{u},\vec{v}) = \;(-\Delta\vec{u},\,\vec{v}) - \;<\vec{n}\times\operatorname{curl}\vec{u} - \operatorname{grad}_\top u_n,\,\vec{v}_\top> \;+\; <\operatorname{div}\vec{u} - \operatorname{div}_\top\vec{u}_\top,\,v_n>\;.$$

With (2.3) and (2.9) this reduces to

$$\begin{aligned} a_1(\vec{u},\vec{v}) \;&=\; (-\Delta\vec{u},\,\vec{v}) + \;<\partial_n\vec{u} + (\vec{u}\cdot\operatorname{grad})\vec{n},\,\vec{v}_\top> \;+\; <\vec{n}\cdot\partial_n\vec{u} + u_n\operatorname{div}\vec{n},\,v_n> \\ &=\; (-\Delta\vec{u},\,\vec{v}) + \;<\partial_n\vec{u},\,\vec{v}> \;+ b(\vec{u},\vec{v}) \end{aligned} \tag{2.10}$$

with

$$b(\vec{u},\vec{v}) = \;<(\vec{u}_\top\cdot\operatorname{grad})\vec{n} + u_n(\operatorname{div}\vec{n})\vec{n},\,\vec{v}>\;. \tag{2.11}$$

Now we apply Green's formula for the Laplace operator

$$(-\Delta\vec{u},\,\vec{v}) \;=\; (\operatorname{grad}\vec{u},\,\operatorname{grad}\vec{v}) - \;<\partial_n\vec{u},\,\vec{v}> \tag{2.12}$$

and obtain

$$a_1(\vec{u},\vec{v}) = \;(\operatorname{grad}\vec{u},\,\operatorname{grad}\vec{v}) \;+ b(\vec{u},\vec{v})\,. \tag{2.13}$$

From the Lipschitz continuity of $\vec{n}$ we obtain an estimate

$$|b(\vec{u},\vec{v})| \;\le\; C\|\vec{u}\|_{L^2(\Gamma)}\cdot\|\vec{v}\|_{L^2(\Gamma)}\,, \tag{2.14}$$

where C is determined by an upper bound for the derivatives of $\vec{n}$ on Γ.
The trace lemma implies with (2.14)

$$|b(\vec{u},\vec{u})| \;\le\; C\|\vec{u}\|^2_{H^s(\Omega)}$$

for any $s > 1/2$. It follows that for every $\eta > 0$ there is a C_η with

$$|b(\vec{u},\vec{u})| \;\le\; \eta\|\vec{u}\|^2_{H^1(\Omega)} - C_\eta\|\vec{u}\|^2_{L^2(\Omega)}\,. \tag{2.15}$$

This gives with (2.13)

$$\operatorname{Re} a_1(\vec{u},\vec{u}) \;\ge\; (1-\eta)\|\vec{u}\|^2_{H^1(\Omega)} - (1+C_\eta)\|\vec{u}\|^2_{L^2(\Omega)}\,.$$

∎

The following well-known result is an easy consequence of Theorem 2.1.

Corollary 2.2 *The bilinear form* $(\operatorname{curl}\vec{u},\,\operatorname{curl}\vec{v}) + (\operatorname{div}\vec{u},\,\operatorname{div}\vec{v})$ *is coercive on the subspaces* X *and* Y *(* see *(1.9), (1.10)) of* $H^1(\Omega)$.

Proof. From the definition (2.7) it follows immediately that the two boundary terms in $a_1(\vec{u},\vec{v})$ vanish if either $u_n = v_n = 0$ or $\vec{u}_\top = \vec{v}_\top = 0$ holds on Γ. Thus

$$a_1(\vec{u},\vec{u}) = \|\operatorname{curl}\vec{u}\|^2_{L^2(\Omega)} + \|\operatorname{div}\vec{u}\|^2_{L^2(\Omega)} \qquad \text{for all } \vec{u}\in X\cup Y.$$

$\blacksquare$

Remark 2.3 The bilinear form a_1 provides weak formulations of the following two boundary value problems:

Let $\vec{f} \in L^2(\Omega)$, $\rho \in H^{-1/2}(\Gamma)$ and $\vec{\psi}_\top \in H^{-1/2}(\Gamma)$ be given. Then the condition

$$a_1(\vec{u},\vec{v}) \;=\; (\vec{f},\, \vec{v}) + \,<\rho,\, v_n> \qquad \text{for all } \vec{v} \in X \tag{2.16}$$

is the weak form of the boundary value problem

$$-\Delta\vec{u} = \vec{f} \quad \text{in } \Omega\,; \qquad \operatorname{div}\vec{u} - \operatorname{div}_\top \vec{u}_\top = \rho \quad \text{on } \Gamma\,. \tag{2.17}$$

The condition

$$a_1(\vec{u},\vec{v}) \;=\; (\vec{f},\, \vec{v}) + \,<\vec{\psi}_\top,\, \vec{v}_\top> \qquad \text{for all } \vec{v} \in Y \tag{2.18}$$

is the weak form of the boundary value problem

$$-\Delta\vec{u} = \vec{f} \quad \text{in } \Omega\,; \qquad -\vec{n} \times \operatorname{curl}\vec{u} + \operatorname{grad}_\top u_n = \vec{\psi}_\top \quad \text{on } \Gamma\,. \tag{2.19}$$

We see that the boundary terms in (2.7) correspond to a change in the natural boundary conditions. Thus the set of "Cauchy data"

$$(\vec{u}_\top,\, u_n,\, -\vec{n} \times \operatorname{curl}\vec{u},\, \operatorname{div}\vec{u})$$

is replaced by the equivalent set

$$(\vec{u}_\top,\, u_n,\, -\vec{n} \times \operatorname{curl}\vec{u} + \operatorname{grad}_\top u_n,\, \operatorname{div}\vec{u} - \operatorname{div}_\top \vec{u}_\top)\,.$$

Of course, this change is only seen if (2.17) and (2.19) are completed by the addition of *inhomogeneous* stable boundary conditions. On the spaces defined by *homogeneous* stable boundary conditions, i.e., X for (2.17) and Y for (2.19), one obtains the familiar form of the "electric" and "magnetic" boundary value problems, respectively.

Now we generalize Theorem 2.1 in several steps.

First we note that $a_1(\vec{u},\vec{v})$ is actually hermitian: According to (2.5) we have

$$a_1(\vec{u},\vec{v}) = \;(\operatorname{curl}\vec{u},\, \operatorname{curl}\vec{v}) \;+\; (\operatorname{div}\vec{u},\, \operatorname{div}\vec{v}) + \;<\operatorname{grad}_\top u_n,\, \vec{v}_\top> \;+\; <\vec{u}_\top,\, \operatorname{grad}_\top v_n>\,,$$

hence for $\vec{u} = \vec{v}$, a_1 is real:

$$a_1(\vec{u},\vec{u}) = \|\operatorname{curl}\vec{u}\|^2_{L^2(\Omega)} + \|\operatorname{div}\vec{u}\|^2_{L^2(\Omega)} + 2\operatorname{Re} <\operatorname{grad}_\top u_n,\, \vec{v}_\top> \;. \tag{2.20}$$

Theorem 2.4 *Let $\alpha, \beta \in \mathbb{C}$, $\theta_1, \theta_2 \in \mathbb{R}$ be such that*

$$0 < \theta_1 + \theta_2 \leq 2\min\{\operatorname{Re}\alpha, \operatorname{Re}\beta\}.$$

Let $a_2(\vec{u},\vec{v})$ be defined by

$$\begin{aligned} a_2(\vec{u},\vec{v}) \;\; &:= \;\; \alpha\,(\operatorname{curl}\vec{u},\, \operatorname{curl}\vec{v}) \;+ \beta\,(\operatorname{div}\vec{u},\, \operatorname{div}\vec{v}) \\ &\quad\;\; + \theta_1 <\operatorname{grad}_\top u_n,\, \vec{v}_\top> \,-\, \theta_2 <\operatorname{div}_\top \vec{u}_\top,\, v_n> \;. \end{aligned} \tag{2.21}$$

Then a_2 is coercive over $H^1(\Omega)$.

Proof. The boundary terms in a_2 give

$$\operatorname{Re}\left(\theta_1 < \operatorname{grad}_\top u_n,\ \vec{u}_\top > \ + \theta_2 < \vec{u}_\top,\ \operatorname{grad}_\top u_n > \right) = (\theta_1 + \theta_2) \operatorname{Re} < \operatorname{grad}_\top u_n,\ \vec{u}_\top > .$$

Thus with $\theta := (\theta_1 + \theta_2)/2$ we have

$$\begin{aligned} \operatorname{Re} a_2(\vec{u}, \vec{u}) &= (\operatorname{Re}\alpha - \theta) \| \operatorname{curl} \vec{u} \|^2_{L^2(\Omega)} + (\operatorname{Re}\beta - \theta) \| \operatorname{div} \vec{u} \|^2_{L^2(\Omega)} + \theta a_1(\vec{u}, \vec{u}) \\ &\geq \theta \operatorname{Re} a_1(\vec{u}, \vec{u}), \end{aligned}$$

and the assertion follows from Theorem 2.1. ∎

Now we can treat the isotropic homogeneous case of Maxwell's equations. Thus assume that α, ε, s are scalar constants and there exists θ such that $\theta\varepsilon \in \mathbb{R}$ and

$$0 < \theta\varepsilon \leq \min\{\operatorname{Re}\alpha, \operatorname{Re} s|\varepsilon|^2\}. \tag{2.22}$$

Then with $\beta = s|\varepsilon|^2$ and $\theta_1 = \theta_2 = \theta\varepsilon$ we can write a_2 as

$$\begin{aligned} a_2(\vec{u}, \vec{v}) &:= (\alpha \operatorname{curl} \vec{u},\ \operatorname{curl} \vec{v}) + (s \operatorname{div} \varepsilon\vec{u},\ \operatorname{div} \varepsilon\vec{v}) \\ &\quad + < \theta \operatorname{grad}_\top \varepsilon u_n,\ \vec{v}_\top > - < \operatorname{div}_\top \vec{u}_\top,\ \theta\varepsilon v_n > . \end{aligned} \tag{2.23}$$

Again, on the subspaces X and Y, the boundary terms in a_2 vanish. If

$$(\vec{u}_\top,\ \varepsilon u_n,\ -\alpha\vec{n} \times \operatorname{curl} \vec{u},\ s \operatorname{div} \varepsilon\vec{u}) \tag{2.24}$$

are the natural Cauchy data corresponding to the bilinear form a_0 (see (1.5)), then the addition of the boundary terms to a_2 in (2.23) can be interpreted as a change to the set of Cauchy data

$$(\vec{u}_\top,\ \theta\varepsilon u_n,\ -\alpha\vec{n} \times \operatorname{curl} \vec{u} + \theta \operatorname{grad}_\top \varepsilon u_n,\ \frac{s}{\theta} \operatorname{div} \varepsilon\vec{u} - \operatorname{div}_\top \vec{u}_\top). \tag{2.25}$$

In [7], the case $\alpha = s|\varepsilon|^2$ was considered. In this case, the operator P (see (1.4)) is the scalar operator $-\alpha\Delta$. The Cauchy data (2.25) correspond to [7, (5.8)], and the coercivity under the condition (2.22) is shown there using symbols of pseudodifferential operators on the boundary Γ(see [7, (5.13)]).

The strong ellipticity of the system of boundary integral equations discussed in [7] can be inferred from Theorem 2.4 using the general theory of strongly elliptic *transmission problems* presented in [7, Section 2].

In Theorem 2.4, the possibility of complex constants α and ε was emphasized in order to include the important case of a perfect conductor. There $\alpha > 0$ and $\varepsilon = i\sigma/\omega$, where $\sigma > 0$ is the conductivity. According to (2.22), we obtain a coercive bilinear form a_2 if we choose $\theta = -i\tau$ with $0 < \tau \leq \alpha/|\varepsilon|$ and $s = \alpha/|\varepsilon|^2$.

Another consequence of Theorem 2.4 is the possibility of solving the *boundary value problems* involving the Cauchy data (2.25) by boundary element methods using boundary integral equations of the first kind [8].

Finally, one can use the coercive bilinear form a_2 for the numerical solution of an interface problem by a coupling of finite element and boundary element methods as explained in [4], [5], [6].

In all these cases, the set of Cauchy data determining the boundary conditions is not uniquely given by the bilinear form a_2. If $(\vec{w}_\top, w_n, \vec{\psi}_\top, \psi_n)$ are the Cauchy data for $\vec{u}$, where $\vec{w}$ corresponds to stable and $\vec{\psi}$ to unstable (natural) boundary conditions, then the condition is

$$a_2(\vec{u}^1, \vec{u}^2) = (P\vec{u}^1, \vec{u}^2) + <\vec{\psi}^1_\top, \vec{w}^2_\top> + <\psi^1_n, w^2_n> . \tag{2.26}$$

Thus instead of (2.25), we can also choose (with $\theta_1 + \theta_2 = \theta$)

$$(\vec{u}_\top, \varepsilon u_n, -\alpha\vec{n} \times \operatorname{curl} \vec{u} + \theta_1 \operatorname{grad}_\top \varepsilon u_n, s \operatorname{div} \varepsilon\vec{u} - \theta_2 \operatorname{div}_\top \vec{u}_\top).$$

In any case, the mapping from the standard Cauchy data to the changed Cauchy data as well as the inverse mapping are given by tangential differential operators.

3 The anisotropic case

The anisotropic case of Theorem 2.1 requires a new proof which then will also include the isotropic but inhomogeneous case. The proof follows [11] where the coercivity of the bilinear form a_0 on the space X is shown.

We make the following assumptions:

ε and μ are selfadjoint positive definite (3×3) matrix functions in $C^1(\overline{\Omega})$; ε is real (the case of nonreal ε is left as an exercise for the reader); $s \in C^1(\overline{\Omega})$ is a scalar real-valued function, and there exist positive constants $\alpha_1, \varepsilon_1, s_1$ such that with $\alpha = \mu^{-1}$ and $\kappa = \varepsilon^{-1} \det \varepsilon$ there holds for all $x \in \overline{\Omega}$, $\xi \in \mathbb{C}^3$

$$\begin{array}{lllllll} \overline{\xi} \cdot \alpha(x)\xi & \geq & \alpha_1 |\xi|^2 ; & \quad & \overline{\xi} \cdot \varepsilon(x)\xi & \geq & \varepsilon_1 |\xi|^2, \\ \overline{\xi} \cdot \alpha(x)\xi & \geq & s(x)\overline{\xi} \cdot \kappa(x)\xi ; & & s(x) & \geq & s_1. \end{array} \tag{3.1}$$

For a vector field $\vec{u}$ we define the tangential vector field

$$\vec{u}_\varepsilon := \frac{-s}{\vec{n} \cdot \varepsilon\vec{n}} \, \vec{n} \times (\kappa(\vec{n} \times \vec{u})). \tag{3.2}$$

We see that $\vec{u}_\varepsilon\big|_\Gamma = 0$ holds if and only if $\vec{u}_\top\big|_\Gamma = 0$ holds. Thus the space X could be defined in terms of $\vec{u}_\varepsilon$ instead of $\vec{u}_\top$. For scalar ε, we have $\vec{u}_\varepsilon = s\varepsilon\vec{u}_\top$.

Instead of the normal component $u_n = \vec{n} \cdot \vec{u}$, we need here the conormal component $\vec{n} \cdot \varepsilon\vec{u}$, and we define the space Y as in (1.10).

Theorem 3.1 *Let $a_3(\vec{u}, \vec{v})$ be defined by*

$$\begin{aligned} a_3(\vec{u}, \vec{v}) \;:=\; & (\alpha \operatorname{curl} \vec{u}, \operatorname{curl} \vec{v}) + (s \operatorname{div} \varepsilon\vec{u}, \operatorname{div} \varepsilon\vec{v}) \\ & + <\operatorname{grad}_\top(\vec{n} \cdot \varepsilon\vec{u}), \vec{v}_\varepsilon> - <\operatorname{div}_\top \vec{u}_\varepsilon, \vec{n} \cdot \varepsilon\vec{v}> . \end{aligned} \tag{3.3}$$

Then a_3 is coercive over $H^1(\Omega)$.

Proof. Let a_0 be defined as in (1.5). Then, according to (1.8) and (3.1), we have

$$\begin{aligned} \operatorname{Re} a_0(\vec{u}, \vec{u}) \;\geq\; & \operatorname{Re} (s\kappa \operatorname{curl} \vec{u}, \operatorname{curl} \vec{u}) + (s \operatorname{div} \varepsilon\vec{u}, \operatorname{div} \vec{u}) \\ =\; & \operatorname{Re} (\operatorname{curl} s\kappa \operatorname{curl} \vec{u} - \varepsilon \operatorname{grad} s \operatorname{div} \varepsilon\vec{u}, \vec{u}) \\ & + \operatorname{Re} <-s\vec{n} \times (\kappa \operatorname{curl} \vec{u}), \vec{u}> + \operatorname{Re} <s \operatorname{div} \varepsilon\vec{u}, \vec{n} \cdot \varepsilon\vec{u}> . \end{aligned} \tag{3.4}$$

Now we use the formula (see [11])

$$\kappa(\vec{a} \times \vec{b}) = (\varepsilon\vec{a}) \times (\varepsilon\vec{b}). \tag{3.5}$$

This implies

$$s \operatorname{curl} \kappa \operatorname{curl} \vec{u} = \varepsilon \operatorname{grad}(s \operatorname{div} \varepsilon\vec{u}) - s(\operatorname{div} \varepsilon \operatorname{grad})(\varepsilon\vec{u}) + d_1(\vec{u}), \tag{3.6}$$

where $d_1(\vec{u})$ contains derivatives of s and ε, but only first order derivatives of $\vec{u}$, and it is linear in $\vec{u}$. We will denote a similar function below by $d_2(\vec{u})$.
From (3.4) and (3.6) we obtain

$$\begin{aligned} \operatorname{Re} a_0(\vec{u}, \vec{u}) \;\geq\; & \operatorname{Re}(-s(\operatorname{div} \varepsilon \operatorname{grad})\varepsilon\vec{u},\, \vec{u}) + \operatorname{Re}(d_1(\vec{u}),\, \vec{u}) \\ & + \operatorname{Re} < -s\vec{n} \times (\kappa \operatorname{curl} \vec{u}),\, \vec{u} > + \operatorname{Re} < s \operatorname{div} \varepsilon\vec{u},\, \vec{n} \cdot \varepsilon\vec{u} > . \end{aligned} \tag{3.7}$$

Now we use partial integration for the strongly elliptic operator $-s \operatorname{div} \varepsilon \operatorname{grad} \varepsilon$. For this purpose we need the positive selfadjoint square root δ of ε:

$$\delta \in C^1(\overline{\Omega}), \quad \delta^2(x) = \varepsilon(x) \quad \text{for all } x \in \overline{\Omega}.$$

Then we have

$$\begin{aligned} (-s(\operatorname{div} \varepsilon \operatorname{grad})\varepsilon\vec{u},\, \vec{u}) \;=\; & (s\varepsilon \operatorname{grad} \delta\vec{u},\, \operatorname{grad} \delta\vec{u}) \\ & + (d_2(\vec{u}),\, \vec{u}) - < s(\vec{n} \cdot \varepsilon \operatorname{grad})\varepsilon\vec{u},\, \vec{u} > . \end{aligned} \tag{3.8}$$

The first term on the right hand side causes the coercivity:

$$\operatorname{Re}(s\varepsilon \operatorname{grad} \delta\vec{u},\, \operatorname{grad} \delta\vec{u}) \geq s_1\varepsilon_1(\|\delta\vec{u}\|^2_{H^1(\Omega)} - \|\delta\vec{u}\|^2_{L^2(\Omega)}), \tag{3.9}$$

and there exists $\gamma_1 > 0$ such that

$$\|\delta\vec{u}\|^2_{H^1(\Omega)} \geq \gamma_1\|\vec{u}\|^2_{H^1(\Omega)} - c\|\vec{u}\|^2_{L^2(\Omega)}.$$

The terms $(d_j(\vec{u}),\, \vec{u})$ can be estimated by

$$|\,(d_j(\vec{u}),\, \vec{u})\,| \leq C\|\vec{u}\|_{H^1(\Omega)} \cdot \|\vec{u}\|_{L^2(\Omega)} \leq \eta\|\vec{u}\|^2_{H^1(\Omega)} + C_\eta\|\vec{u}\|^2_{L^2(\Omega)}$$

for any $\eta > 0$. Thus they do not disturb the coercivity.
From (3.4)–(3.9) we obtain

$$\begin{aligned} \operatorname{Re} a_0(\vec{u}, \vec{u}) \;\geq\; & \gamma\|\vec{u}\|^2_{H^1(\Omega)} - c\|\vec{u}\|^2_{L^2(\Omega)} \\ & + \operatorname{Re}\{ < -s\vec{n} \times (\kappa \operatorname{curl} \vec{u}),\, \vec{u} > + < s \operatorname{div} \varepsilon\vec{u},\, \vec{n} \cdot \varepsilon\vec{u} > \\ & \quad - < s(\vec{n} \cdot \varepsilon \operatorname{grad})\varepsilon\vec{u},\, \vec{u} > \}. \end{aligned} \tag{3.10}$$

We have to show that the boundary terms on the right hand side of (3.10) coincide up to compact terms with the negative of the boundary terms in the definition (3.3) of a_3. We denote by r_1, r_2, etc., expressions containing derivatives of ε, s and $\vec{n}$, but no derivatives of $\vec{u}$. Then for the terms $< r_j(\vec{u}), \vec{u} >$, we will have estimates similar to (2.15) above, hence these compact terms will not disturb the validity of Gårding's inequality.
From (3.5) above we obtain

$$\begin{aligned} \vec{n} \times (\kappa \operatorname{curl} \vec{u}) &= \vec{n} \times ((\varepsilon \operatorname{grad}) \times (\varepsilon\vec{u})) + r_1(\vec{u}) \\ &= (\varepsilon \operatorname{grad})(\vec{n} \cdot \varepsilon\vec{u}) - (\vec{n} \cdot \varepsilon \operatorname{grad})(\varepsilon\vec{u}) + r_2(\vec{u}). \end{aligned} \tag{3.11}$$

This gives for the boundary term

$$\begin{aligned} & <-s\vec{n} \times (\kappa \operatorname{curl} \vec{u}),\, \vec{u}> \;+\; <s \operatorname{div} \varepsilon\vec{u},\, \vec{n}\cdot\varepsilon\vec{u}> \;-\; <s(\vec{n}\cdot\varepsilon \operatorname{grad})\varepsilon\vec{u},\, \vec{u}> \\ & \quad = <s \operatorname{div} \varepsilon\vec{u},\, \vec{n}\cdot\varepsilon\vec{u}> \;-\; <s \operatorname{grad}(\vec{n}\cdot\varepsilon\vec{u}),\, \varepsilon\vec{u}> \;-\; <r_2(\vec{u}),\, \vec{u}> \\ & \quad = <\operatorname{div}_\top(s\varepsilon\vec{u})_\top,\, \vec{n}\cdot\varepsilon\vec{u}> \;-\; <\operatorname{grad}_\top(\vec{n}\cdot\varepsilon\vec{u}),\, s(\varepsilon\vec{u})_\top> \;+\; <r_3(\vec{u}),\, \vec{u}> . \end{aligned} \tag{3.12}$$

In the latter equality we wrote $\varepsilon\vec{u} = (\varepsilon\vec{u})_\top + \vec{n}(\vec{n}\cdot\varepsilon\vec{u})$ in the second term and used the definition (2.3) of $\operatorname{div}_\top$ which shows that the terms $<s\partial_n(\varepsilon\vec{u})_n,\, (\varepsilon\vec{u})_n>$ cancel.
Now the form of the boundary terms achieved in (3.12) is already similar to those in the definition (2.7) of a_1. In fact, for $s = \varepsilon = 1$, they coincide with those in (2.7). We could have defined a_3 using the boundary terms from (3.12) which are simpler in form than those of (3.3), and they contain only tangential derivatives, too. We would not consider this satisfactory, however, because the tangential components $(\varepsilon\vec{u})_\top$ appearing in (3.12) also contain the normal component of $\vec{u}$. Thus it is not true in general that the boundary terms in (3.12) vanish on the space X.
By definition of $\vec{u}_\varepsilon$ and (3.5) we have

$$\vec{u}_\varepsilon = \frac{-s}{\vec{n}\cdot\varepsilon\vec{n}}\,\vec{n} \times ((\varepsilon\vec{n}) \times (\varepsilon\vec{u})) = \frac{-s}{\vec{n}\cdot\varepsilon\vec{n}}\,(\vec{n}\cdot\varepsilon\vec{u})\varepsilon\vec{n} + s\varepsilon\vec{u}. \tag{3.13}$$

This gives for the first boundary term in (3.3) for $\vec{u} = \vec{v}$

$$<\operatorname{grad}_\top(\vec{n}\cdot\varepsilon\vec{u}),\, \vec{u}> = <\frac{-s}{\vec{n}\cdot\varepsilon\vec{n}}(\vec{n}\cdot\varepsilon \operatorname{grad})(\vec{n}\cdot\varepsilon\vec{u}),\, \vec{n}\cdot\varepsilon\vec{u}> \;+\; <s \operatorname{grad}(\vec{n}\cdot\varepsilon\vec{u}),\, \varepsilon\vec{u}> .$$

From (3.13) follows for the term $\operatorname{div}_\top \vec{u}_\varepsilon$:

$$\begin{aligned} \operatorname{div}_\top \vec{u}_\varepsilon &= \operatorname{div} \vec{u}_\varepsilon + r_4(\vec{u}) \\ &= \frac{-s}{\vec{n}\cdot\varepsilon\vec{n}} \operatorname{div}((\vec{n}\cdot\varepsilon\vec{u})\varepsilon\vec{n}) + s \operatorname{div} \varepsilon\vec{u} + r_5(\vec{u}) \\ &= \frac{-s}{\vec{n}\cdot\varepsilon\vec{n}}\,(\vec{n}\cdot\varepsilon \operatorname{grad})(\vec{n}\cdot\varepsilon\vec{u}) + s \operatorname{div} \varepsilon\vec{u} + r_6(\vec{u}). \end{aligned} \tag{3.14}$$

Hence the two boundary terms in (3.3) together give

$$\begin{aligned} & <\operatorname{grad}_\top(\vec{n}\cdot\varepsilon\vec{u}),\, \vec{u}_\varepsilon> \;-\; <\operatorname{div}_\top \vec{u}_\varepsilon,\, \vec{n}\cdot\varepsilon\vec{u}> \\ & \quad = <s \operatorname{grad}(\vec{n}\cdot\varepsilon\vec{u}),\, \varepsilon\vec{u}> \;-\; <s \operatorname{div} \varepsilon\vec{u},\, \vec{n}\cdot\varepsilon\vec{u}> \;+\; <r_7(\vec{u}),\, \vec{u}> , \end{aligned} \tag{3.15}$$

and this coincides with the negative of (3.12) up to compact terms. Therefore, taking (3.10), (3.12) and (3.15) together, we obtain the desired Gårding inequality for a_3. ∎

4 Polyhedra and piecewise smooth domains

In this section we want to show that all previous theorems remain true for piecewise smooth domains.

By a piecewise smooth domain we mean here the image of a polyhedron in $\mathbb{R}^3$ under a $C^{1,1}$ mapping. The statement needs some explanation, because on a piecewise smooth domain the tangential and normal components of even a smooth vector field are in general discontinuous and therefore the tangential derivatives appearing in the definitions of the various bilinear forms need to be explained. Of course, also the proofs as given above will not work, because one of the main tools, namely the extension of the normal vector field $\vec{n}$ to a neighborhood of Γ, is in general not available.

The piecewise smooth boundary Γ is, however, composed of smooth ($C^{1,1}$) faces Γ^j, $j = 1, \ldots, J$: $\Gamma = \bigcup_{j=1}^J \overline{\Gamma^j}$, such that $\Gamma \setminus \bigcup \Gamma^j$ is the union of all corners and edges of Γ. On each face Γ^j, the normal $\vec{n}$ is Lipschitz continuous and can be extended to a neighborhood of Γ^j. Thus on each face separately, the quantities needed in the statements of the theorems make sense. For example, in the definition (2.7) of $a_1(\vec{u}, \vec{v})$, we now interpret $<\operatorname{grad} u_n, \vec{v}_\top>$ as $\sum_{j=1}^J <\operatorname{grad} u_n, \vec{v}_\top>_j$, where $<\cdot, \cdot>_j$ denotes the extension of the L^2 scalar product on Γ^j:

$$<\operatorname{grad} u_n, \vec{v}_\top>_j := \int_{\Gamma^j} \operatorname{grad}_\top u_n \cdot \overline{\vec{v}_\top}\, ds. \tag{4.1}$$

The first Green formulas (1.6), (1.7), (1.8) are, of course, valid for any Lipschitz domain. The only formula that is definitely not true in general is the formula (2.5) for partial integration on the boundary.

Instead of repeating the proofs of all the theorems for piecewise smooth domains, we present a stronger version of Theorem 2.1 for the case of a polyhedron and leave its generalization to $C^{1,1}$ images of polyhedra as well as the generalizations of Theorems 2.4 and 3.1 to the reader.

If Ω is a polyhedron, then the faces Γ^j are subsets of planes. Therefore the normal on each Γ^j is constant.

Theorem 4.1 *Let Ω be a polyhedron and let $a_4(\vec{u}, \vec{v})$ be defined by*

$$\begin{aligned} a_4(\vec{u}, \vec{v}) \;&:=\; (\operatorname{curl} \vec{u}, \operatorname{curl} \vec{v}) + (\operatorname{div} \vec{u}, \operatorname{div} \vec{v}) \\ &\quad + \sum_{j=1}^J \left\{ <\operatorname{grad}_\top u_n, \vec{v}_\top>_j - <\operatorname{div}_\top \vec{u}_\top, v_n>_j \right\} . \end{aligned} \tag{4.2}$$

Then for all $\vec{u}, \vec{v} \in H^1(\Omega)$

$$a_4(\vec{u}, \vec{v}) = (\operatorname{grad} \vec{u}, \operatorname{grad} \vec{v}) . \tag{4.3}$$

Proof. The right hand side of (4.3) is continuous on $H^1(\Omega)$, and the left hand side is, according to (4.1), defined by continuous extension from the case of smooth functions. Therefore it suffices to show (4.3) for $\vec{u}, \vec{v} \in C^2(\overline{\Omega})$. Since the Green formulas (1.8) and (2.12) hold and the boundary data are continuous on each face Γ^j, we obtain

$$\begin{aligned} a_4(\vec{u}, \vec{v}) &= (-\Delta \vec{u}, \vec{v}) - <\vec{n} \times \operatorname{curl} \vec{u} - \operatorname{grad}_\top u_n, \vec{v}_\top> + <\operatorname{div} \vec{u} - \operatorname{div}_\top \vec{u}_\top, v_n> \\ &= (\operatorname{grad} \vec{u}, \operatorname{grad} \vec{v}) - <\partial_n \vec{u}, \vec{v}> \\ &\quad - <\vec{n} \times \operatorname{curl} \vec{u} - \operatorname{grad}_\top u_n, \vec{v}_\top> + <\operatorname{div} \vec{u} - \operatorname{div}_\top \vec{u}_\top, v_n> \;. \end{aligned} \tag{4.4}$$

Now on each face Γ^j, the normal $\vec{n}$ is a constant vector. Therefore on Γ^j (compare (2.4), (2.9))

$$\operatorname{div}_\top \vec{u}_\top = \operatorname{div} \vec{u} - \partial_n u_n$$

and

$$\operatorname{grad} u_n = \vec{n} \times \operatorname{curl} \vec{u} + \partial_n \vec{u}\,,$$

hence with $\vec{n} \cdot \partial_n \vec{u} = \partial_n u_n$ on Γ^j we obtain

$$\begin{aligned} -(\vec{n} \times \operatorname{curl} \vec{u} - \operatorname{grad}_\top u_n) \cdot \overline{\vec{v}_\top} &= \partial_n \vec{u} \cdot \overline{\vec{v}_\top} \;=\; \partial_n \vec{u} \cdot \overline{\vec{v}} - (\partial_n u_n) \overline{v_n} \\ &= \partial_n \vec{u} \cdot \overline{\vec{v}} - (\operatorname{div}_\top \vec{u} - \operatorname{div} \vec{u}) \overline{v_n}\,. \end{aligned}$$

Therefore the boundary terms cancel on each Γ^j, and thus (4.4) implies (4.3). ∎

The identity (4.3) implies of course coercivity:

$$a_4(\vec{u}, \vec{u}) \;=\; \|\vec{u}\|^2_{H^1(\Omega)} - \|\vec{u}\|^2_{L^2(\Omega)}. \tag{4.5}$$

Since the bilinear form a_4 coincides with

$$a_0(\vec{u}, \vec{v}) = \;(\operatorname{curl} \vec{u},\, \operatorname{curl} \vec{v}) \;+\; (\operatorname{div} \vec{u},\, \operatorname{div} \vec{v})$$

on the subspaces X and Y of $H^1(\Omega)$, one obtains as a corollary that a_0 is coercive over X and Y for every polyhedron Ω (and then also for every piecewise smooth domain Ω). One must be careful, however, not to mistake this coercivity result for a regularity result. It is, in general, for polyhedral Ω, not true that every distribution $\vec{u} \in L^2(\Omega)$ for which $\operatorname{curl} \vec{u} \in L^2(\Omega)$ and $\operatorname{div} \vec{u} \in L^2(\Omega)$ hold and either $\vec{u}_\top = 0$ or $\vec{n} \cdot \vec{u}$ holds on Γ (so that

$$a_4(\vec{u}, \vec{u}) = a_0(\vec{u}, \vec{u}) = \|\operatorname{curl} \vec{u}\|^2_{L^2(\Omega)} + \|\operatorname{div} \vec{u}\|^2_{L^2(\Omega)}$$

is defined), is contained in $H^1(\Omega)$. If one denotes the Hilbert spaces of these distributions by $H(\operatorname{div}) \cap H_0(\operatorname{curl})$ and $H_0(\operatorname{div}) \cap H(\operatorname{curl})$, respectively (see [9]), then the coercivity of a_4 implies that X (with the $H^1(\Omega)$ norm) is a closed subspace of $H(\operatorname{div}) \cap H_0(\operatorname{curl})$ and Y is a closed subspace of $H_0(\operatorname{div}) \cap H(\operatorname{curl})$, and that the two norms are equivalent on these subspaces, but in general these are genuine subspaces of infinite codimension due to edge singularities.

The situation is analogous to the well-known fact (see [10]) that the quadratic form $\|\Delta u\|^2_{L^2(\Omega)}$ is coercive over $H^2(\Omega)$ for every polyhedron and every convex domain in $\mathbb{R}^n$, whereas the corresponding H^2 regularity result holds for convex and smooth domains, but not for general polyhedra.

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