

A PROOF OF THE RIEMANN HYPOTHESIS

enabled by

**a new integral and series representation
of the meromorphic Zeta function occurring in the
symmetrical form of the Riemann functional equation**

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Dedicated to my wife Vibhuta
on the occasion of her 62th birthday, August 25, 2023 (*)

Abstract

For $s \neq \nu, \nu \in \mathbb{Z}$, Riemann's meromorphic Zeta function

$$\xi^*(s) := \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \int_1^\infty \psi(x^2) [x^s + x^{1-s}] \frac{dx}{x} - \frac{1}{2} \frac{1}{s(1-s)} = \xi^*(1-s)$$

is represented in the form

$$\xi^*(s) = \frac{\zeta(s) \sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s) \sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s} \right] - 2 \sum_{n=0}^{\infty} b_{2n} \left(s - \frac{1}{2}\right)^{2n}$$

with

$$b_{2n} := \int_1^\infty \Phi(x) \left[\sum_{n=0}^{\infty} \frac{\log^{2n}(x)}{(2n)!} \right] \frac{dx}{\sqrt{x}} \quad \text{and} \quad \Phi(x) := \sum_{n=1}^{\infty} (e^{-2\pi n x} - e^{-\pi n^2 x^2}).$$

The non-trivial zeros $\left\{s_n = \frac{1}{2} + it_n\right\}$ of the Zeta function are characterized by the identity of two convergent series representations in the form

$$\sum_{n=0}^{\infty} (-1)^n b_{2n} t_n^{2n} = \frac{1}{2\pi} \sum_{n=0}^{\infty} (-1)^n \zeta(2n) \left[\frac{4n-1}{(2n-\frac{1}{2})^2 + t_n^2} \right],$$

which do not allow negative values $t_n^{2n} < 0$. The correspondingly defined entire Zeta function $\xi^{**}(s) := \sin(\pi s) \xi^*(s)$ can be represented in the form

$$\xi^{**}(s) = \xi^*(s) \sin(\pi s) = (1-s) \frac{\sin(\pi s)}{\pi s} \left[\pi^{-\frac{s}{2}} \frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s} \right] \zeta(s) \quad (**).$$

In the critical stripe the term in bracket is the Mellin transform of a Kummer function

$$M \left[{}_1F_1 \left(\frac{1}{2}; \frac{3}{2}, -\pi x^2 \right) \right] (s) = \pi^{-\frac{s}{2}} \frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s}, \quad 0 < \operatorname{Re}(s) < 1 \quad (***) .$$

(*) error correction in December 2024: the term (**)

(**) the representation is in line with the concept of „a self-adjoint operator with transform $\bar{\xi}(s)$ “, (EdH) 10.3. The concept is also in line with the proposed Kummer function based Zeta function theory and a related alternatively proposed two-semicircle method to the Hardy-Littewood (major/minor arcs based) circle method in (BrK).

(***) the Kummer function ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}, z\right)$ is accompanied by the product representation ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}, z\right) = \frac{\sqrt{\pi}}{2} e^{\frac{z}{2}} \prod \left(1 - \frac{z}{z_n}\right) e^{z/z_n}$ with only complex valued zeros with $\operatorname{Re}(z_n) > 1/2$ and imaginary parts lying in the horizontal stripes $(2n-1)\pi < |\operatorname{Im}(z_n)| < 2\pi n, n \in \mathbb{N}$.

1. Notations and Main Theorem

For the notations we refer to (EdH). The baseline function for the Zeta function theory is given by $\psi(x) := \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, (EdH) 1.7. It is related to Jacobi's functional equation of the theta function ϑ enabling the symmetrical form of Riemann's functional equation in the form, (EdH) 1.7,

$$\xi^*(s) := \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \int_1^{\infty} \psi(x^2) [x^s + x^{1-s}] \frac{dx}{x} - \frac{1}{2} \frac{1}{s(1-s)} = \xi^*(1-s).$$

Riemann's related entire Zeta function is given by $\xi(s) = \frac{s}{2}(s-1)\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s)$, (EdH) 1.8. Alternatively to $\psi(x^2)$ we shall apply the function

$$\Phi(x) := \varphi(x) - \psi(x^2) := \sum_{n=1}^{\infty} \Phi_n(x) := \sum_{n=1}^{\infty} (e^{-2\pi n x} - e^{-\pi n^2 x^2}), \quad x \geq 1 \quad (*).$$

The main result of our paper is an alternative $\xi^*(s)$ –function representation with three $s \leftrightarrow (1-s)$ symmetric summands in the form

$$\xi^*(s) = \frac{\zeta(s) \sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s) \sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s} \right] - 2 \sum_{n=0}^{\infty} b_{2n} \left(s - \frac{1}{2}\right)^{2n}$$

with

$$b_{2n} := \int_1^{\infty} \Phi(x) \left[\sum_{n=0}^{\infty} \frac{\log^{2n}(x)}{(2n)!} \right] \frac{dx}{\sqrt{x}}.$$

A correspondingly defined entire Zeta function $\xi^{**}(s)$ is built by multiplication of $\xi^*(s)$ with $\frac{\sin(\pi s)}{\pi}$ (**), i.e.

$$\xi^{**}(s) = \xi^*(s) \sin(\pi s) = (1-s) \frac{\sin(\pi s)}{\pi s} \left[\pi^{-\frac{s}{2}} \frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s} \right] \zeta(s).$$

The term $(1-s)$ corresponds to the principle term of the Riemann density function $J(x)$ (***); the term $\frac{\sin(\pi s)}{\pi s}$ is an entire function of order one and order type $\sigma = \pi/2$ with $\frac{\sin(\pi s)}{\pi s} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right) = \prod_{n=-\infty}^{\infty} \left(1 - \frac{s}{n}\right) e^{s/n}$, (LeB) p. 32. For $0 < \text{Re}(s) < 1$ the term in bracket is the Mellin transform of the Kummer function ${}_1F_1\left(\frac{1}{2}; \frac{3}{2}, -\pi x^2\right)$ (****) i.e., *formally* with $\omega(x) := \sum_{n=1}^{\infty} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}, -n^2 x^2\right)$ the zeta function $\zeta(s)$ can be represented in the form $\zeta(s) := \frac{1-s}{\Gamma\left(1+\frac{s}{2}\right)} \int_0^{\infty} x^s \omega(x) \frac{dx}{x}$ (*****). For further details including the non-negative zeros of the Digamma function we refer to the concept of „telescoping product representations“ in (BrK).

(*) Note: $\Phi_n(x) \geq 0$ for $n \geq 2, x \geq 1$; $\Phi_1(x) < 0$ for $1 \leq x < 2$; $\Phi_1(2) = 0$; $\Phi_1(x) > 0$ for $x > 2$. Putting $\Phi_{1,2}(x) := -\Phi_1(x)$, $\Phi_{0,1}^*(x) := -\Phi_1\left(\frac{1}{x}\right)$, for $1 \leq x < 2$, $\Phi_{1,2}^*(x) = \Phi_{0,1}^*(x) = 0$ for $x \geq 2$, $\Phi_{2,\infty}^*(x) := \Phi(x)$ for $x \geq 2$, $\Phi_{2,\infty}^*(x) = 0$ for $x < 2$, the three terms of the sum $\Phi_{0,1}^*(x) + \Phi_{1,2}^*(x) + \Phi_{2,\infty}^*(x) > 0$ have the disjoint domains $0 < x < 1$, $1 \leq x < 2$, $2 \leq x < \infty$; we note Riemann's density function connection between $\zeta(s)$ and the primes given by $J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s}$, ($a > 1$), which is zero for $0 \leq x < 2$, (EdH) 1.11;

$$\varphi(x) = \sum_{n=1}^{\infty} e^{-2\pi n x} = \frac{1}{1-e^{-2\pi x}} = \frac{1}{2} \frac{e^{-\pi x}}{\sinh(\pi x)} \quad \text{and} \quad \int_0^{\infty} x^{2m} \frac{e^{-\pi x}}{\sinh(\pi x)} \frac{dx}{x} = \frac{|B_{2m}|}{2m}, \quad \int_0^{\infty} x^{2m} \frac{e^{-\pi x}}{\cosh(\pi x)} \frac{dx}{x} = (1-2^{1-2m}) \frac{|B_{2m}|}{2m}, \quad (\text{GrI}) 3.552$$

(**) Riemann built his famous power series representation of his entire Zeta function $\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) (s-1)\zeta(s)$ by multiplication of $\xi^*(s) = \int_1^{\infty} \psi(x^2) [x^s + x^{1-s}] \frac{dx}{x} - \frac{1}{2} \frac{1}{s(1-s)}$ with $s(1-s)$ to govern the two poles of the term $-\frac{1}{2} \frac{1}{s(1-s)} = -\frac{1}{2} \frac{1}{s(1-s)} = -\frac{1}{2} \left[\frac{1}{s} + \frac{1}{1-s} \right]$; multiplication of this term with $\sin(\pi s)$ gives $\frac{1}{2} \frac{\sin(\pi s)}{s(1-s)} = \sin\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{2}s\right) \left[\frac{1}{s} + \frac{1}{1-s} \right] = \cos\left(\frac{\pi}{2}s\right) \left[\frac{\sin\left(\frac{\pi}{2}s\right)}{s} + \frac{\sin\left(\frac{\pi}{2}(1-s)\right)}{1-s} \right]$; see also (EdH) 10.3 „A self-adjoint operator with transform $\xi(s)$ “, and (EdH) 10.5 „ $\frac{2\xi(s)}{s(s-1)}$ as a transform“.

(***) For Riemann's density function $J(x)$ the term $\log(s-1)$ gives $\text{li}_1(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{d[\log(s-1)]}{s} x^s ds$ ($a > 1$), (EdH) 1.14.

(****) the Mellin transform of the Kummer function in the critical stripe is given by $M\left[{}_1F_1\left(\frac{1}{2}; \frac{3}{2}, -\pi x^2\right)\right](s) = \pi^{-\frac{s}{2}} \frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)}$, (GrI) 7.612;

${}_1F_1\left(\frac{1}{2}; \frac{3}{2}, x\right) \sim \frac{e^x}{\sqrt{\pi x}}$ as $x \rightarrow \infty$, (OIF) p. 257); the zeros $z_\nu, \nu \in \mathbb{Z} - \{0\}$, of ${}_1F_1\left(\frac{1}{2}; \frac{3}{2}, z\right)$ are all simple, complex valued with $\text{Re}(z_\nu) > 1/2$, and $(2n-1)\pi < |\text{Im}(z_\nu)| < 2\pi n, n \in \mathbb{N}$, (SeA); The related product representation is given by ${}_1F_1\left(\frac{1}{2}; \frac{3}{2}, z\right) = \frac{\sqrt{\pi}}{2} e^{\frac{z}{2}} \prod \left(1 - \frac{z}{z_n}\right) e^{z/z_n}$, (BuH) p.184; we note that the „density“ properties of the zeros are very much related to the negative zeros w_n of the Digamma function accompanied by the additional asymptotics $y_n := n + w_n \sim \ln(n)^{-1}$; the latter property implies that the negative zeros of the Digamma function fulfill the Kadec-1/4 condition, enabling the definition of non-harmonic Fourier series based Riesz bases and related Paley-Wiener functions, (BrK), (LeN), (YoR); regarding Riemann's method deriving the formula for his prime number density function $J(x)$, (EdH) 1.13.

(*****) $M[-\text{sh}'(s)] = sM[h(s)]; M[(\text{sh})'(s)] = (1-s)M[h(s)];$ (TiE) 2.12.6: $(s-1)\zeta(s) = e^{bs} \frac{1}{\Gamma\left(1+\frac{s}{2}\right)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$, with $b := \log(2\pi) - 1 - \gamma/2$.

Main Theorem: For $s \neq \nu, \nu \in \mathbb{Z}$, it holds

$$\xi^*(s) = \frac{\zeta(s) \sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s) \sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s} \right] - 2 \sum_{n=0}^{\infty} b_{2n} \left(s - \frac{1}{2}\right)^{2n}$$

and

$$b_{2n} = \int_1^{\infty} \Phi(x) \left[\sum_{n=0}^{\infty} \frac{\log^{2n}(x)}{(2n)!} \right] \frac{dx}{\sqrt{x}}.$$

In proving the Main Theorem the essential step (which is proven in the next section) is

Lemma MT: For $s \neq \nu, \nu \in \mathbb{Z}$, it holds

$$-\frac{1}{2} \frac{1}{s(1-s)} = \frac{1}{2} \left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)} \right] + \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s} \right] - \int_1^{\infty} [x^s + x^{1-s}] \frac{1}{2} \frac{e^{-\pi x}}{\sinh(\pi x)} \frac{dx}{x}.$$

Proof of the Main Theorem: With

$$\xi^*(s) = \int_1^{\infty} \psi(x^2) [x^s + x^{1-s}] \frac{dx}{x} - \frac{1}{2} \frac{1}{s(1-s)}$$

and

$$\Phi(x) := \varphi(x) - \psi(x^2) = \sum_{n=1}^{\infty} (e^{-2\pi n x} - e^{-\pi n^2 x^2}), \quad x > 1$$

the Lemma MT gives

$$\xi^*(s) = - \int_1^{\infty} \Phi(x) [x^s + x^{1-s}] \frac{dx}{x} + \frac{1}{2} \left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)} \right] + \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s} \right].$$

Analogue to Riemann's approach deriving his famous power series representation for $\xi(s)$, (EdH) 1.8, (*), the first term can be represented in the form

$$- \int_1^{\infty} \Phi(x) [x^s + x^{1-s}] \frac{dx}{x} = -2 \sum_{n=0}^{\infty} b_{2n} \left(s - \frac{1}{2}\right)^{2n}.$$

Corollary: The set of non-trivial zeros $\{s_n = \frac{1}{2} + it_n\}$ of the zeta function are characterized by the identity of two convergent series representations

$$\sum_{n=0}^{\infty} b_{2n} \left(s_n - \frac{1}{2}\right)^{2n} = \frac{1}{2\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{\zeta(2n)}{2n-z_n} + \frac{\zeta(2n)}{(2n-1)+s_n} \right]$$

resp.

$$\sum_{n=0}^{\infty} (-1)^n b_{2n} t_n^{2n} = \frac{1}{2\pi} \sum_{n=0}^{\infty} (-1)^n \zeta(2n) \left[\frac{4n-1}{(2n-\frac{1}{2})^2 + t_n^2} \right].$$

Note: In case there would exist a negative value $t_n^{2n} < 0$, the affected term on the left side changes its sign, while the corresponding term on the right side would not.

(*) $[x^s + x^{1-s}] = 2\sqrt{x} \left[\cosh\left(s - \frac{1}{2}\right) \log x \right]$ and $\cosh(y) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!}$ with $y := \left(s - \frac{1}{2}\right) \log x$.

2. Proof of the Lemma MT

With

$$\varphi(x) = \frac{1}{2} \frac{e^{-\pi x}}{\sinh(\pi x)} = \frac{1}{e^{2\pi x} - 1} = \sum_{n=1}^{\infty} e^{-2\pi n x}, \quad x > 1$$

the Lemma MT takes the form

Lemma MT: For $s \neq \nu$, $\nu \in \mathbb{Z}$, it holds

$$-\frac{1}{2} \frac{1}{s(1-s)} = -\int_1^{\infty} [x^s + x^{1-s}] \varphi(x) + \frac{1}{2} \left[\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} + \frac{\zeta(1-s)}{\cos(\frac{\pi}{2}s)} \right] + \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s} \right].$$

Proof:

As $\frac{1}{s-1} + \frac{1}{-s} = \frac{1}{s(s-1)}$ the Lemma MT is a consequence of the integral and series representations as provided in (MiM) in section 4 (*) in the form

$$\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} = \frac{1}{s-1} - \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n)}{2n-s} + \int_1^{\infty} x^{1-s} \frac{e^{-\pi x}}{\sinh(\pi x)} \frac{dx}{x}$$

resp.

$$\frac{\zeta(1-s)}{\sin(\frac{\pi}{2}(1-s))} = \frac{1}{-s} - \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n)}{(2n-1)+s} + \int_1^{\infty} x^s \frac{e^{-\pi x}}{\sinh(\pi x)} \frac{dx}{x}.$$

Note (Gr1) 3.552, 9.521: For $\text{Re}(s) > 0$ it holds

$$\begin{aligned} \int_0^{\infty} x^s \varphi(x) \frac{dx}{x} &= \frac{\Gamma(s)}{(2\pi)^s} \zeta(s, 1) \\ &= \frac{1}{\sin(\pi s)} \left[\sin\left(\frac{\pi}{2}s\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi n)}{n^{1-s}} + \cos\left(\frac{\pi}{2}s\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi n)}{n^{1-s}} \right] \\ &= \frac{1}{2} \left[\frac{1}{\cos(\frac{\pi}{2}s)} \sum_{n=1}^{\infty} \frac{\cos(2\pi n)}{n^{1-s}} + \frac{1}{\sin(\frac{\pi}{2}s)} \sum_{n=1}^{\infty} \frac{\sin(2\pi n)}{n^{1-s}} \right]. \end{aligned}$$

(*) (MiM) p. 7: *Special cases, 4.1 The case $c = 0$*

For the special case $c = 0$ the integral

$$\zeta(s) = -\pi^{s-1} \frac{\sin(\frac{\pi}{2}s)}{s-1} \int_0^{\infty} \frac{x^{1-s}}{\sinh^2(x)} dx, \quad \text{Re}(s) < 0 \quad (\text{MiM}) (4.1)$$

can be broken into two parts $\zeta(s) = \zeta_0(s) + \zeta_1(s)$ where

$$\zeta_1(s) = \frac{\sin(\frac{\pi}{2}s)}{s-1} + \sin\left(\frac{\pi}{2}s\right) \int_1^{\infty} x^{1-s} \frac{e^{-\pi x}}{\sinh(\pi x)} \frac{dx}{x} \quad (\text{MiM}) (4.6)$$

$$\zeta_0(s) = -\frac{2}{\pi} \sin\left(\frac{\pi}{2}s\right) \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n)}{2n-s} \quad (\text{MiM}) (4.8)$$

which are both valid for all s .

3. References

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