Asymptotics for the Kummer function of Bose plasmas

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Abstract

The asymptotic expansions for the Kummer function obtained in the study of the linear response of magnetised Bose plasmas at $T = 0$ K are presented for large and small values of its parameter, thereby displaying the function's asymptotic non-uniformity. The large parameter expansion plays a determining role in the behaviour of these Bose systems in the limit that the external magnetic field $B \to 0$. This particular expansion is generalised herein and its validity tested by determining the asymptotic expansion for the Hurwitz zeta function.

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I. INTRODUCTION

Series of the form. $S(a,b,x) = \sum_{n=0}^{\infty} x^n/\Gamma(n+1)(a+bn)$, arise in the study of the linear response of both the charged Bose gas and the relativistic boson-anti-boson (pair boson) plasma when these systems are immersed in an external magnetic field at $T = 0$ K. Hore and Frankel, hereafter referred to as HF [1], found that the longitudinal dielectric response function for the charged Bose system could be written as

$$\varepsilon^{NR}(\vec{k},\omega,T=0) = 1 + \frac{m\omega_p^2}{k^2}(H(x,\hbar^2k_z^2/2m+\hbar\omega,\hbar\omega_c) - H(x,-\hbar^2k_z^2/2m+\hbar\omega,\hbar\omega_c)) \quad , \tag{1}$$

where $x = \hbar k_\perp^2/2m\omega_c$, $k_\perp^2 = k_x^2 + k_y^2$, and

$$H(x,a,b) = e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!\,(a+bn)} \quad . \tag{2}$$

In the above $\omega_c = |e|B/mc$ and ω_p are respectively the cyclotron frequency for the charged Bose gas in a uniform magnetic field of magnitude B and the plasma frequency while ω and $\vec{k}$ are respectively the frequency and wave number of a small oscillation of the system about equilibrium. In a more recent paper, Witte, Kowalenko and Hines, WKH [2], found that the longitudinal dielectric response function for the relativistic pair boson plasma at $T = 0$ K in an external magnetic field assumed the following form

$$\varepsilon^{R}(\vec{q},\Omega,T=0) = 1 + \frac{\Omega_p^2}{2q^2}(2 - \frac{(\Omega-2E_0)^2}{u}\Phi(1,1-u/\beta;-z) - \frac{(\Omega+2E_0)^2}{v}\Phi(1,1-v/\beta;-z)) \quad , \tag{3}$$

where $u = -y - 2E_0\Omega$, $v = -y + 2E_0\Omega$, $E_0^2 = 1 + \beta/2$, $q^2 = (\hbar k/mc)^2$, $y = q_z^2 - \Omega^2$, $\Omega = \hbar\omega/mc^2$, and $z = \mu_\perp^2/2$. In addition, $\mu_\perp^2 = \hbar k_\perp^2/eB$, $\beta = 2b^2 = 2e\hbar B/m^2c^2$ and Ω_p is the dimensionless form of the plasma frequency, whilst the Φ-functions are known as Kummer or confluent hypergeometric functions. For those with the specific parameters in Eq. 3. however, we can write

$$\Phi(1,1+x;-z) = xe^{-z}\sum_{n=0}^{\infty}\frac{z^n}{n!(x+n)} \quad , \tag{4}$$

and once again, we have a similar series to Eq. 2. Hence, although not unexpected, both systems have a common mathematical structure.

In this paper, we aim to develop the asymptotic expansion for the related series in Eqs. 2 and 4 in the limit as $B(\beta) \to 0$. In so doing, we shall extend a well-known theorem in Laplace integrals to discuss the case when either b in Eq. 2 or x in Eq. 4 is negative. Although we shall utilise standard techniques, we shall present a new asymptotic expansion for the above confluent hypergeometric functions, which do not appear in the standard texts [3,4]. To establish the validity of our main result, we shall generalise it in order to develop the asymptotic expansion of the Hurwitz zeta function, which we will then show agrees with the asymptotic expansion obtained via Mellin transform techniques. In a future publication our main result will be used to obtain asymptotic expansions for more complicated but, nevertheless, related series and special functions. In this publication we shall also derive asymptotic expansions for other confluent hypergeometric functions based again on the main result presented here.

The need to develop the asymptotic expansions mentioned above is required in order to study the small B behaviour of the two systems at $T = 0$ K. Specifically, the small B-asymptotic expansions are required for the evaluation of the longitudinal modes of oscillation and the screening potentials of test particles immersed in the systems. In addition, the transverse response functions should possess similar confluent hypergeometric functions and thus the asymptotic expansions displayed here should be applicable to the study of the transverse oscillations of the two systems. We should also see that as $B \to 0$, we recover the $B = 0$ results given in HF, WKH and Kowalenko, Frankel and Hines (KFH) [5], which both HF and WKH were unable to do correctly. As a consequence, the studies of the physical properties of both systems will follow in separate publications. As an aside, we mention the recent work of Daicic and Frankel [6], who have shown that WKH and KFH introduced spurious modes in their studies of the plasmon oscillations of the relativistic pair boson gas.

The task at hand in this paper, therefore, is not to consider the interesting physical properties which these systems display but to establish the mathematical machinery for such studies to follow. We should mention that whilst our task does not require intricate mathematical techniques, it is not by any means a trivial assignment. The reader should observe that both the response functions in Eqs. 1 and 3 are highly anisotropic in form and as a consequence, the passage to the $B = 0$ isotropic result must somehow transform the exponential dependence and powers of $k_\perp^2$ in the numerator and k_z^2 in the denominator of the H- or Φ-functions into a function dependent on the variable k^2. It is, therefore, because of the requirement to establish isotropy as $B \to 0$, that we have been able to produce a different asymptotic behaviour for the above confluent hypergeometric functions.

This paper is organised as follows. In Sec. 2, we develop asymptotic expansions for $|a/b| > 1$ for the series in Eqs. 2 and 4 by utilising the method of expanding the exponential after we have written both series as Laplace integrals. Initially, our result will only be valid for $\mathrm{Re}\, a$ and $\mathrm{Re}\, b$ both greater than or less than zero in Eq. 2 but by using the technique of analytic continuation, we shall extend our result to the case where either $\mathrm{Re}\, a$ or $\mathrm{Re}\, b$ is less than zero. We shall show that series given in Eqs. 2 and 4 is a meromorphic function in a and b with simple poles occurring whenever a/b is equal to a negative integer or to zero, which is not very surprising since the series can be related to an incomplete gamma function. We also make contact with existing literature on the asymptotic expansions of incomplete gamma functions and show that the unwieldy expansion given for $\gamma(\alpha, x)$ in these papers can be reduced to the main result presented in this section. We conclude the section by presenting the small B-asymptotic expansions for the longitudinal dielectric response functions for both relativistic and non-relativistic Bose systems, which will be used in future publications to study the physical properties of these systems. In Sec. 3 we consider the case where $|a/b| < 1$, thereby revealing the non-uniformity in developing an asymptotic expansion for $S(a, b, x)$. However, we shall show that this asymptotic expansion is only significant either for extremely large magnetic fields or for a very narrow region of wavenumber space and hence, this asymptotic expansion is of very limited use when determining the physical properties of Bose systems. In the conclusion we present a brief summary and discuss the applicability of the main result.

II. ASYMPTOTICS FOR LARGE a/b

As can be inferred from the introduction, in the non-relativistic case a would correspond to $\hbar k_z^2/2m \pm \hbar\omega$ while b would be equal to $\hbar\omega_c$. Therefore, as long as the denominator does not vanish identically, we can see immediately that the ratio of a/b becomes very large in the limit as the magnetic field goes to zero. A similar scenario can be inferred for the relativistic system. Since our ultimate goal is to study the small B-behaviour of both systems in future publications, we require the asymptotic expansion of the series $S(a, b, x)$ for $a/b \gg 1$ where b may be either positive or negative. The expansion that we present will be valid for all values of x, which equals $\hbar k_\perp^2/2m\omega_c$ for the non-relativistic system and z for the relativistic case. By utilising this asymptotic expansion, we conclude the section by presenting the small magnetic field expansions of the longitudinal dielectric response functions for both the relativistic and non-relativistic charged Bose gases.

For the moment we consider the series in Eq. 2 where $\mathrm{Re}\, a$ and $\mathrm{Re}\, b$ are greater than zero. The situation where $a = 0$ is singular for $n = 0$. If, however, we were to consider the sum for $n = 1$ to ∞ with $a = 0$, then we would obtain the series for the exponential integral $Ei(z)$, which we need not consider here since it is studied extensively in Ref. [4]. We now write the series in Eq. 2 in the following form

$$S(a, b, x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)(a+bn)} = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)} \int_0^{\infty} dt\; e^{-(a+bn)t} \quad . \tag{5}$$

Noting that the series in n is absolutely convergent, we can interchange the order of the summation and the integration to obtain

$$S(a, b, x) = \int_0^{\infty} dt\; e^{-at + xe^{-bt}} \quad . \tag{6}$$

Our concern here is the development of an asymptotic series for S in terms of small b. We note that for small t, the integrand is $O(e^x)$ whilst for large t, it is $O(e^{-at})$. Hence, we can see that the dominant contribution to the asymptotics from b occurs at $t = 0$.

The integral in Eq. 6 is of the form studied by Dingle in Ch. V of Ref. [7], where $F(t) = at + \exp(e^{-bt})$ and $G(t) = 1$. As a result, we can use the method of expanding most of the exponential, whereby we retain only the linear term in $F(t)$ in the original exponential and expand the rest in rising powers of t. In general, this method is the least satisfactory of the methods discussed by Dingle. However, the method comes into its own when additional complicated factors are introduced into the integrand. Nevertheless, the method is applicable to Eq. 6 and we find

$$S(a,b,x) = e^x \int_0^\infty dt\, e^{-(a+bx)t} \Big(1 + \frac{xb^2t^2}{2!} - \frac{xb^3t^3}{3!} + \frac{xb^4t^4}{4!} + \frac{x^2b^4t^4}{(2!)^3} - \frac{xb^5t^5}{5!} - \frac{x^2b^5t^5}{2!3!} + O((bt)^6)\Big) \quad . \tag{7}$$

All the integrals in Eq. 7 yield Γ-functions and hence, we find directly

$$S(a,b,x) \sim \frac{e^x}{a+bx}\Big(1 + \frac{xb^2}{(a+bx)^2} - \frac{xb^3}{(a+bx)^3} + \frac{xb^4}{(a+bx)^4} + \frac{3x^2b^4}{(a+bx)^4} - \frac{xb^5}{(a+bx)^5} - \frac{10x^2b^5}{(a+bx)^5} + O\Big(\big(\frac{b}{a+bx}\big)^6\Big)\Big) \quad . \tag{8}$$

In Appendix A we describe the methodology for determining the coefficients of the higher order terms for this asymptotic series. Since the method employed to obtain the above expansion only requires that $a/b \gg 0$ and thus is not restricted by the value of x, save that it is positive, we stress again that x does not need to be fixed. For the Bose systems mentioned in the introduction, this means that as the magnetic field goes to zero, $k_\perp^2$ can also go to zero much faster than B and Eq. 8 will still be valid.

We should add at this stage that it is possible to generalise the above result to the situation where the factor in the denominator is $(a+bn)^s$ and to even more general cases. In applying his special technique for handling integrals, Ramanujan [8] also developed the asymptotic expansion for $(a+bn)^s$-case, and as expected, his result for $s=1$ agrees with our Eq. 8. However, he did not examine the case where $a/b < 0$, which we require to evaluate the physical properties of the Bose systems mentioned earlier. In Appendix B we show how to extend the above analysis to the series considered by Ramanujan and as a consequence, develop the asymptotic expansion of this series for large a/b.

We can see immediately that the above asymptotic series is only valid for $a/b \gg 1$. If we put $a = 0$ into the series of Eq. 5, then the series is divergent whereas no such singularity arises when $a = 0$ in Eq. 8. Hence, our asymptotic series is non-uniformly valid for different values of a/b. In the next section we shall derive the asymptotic expansion for the case where $a/b \ll 1$. As we shall see, this expansion will only become of use in Bose plasmas when the magnetic field is extremely large which is not the concern of this paper. Furthermore, we should mention that the series expansion in Eq. 5 could also be used in studying magnetised Bose plasmas but because it is only valid in an extremely small region of wavenumber-space, i.e. $|q_\perp| \ll 1$ and $|q_z| \ll |q_\perp|$ for small magnetic fields, it is of limited value when evaluating the physical properties of weakly-magnetised Bose systems. It is, however, useful when evaluating the physical properties of Bose systems in the strong field limit.

Although Eq. 8 has been determined for complex a and b with both Rea and Re$b > 0$, we can extend the method to the case where one of a and b is real and not necessarily positive and the other is imaginary. For the case of a real and positive and b imaginary, the series given in Eq. 5 can also be written in the form of Eq. 6 and hence the result given by Eq. 8 once again holds except that b is now complex. For the case of a real and negative and b imaginary, all that is required is to take a factor of -1 from the denominator and then to apply the same method used to derive Eq. 8. When this is done, we get

$$S(a,b,x) \sim \frac{-e^x}{|a| - bx}\Big(1 + \frac{xb^2}{(|a|-bx)^2} + \frac{xb^3}{(|a|-bx)^3} + \frac{xb^4}{(|a|-bx)^4} + \frac{3x^2b^4}{(|a|-bx)^4} - .....\Big) \quad , \tag{9}$$

which is identical to Eq. 8. The cases of b real and positive with a imaginary and b real and negative with a imaginary will be covered by the following extension of Eq. 8 into complex plane.

Before we can go any further, we need to consider the case where the real part of either a or b is negative for $S(a,b,x)$. To examine this case we need to extend Thm. 1 on p. 383 of Ref. [9]. For the sake of simplicity we shall consider $e^{-x}S(a,b,x)$, which is the expression that appears in the response theory of the Bose systems described in the introduction. Then for the case of either Rea and Re$b > 0$ or Rea and Re$b < 0$, we can write

$$e^{-x}S(a,b,x) = b^{-1}\int_0^\infty dt\; e^{(-\alpha_1+x)t+x(e^{-t}-1+t)} \quad . \tag{10}$$

where $\alpha_1 = a/b$. Our aim is now to seek the analytic continuation of the Laplace integral in Eq. 10 to negative real values of α_1, i.e where $|\arg(\alpha_1+x)| > \pi/2$.

To apply the theorem given on p. 383 of Nikiforov and Uvarov [9], let $f(t) = e^{x(e^{-t}-1+t)}$. For $t \to 0$,

$$f(t) = 1 + xt^2/2! - xt^3/3! + xt^4/4! + x^2t^2/8 + \dots \tag{11}$$

but for $t \to \infty$, $f(t) \sim e^{xt}$. The theorem states explicitly that $f(t)$ must be $O(t^\beta)$ as $t \to \infty$, where β is a constant. However, we shall see that the extra factor of $\exp(-xt)$ in Eq. 10 will counter the exponential growth of $f(t)$. So, if we put $\alpha_1 = re^{i\phi}$, then the integral in Eq. 10 becomes

$$I = \int_0^\infty dt\; e^{-(re^{i\phi}+x)t} f(t) \quad . \tag{12}$$

and to ensure that it is always analytic, $|\phi| \le \pi/2$, provided $x > 0$. Otherwise, the integral will only be analytic for $\cos\phi > |x|/r$.

Now consider I as a complex integral, i.e.

$$I = \int_c dt\; e^{-(\alpha_1+x)t} f(t) \quad . \tag{13}$$

Following Nikiforov and Uvarov, we evaluate Eq. 13 along the ray given by $t = \rho e^{i\theta}$, so that

$$I(\theta) = e^{-x}\int_0^\infty d\rho\; \exp(i\theta - \alpha_1\rho e^{i\theta} + xe^{-\rho e^{i\theta}}) \quad . \tag{14}$$

Eq. 14 is analytic for $|\arg(\alpha_1 e^{i\theta})| = |\phi+\theta| < \pi/2$ and for $\mathrm{Re}\,\alpha_1\rho e^{i\theta} > xe^{-\rho\cos\theta}$ as $\rho \to \infty$. The latter condition is satisfied by $|\arg\theta| < \pi/2$. To prove that $I(\theta)$ is the analytic continuation of I, we must show that the two forms are equal along the ray $\phi = -\theta/2$. This is done by letting the contour C in Eq. 13 consist of the line from the origin along the real axis, a semi-circular arc rotated by an angle θ and the line from infinity to the origin with $\arg t = \theta$. Then by Cauchy's theorem, we obtain

$$I + I_{arc} = I(\theta) \quad . \tag{15}$$

where $\phi = -\theta/2$ for I and $I(\theta)$ and the contribution along the arc is given by

$$I_{arc} = \lim_{R\to\infty} e^{-x} \int_0^\infty d\beta \, iR \exp(i\beta - rRe^{i(\beta-\theta/2)} + xe^{-Re^{i\beta}}) \quad . \tag{16}$$

Since $\beta \in [0,\theta]$, $|\beta-\theta/2| \leq |\theta/2| < \pi/2$. Thus, $\cos(\beta-\theta/2) \geq \cos\theta/2 > 0$. As a consequence, I_{arc} tends to zero as $R \to \infty$, and hence, $I = I(\theta)$ for $\phi = -\theta/2$.

We note that $f(t)$ is analytic in the region $|\arg t| < \pi$. Since $|\phi + \theta| < \pi/2$ for $I(\theta)$, $I(\theta)$ is the analytic continuation of I into the region $-\pi < \phi < \pi$. Thus, we can expand $f(\rho e^{i\theta})$ as follows

$$f(\rho e^{i\theta}) \approx 1 + x\rho^2 e^{2i\theta}/2! - x\rho^3 e^{3i\theta}/3! + \ldots \quad . \tag{17}$$

Hence, $I(\theta)$ becomes

$$\begin{aligned} I(\theta) = e^{i\theta}\Big(&\frac{1}{(\alpha+x)e^{i\theta}} + \frac{x\Gamma(3)e^{2i\theta}}{2!(\alpha+x)^3 e^{3i\theta}} - \frac{x\Gamma(4)e^{3i\theta}}{3!(\alpha+x)^4 e^{4i\theta}} + \frac{x\Gamma(5)e^{4i\theta}}{4!(\alpha+x)^5 e^{5i\theta}} \\ &+ \frac{x^2\Gamma(5)e^{4i\theta}}{8(\alpha+x)^5 e^{5i\theta}} + O((\alpha+x)^{-6})\Big) \quad , \end{aligned} \tag{18}$$

which yields the same result as Eq. 8 with $\alpha_1 = a/b$ and $b = 1$ for $|\phi| < \pi$.

To show that Eq. 8 is also valid for $\phi = \pi$ except when a/b is a negative integer, we need firstly to prove that $S(a,b,x)$ or $b^{-1}S(\alpha_1,1,x)$ can be analytically continued into the negative real half of the complex plane. This can be done by integrating Eq. 6 by parts to establish

$$(c/b - 1)S(c/b - 1, 1, x) = e^x - b^{-1}cxS(c/b, 1, x) \quad . \tag{19}$$

As a consequence of Eq. 19, we can analytically continue $S(c/b, 1, x)$ into the negative real half of the complex plane whereupon we notice that simple poles occur at the origin and for negative integers. Since $S(c/b, 1, x)$ is defined for all values in the complex plane except at the origin and for negative integers, it is, therefore, continuous for all points along the negative real axis except for negative integers and the origin. Thus, the asymptotic expansion given by Eq. 8 must be valid for values along the negative real axis except for negative integers and the origin, or else $S(c/b, 1, x)$ would be discontinuous, which would contradict our earlier remarks about the continuity of Eq. 19. It should also be mentioned that for $a/b < 0$ Eq. 8 must have a removable singularity when $x = |a/b|$ since the series representation for $S(a, b, x)$ given in Eq. 5 is always defined provided a/b is not equal to a negative integer.

At this point we should mention that $S(a, b, x)$ can also be expressed in terms of the incomplete gamma function by making the substitution $y = -x\exp(-bt)$ so that $S(a, b, x) = b^{-1}x^{-a/b}\exp(-i\pi a/b)\gamma(a/b, xe^{i\pi})$. As a consequence, we now turn our attention to the work of Temme, who has evaluated asymptotic expansions for the incomplete gamma functions in Refs. [10] to [12] by using the method of steepest descent. Temme [12] gives the following asymptotic expansion for $\gamma(a, x)$, valid only for $a \to \infty$

$$\gamma(a, x) = \frac{1}{2}\, erfc(-\eta\sqrt{a/2}) - R_a(\eta) \quad . \tag{20}$$

where $erfc(z)$ is the complementary error function, the real parameter $\eta^2/2 = \lambda - 1 - ln\lambda$, $\lambda = x/a$ and

$$R_a(\eta) = \frac{e^{-a\eta^2/2}}{\sqrt{2\pi a}} \sum_{n=0}^{\infty} C_n(\eta)a^{-n} \quad . \tag{21}$$

The coefficient $C_0(\eta)$ equals $1/(\lambda - 1) - 1/\eta$ while the other coefficients are obtained via the following recursion relation

$$\eta C_n(\eta) = \frac{d}{d\eta}C_{n-1}(\eta) + \frac{\eta}{\lambda - 1}\gamma_n' \quad , \tag{22}$$

where the numbers γ_n' are related to γ_n, which appear in the asymptotic expansion of the gamma function. The γ_n are given by

$$\Gamma(a) = \sqrt{2\pi}\, a^{a-1/2}\, e^{-a} \sum_{n=0}^{\infty} \gamma_n\, a^{-n} \quad . \tag{23}$$

Thus $\gamma_n' = (-1)^n\gamma_n$ and using No. 8.327 from Gradshteyn and Ryzhik [13], we find that the first few γ_n' are $\gamma_0' = 1$, $\gamma_1' = -1/12$, $\gamma_2' = 1/288$ and $\gamma_3' = 139/51840$.

We should make the following remark concerning the above result. It has been determined by taking the limit as $a \to \infty$ of an integral representation for the incomplete gamma function given in Dingle's book [7]. The complementary error function arises after taking the path of steepest descent. Therefore, to be consistent, Temme should express the complementary error function in terms of its large a-asymptotic expansion.

We now show that although Temme's result is extremely unwieldy, it can be simplified to yield our result for $S(a, b, x)$. First we note that Temme's a corresponds to a/b in our result, so that λ above corresponds to our $bxe^{i\pi}/a$. Then we note that the C_n in Temme's result consist of a term dependent only on powers of η^{-1} and a term dependent only on powers of $(\lambda - 1)^{-1}$. If we isolate the η^{-1} term of $C_n(\eta)$ and denote it by $C_n'(\eta)$, then by using Temme's recursion relation we find that

$$C_n'(\eta) = (-1)^{n+1}(2n-1)!!/\eta^{2n+1} \quad . \tag{24}$$

where $n \geq 1$. Hence,

$$R_a(\eta) = -\frac{e^{-a\eta^2/2}}{\eta\sqrt{2\pi a}} \left(1 + \sum_{n=1}^{\infty}(-1)^n(2n-1)!!\,(a\eta^2)^{-n}\right) = (1/2)\, erfc(-\eta\sqrt{a/2}). \tag{25}$$

where we have utilised No. 7.1.23 from Abramowitz and Stegun [4]. Hence, the η^{-1} part of Temme's C_n cancels the complementary error function and we are left with a result that depends only on inverse powers of $\lambda - 1$ or $bxe^{i\pi}/a - 1$ in our terms.

Since the η^{-1} terms occurring in all C_n can be removed, the recursion relation reduces to

$$C_n(\eta) = \sum_{j=0}^{n}\left(\frac{\lambda}{\lambda - 1}\frac{d}{d\lambda}\right)^{n-j}\left(\frac{(-1)^j\gamma_j}{\lambda - 1}\right) \quad . \tag{26}$$

Now by substituting a by a/b and λ by $bxe^{i\pi}/a$ in Temme's result, we find that

$$\gamma(a/b, xe^{i\pi}) = -\Gamma(a/b)R_{a/b}(\eta) = \frac{\Gamma\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)^{a/b+1/2}(xe^{i\pi})^{a/b}}{\sqrt{2\pi}\,(a+bx)} \times a\, e^{x+a/b}\sum_{n=0}^{\infty} C_n(\eta)(a/b)^{-n} \quad . \tag{27}$$

where the C_n are now given by Eq. 26. If we introduce the large a/b-expansion for the gamma function into Eq. 27 and use the relationship between $\gamma(a/b, xe^{i\pi})$ and $S(a, b, x)$ given above, then we find that according to Temme

$$S(a, b, x) = \frac{-e^x}{a} \sum_{n=0}^{\infty} \gamma_n (a/b)^{-n} \sum_{n=0}^{\infty} (a/b)^{-n} \sum_{j=0}^{n} \left(\frac{\lambda}{\lambda - 1} \frac{d}{d\lambda}\right)^{n-j} \left(\frac{(-1)^j \gamma_j}{\lambda - 1}\right) \quad . \tag{28}$$

Thus we need to show that the product of the sums yields the result given by Eq. 8. After evaluation of some of the terms in the second sum in Eq. 28, $S(a, b, x)$ becomes

$$\begin{aligned} S(a, b, x) = & \left(\frac{e^x}{a + bx}\right)\left(\sum_{n=0}^{\infty} \gamma_n (a/b)^{-n}\right)\Big[1 + \frac{b^2 x}{(a + bx)^2} + \frac{3b^4 x^2}{(a + bx)^4} - \frac{b^3 x}{(a + bx)^3} + \\ & \frac{15b^6 x^3}{(a + bx)^6} - \frac{10b^5 x^2}{(a + bx)^5} + \frac{b^4 x}{(a + bx)^4} + \frac{b\gamma_1}{a} + \frac{b^2 \gamma_2}{a^2} + \frac{b^3 x \gamma_1}{a(a + bx)^2} - \\ & \frac{b^4 x \gamma_1}{a(a + bx)^3} - \frac{x b^5}{(a + bx)^5} + \frac{3b^5 x^2 \gamma_1}{a(a + bx)^4} + \frac{b^4 x \gamma_2}{a^2 (a + bx)^2} + \frac{b^3 \gamma_3}{a^3} + \ldots\Big] \quad . \end{aligned} \tag{29}$$

It can be clearly seen that our main result given by Eq. 8 appears in Temme's expansion directly above but there are also additional terms, not appearing in our result. We must, therefore, show that these additional terms cancel one another. The terms in Eq. 29, which appear in our Eq. 8, all come from the part of C_n related to C_0. That is Eq. 8 can be written as

$$-a\, e^{-x} S(a, b, x) = \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^{-n} \left(\frac{\lambda}{\lambda - 1} \frac{d}{d\lambda}\right)^n \frac{1}{\lambda - 1} \quad . \tag{30}$$

Hence, to show that the remaining terms in Temme's expansion cancel, we need to prove that

$$\sum_{n=0}^{\infty} \gamma_n \left(\frac{a}{b}\right)^{-n} \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^{-n} \sum_{j=0}^{n} \left(\frac{\lambda}{\lambda - 1} \frac{d}{d\lambda}\right)^{n-j} \frac{(-1)^j \gamma_j}{\lambda - 1} = \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^{-n} \left(\frac{\lambda}{\lambda - 1} \frac{d}{d\lambda}\right)^n \frac{1}{\lambda - 1} \quad . \tag{31}$$

By isolating powers of b/a it can be shown that the above holds for small powers and thus by induction, it can be shown that the above statement is equivalent to proving

$$\sum_{m=0}^{n} (-1)^{n-m} \gamma_m \gamma_{n-m} = 0 \quad . \tag{32}$$

Eq. 32 can be shown quite easily for odd values of n and holds for $n = 2$ and $n = 4$ using the values of γ_n given by No. 6.1.37 in Abramowitz and Stegun [4]. Higher values of γ_n are difficult to evaluate as demonstrated on p. 414 of Morse and Feshbach [14], so we shall use the following result to prove Eq. 32 for large a/b

$$\Gamma(a/b)^{-1} = \sqrt{\frac{a}{2\pi b}}\, e^{a/b}\Big(\frac{a}{b}\Big)^{-a/b} \sum_{n=0}^{\infty}(-1)^n\gamma_n\Big(\frac{a}{b}\Big)^{-n} \quad . \tag{33}$$

Multiplying Eq. 33 by Eq. 23 with a set equal to a/b we get

$$\Big(\sum_{n=0}^{\infty} \gamma_n\Big(\frac{a}{b}\Big)^{-n}\Big)\Big(\sum_{n=0}^{\infty} (-1)^n\gamma_n\Big(\frac{a}{b}\Big)^{-n}\Big) = 1 \quad . \tag{34}$$

Since $\gamma_0^2 = 1$, we see that all powers of $(a/b)^{-n}$ must vanish and thus we finally arrive at Eq. 32.

We should add that if the additional terms appearing in Temme's expansion, as given by Eq. 29, did not cancel, and that since a is equal to $\pm\hbar^2k_z^2/2m + \hbar\omega$ for the magnetised charged Bose and a/b equals $-y \pm 2E_0\Omega$ for its relativistic analogue, then $S(a,b,x)$ would become anisotropic, which, in turn, would defeat the aim of this paper as we have outlined in the introduction. In addition, in Appendix B we have shown how to generalise $S(a,b,x)$ to the series $S_s(a,b,x)$ considered by Ramanujan [8]. We could take Temme's expansion above and obtain the equivalent of $S_s(a,b,x)$. If the additional terms remain, then Temme's expansion would not yield Ramanujan's result and furthermore, we would not recover the asymptotic expansion for the Hurwitz zeta function, which we have done in Appendix C by using the Ramanujan generalisation of our Eq. 8.

In Appendix A we have presented the methodology for determining the coefficients c_k in the asymptotic expansion given by Eq. 8. Denoting c_k^j as the coefficient of x^jb^k in the accompanying table, we can use our analysis of Temme's expansion to arrive at the following interesting identity

$$\Big(\frac{\lambda}{\lambda-1}\frac{d}{d\lambda}\Big)^k \frac{1}{\lambda-1} = \sum_{j=1}^{k} \frac{(-1)^k\, c_{j+k}^j\Gamma(j+k+1)\lambda^j}{(\lambda-1)^{j+k+1}} \quad . \tag{35}$$

In Appendix C we have shown that combinations of c_k^j are related to the Bernoulli numbers. Hence, utilising the above and Eq. 73, we find that

$$(\lambda-1)^{2k+1}\Big(\frac{\lambda}{\lambda-1}\frac{d}{d\lambda}\Big)^{2k}\frac{1}{\lambda-1} \overset{\lambda\to\infty}{\sim} \frac{B_{2k}}{\Gamma(2k+1)} \quad . \tag{36}$$

The analysis presented in this section shows that the asymptotic expansion of $S(a,b,x)$ or $b^{-1}S(\alpha_1,1,x)$ given by Eq. 8 is valid for all complex values of a and b provided $\alpha_1 \notin Z^- \cup \{0\}$. For $\alpha_1 = -k$, where $k = 0,1,2,\ldots$, $S(\alpha_1,1,x)$ has a simple pole with residue $x^k/\Gamma(k+1)$. These poles, however, do not arise when evaluating the response functions given by Eqs. 1 and 3 because the introduction of a Landau infinitesimal in the denominators to ensure that fluctuations vanish as $t \to -\infty$ means that only the principal values of the integrals leading to Eqs. 1 and 3 need be considered.

To conclude this section we now give the small B-expansions for both response functions using Eq. 8. For the non-relativistic case, consider $H(x,\hbar\omega - \hbar^2k_z^2/2m, -\hbar\omega_c)$ with x given immediately after Eq. 1. For the situation where $\hbar^2k_z^2/2m > \hbar\omega$, we can factor out the minus sign and then use Eq. 8 directly. For $\hbar^2k_z^2/2m < \hbar\omega$, the H-function becomes

$$H\left(x,\hbar\omega - \hbar^2k_z^2/2m, -\hbar\omega_c\right) = -(\hbar\omega_c)^{-1}e^xS\left((\hbar^2k_z^2/2m-\omega)\omega_c^{-1},1,x\right) \ , \tag{37}$$

where S is now given by Eq. 18 and ω cannot equal $\hbar k^2/2m$. Then we can use Eq. 18 to obtain the asymptotic expansions for the H-functions, which will ultimately yield the result given by Eq. 8. Hence, we find

$$H\left(x,\hbar\omega - \hbar^2k_z^2/2m, -\hbar\omega_c\right) \sim -\left[\frac{1}{\hbar^2k_z^2/2m-\hbar\omega} + \frac{k_\perp^2\omega_c}{2m(\hbar k^2/2m-\omega)^3} - \frac{k_\perp^2\omega_c^2}{2m(\hbar k^2/2m-\omega)^4} + \frac{3\hbar k_\perp^2\omega_c^2}{4m^2(\hbar k^2/2m-\omega)^5} + \mathrm{O}(\omega_c^3)\right] \ , \tag{38}$$

where we have ordered the H-function in powers of ω_c since we are interested in the small field behaviour. For $H(x,\hbar\omega+\hbar^2k_z^2/2m,-\hbar\omega_c)$ one obtains the same results except that $-\omega$ is replaced by ω and there is no minus sign outside of the corresponding square-bracketed expression. Combining the expansions, we get the following results for the longitudinal dielectric response function

$$\varepsilon^{NR}(\vec{k},\omega,T=0) \sim 1 + \frac{\omega_p^2}{(\hbar^2k^4/4m^2-\omega^2)}\left(1 + \frac{\hbar k_\perp^2\omega_c(\hbar^2k^4+12m^2\omega^2)}{8m^3(\hbar^2k^4/4m^2-\omega^2)^2} - \frac{k_\perp^2\omega_c^2(\hbar^4k^8/16m^4+3\hbar^2k^4\omega^2/m^2+\omega^4)}{k^2(\hbar^2k^4/4m^2-\omega^2)^3} + \frac{3\hbar k_\perp^2\omega_c^2(\hbar^5k^{10}/32m^5+5\hbar^2k^4\omega^3/2m^2+\omega^5)}{2mk^2(\hbar^2k^4/4m^2-\omega^2)^4} + \mathrm{O}(\omega_c^3)\right) \ . \tag{39}$$

Finally, using our asymptotic results for $S(a, b, x)$ we can give the asymptotic expansions for the specific confluent hypergeometric functions appearing in the longitudinal dielectric response function for the relativistic pair boson plasma given by Eq. 3. Utilising Eq. 4, we can write

$$\Phi(1, 1+x; -z) = xe^{-z}S(x, 1, z) \sim \frac{x}{x+z}\Big(1 + \frac{z}{(x+z)^2} - \frac{z}{(x+z)^3} + \frac{z}{(x+z)^4} + \frac{3z^2}{(x+z)^4} + O((x+z)^{-5})\Big) \quad . \tag{40}$$

As a consequence, we can utilise the above result to produce a small $B(\beta)$-expansion for the longitudinal dielectric response function for the relativistic pair boson system, which is given by

$$\varepsilon^R(\vec{q}, \Omega, T=0) \sim 1 + \frac{\Omega_p^2}{q^2}\Big[1 + \frac{\chi\Delta + 2\kappa^2}{\Delta^2 - \kappa^2} + \frac{\beta q_\perp^2(\chi(\Delta^3 + 3\kappa^2\Delta) + 2\kappa^2(\kappa^2 + 3\Delta^2))}{(\Delta^2 - \kappa^2)^3} - \frac{\beta^2 q_\perp^2(\chi(\Delta^4 + 6\kappa^2\Delta^2 + \kappa^4) + 8\kappa^2\Delta(\Delta^2 + \kappa^2))}{(\Delta^2 - \kappa^2)^4} + \frac{3\beta^2 q_\perp^4(\chi(\Delta^5 + 10\Delta^3\kappa^2 + 5\Delta\kappa^4) + 5\kappa^2(\chi^4 + 2\chi^2\kappa^2))}{(\Delta^2 - \kappa^2)^5} + O(\beta^3)\Big] \quad , \tag{41}$$

where $\Delta = q^2 - \Omega^2, \kappa = 2E_0\Omega$ and $\chi = \Omega^2 + 4E_0^2$. Noting that $E_0^2 = 1 + \beta/2$, we see that the relativistic result given by Eq. 41 possesses considerably more intricate magnetic field behaviour than the non-relativistic longitudinal dielectric response function given by Eq. 38 since both the numerators and denominators of all the terms in the former possess a more complex dependence on the magnetic field, especially when $\Omega \neq 0$.

In future publications, we shall utilise the results presented in this section to study the modes and screening potentials of both systems for small/weak magnetic fields in addition to developing expressions for the transverse dielectric response functions of both systems for electromagnetic propagation parallel and perpendicular to the magnetic field. Using these results we shall derive the dispersion relationships for the transverse modes supported by both systems. We should also mention that Bardos et al. [15] have utilised the asymptotic expansions presented here in their study of the dielectric response of a planar Bose plasma in the presence of a constant external magnetic field.

III. ASYMPTOTICS FOR SMALL a/b

In this section we demonstrate the non-uniformity of the asymptotic expansion for $S(a,b,x)$ by considering the small a/b limit which, in turn, corresponds to the large magnetic field limit ($B \to \infty$) for the Bose systems mentioned in the introduction. The expression that we derive here will once again be valid for all values of x. However, for small values of x the expansion will not be much better than the original series for $S(a,b,x)$ and hence, we shall concentrate on large values of x after we derive the general expansion.

As mentioned previously for very small values of a/b, $S(a,b,x)$ becomes singular, which is not reflected in the asymptotic expansion given by Eq. 8 since the method of expanding most of the exponential was only valid for a/b not equal to zero. Insight into the nature of the singularity can be obtained by putting b equal to a positive real number and a equal to an imaginary number. Then the series $S(a,b,x)$ can be expressed as a Fourier transform of which a part can be identified as the Fourier transform of the step-function. This, in turn, yields a delta function. The remaining part yields our expansion given by Eq. 8 for the new values of a and b. Since the delta function concerns only the point $a/b = 0$, we may infer that the asymptotic expansion given by Eq. 8 may be valid for not so large values of a/b. In fact, Temme [12] has found that his version of the asymptotic expansion for $S(a,b,x)$ is accurate to four figures for $a/b = 2$. It is, therefore, expected that our asymptotic expansion for $S(a,b,x)$ given in the previous section will become inaccurate only for very small values of a/b.

We shall begin our study of the small a/b case by considering the more general series $S_s(a,b,x)$ as defined in Appendix B but will find that the intractability of the problem will force us to consider integer values of s only. By putting $s = 1$, we shall then be able to determine the asymptotic expansion for the Bose systems mentioned in the introduction. For $a/b \ll 1$, we may write $S_s(a,b,x)$ as

$$S_s(a,b,x) = \frac{1}{a^s} + \sum_{n=1}^{\infty} \frac{x^n}{n!\,(bn)^s}\, {}_1F_0(s;;-a/nb) \quad . \tag{42}$$

Utilising the series expansion for hypergeometric functions, we can rewrite Eq. 42 as

$$S_s(a,b,x) = \frac{1}{a^s} + \frac{b^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \left(\frac{-a}{b}\right)^m \frac{x}{(s+m)\Gamma(m+1)} \int_0^{\infty} dt\, t^{s+m} e^{-t} \exp(xe^{-t}) \quad . \quad (43)$$

where we have used the integral representation of the gamma function and have integrated by parts. Making the substitution $y = xe^{-t}$ in Eq. 43, we get

$$S_s(a,b,x) = \frac{1}{a^s} + \frac{b^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-a/b)^m}{(s+m)\Gamma(m+1)} \int_0^x dy\, \ln^{s+m}(x/y)\, e^y \quad . \quad (44)$$

The integral in Eq. 44 is intractable for real values of s except for positive integer values. For s equal to a positive integer the integral can be evaluated by integrating by parts and follows from the case of putting $s = 1$ and evaluating for all values of m. Hence, we need only consider the $s = 1$ case.

For $s = 1$ and $m = 0$ the integral in Eq. 44 can be evaluated exactly to yield

$$\int_0^x dy\, \ln(x/y)\, e^y = Ei(x) - C - \ln x \quad , \quad (45)$$

where $Ei(x)$ is the exponential integral function and C is Euler's constant. For other integer values of $s+m$, we note that

$$\int_0^x dy\, \ln^{s+m}(x/y)\, e^y = (-1)^{s+m} x\, e^x \frac{\partial^{s+m}}{\partial\alpha^{s+m}} \int_0^1 ds\, (1-s)^{\alpha}\, e^{-xs} \Big|_{\alpha=0} =$$
$$(-1)^{s+m+1} \frac{\partial^{s+m}}{\partial\alpha^{s+m}} (-x)^{-\alpha}\, \gamma(\alpha+1,-x) \Big|_{\alpha=0} \quad , \quad (46)$$

where we have utilised No. 2.3.5.2 from Prudnikov et al [16]. Thus, if we put $s = k$, where k is a positive integer, then Eq. 44 can be rewritten as

$$S_k(a,b,x) = \frac{1}{a^k} + \frac{(-1)^{k+1}}{b^k\,\Gamma(k)} \sum_{m=0}^{\infty} \frac{(a/b)^m}{(k+m)\Gamma(m+1)} \frac{\partial^{k+m}}{\partial\alpha^{k+m}} (-x)^{-\alpha}\, \gamma(\alpha+1,-x) \Big|_{\alpha=0} \quad . \quad (47)$$

Although one can introduce the small x-series expansion for $\gamma(\alpha, x)$ into the above, the result would not be more useful than our original series for $S_s(a,b,x)$. Hence, we need only concentrate on the large x-expansion for Eq. 47. The asymptotic expansion for the incomplete gamma function given on p. 31 of Dingle's book [7] has to be modified according to the prescription given in Sec. 1.3 of the same reference since a Stokes discontinuity occurs at $\mathrm{ph}\, x = \pi$. Hence, the expansion we will use is

$$(-x)^{-\alpha}\,\gamma(\alpha+1,-x)=\cos(\pi\alpha)\,x^{-\alpha}\Gamma(\alpha+1)-e^{x}\sum_{l=0}^{\infty}\frac{\Gamma(l-\alpha)}{\Gamma(-\alpha)\,x^{l}}\quad . \tag{48}$$

Thus for $k=1$ and $m=0$, we obtain

$$\frac{\partial}{\partial\alpha}(-x)^{-\alpha}\,\gamma(\alpha+1,-x)\Big|_{\alpha=0}=-\ln x-C+e^{x}\sum_{l=1}^{\infty}\frac{(l-1)!}{x^{l}}\quad , \tag{49}$$

which is just the large x-asymptotic expansion of Eq. 45. For $k=1$ and $m=1$, we get

$$\frac{\partial^2}{\partial\alpha^2}(-x)^{-\alpha}\,\gamma(\alpha+1,-x)\Big|_{\alpha=0}=\ln^2 x+2C\ln x-2\zeta(2)+C^2\ -$$
$$2e^{x}\sum_{l=2}^{\infty}\frac{(l-1)!}{x^{l}}\,(\Psi(l)+C)\quad , \tag{50}$$

while for $k=1$ and $m=2$, we get

$$\frac{\partial^3}{\partial\alpha^3}(-x)^{-\alpha}\,\gamma(\alpha+1,-x)\Big|_{\alpha=0}=-\ln^3 x-3C\ln^2 x+3\pi^2\ln x-3C^2\ln x-2\zeta(3)\ +$$
$$3\pi^2 C-C^3-3C\zeta(2)+3e^{x}\sum_{l=2}^{\infty}\frac{(l-1)!}{x^{l}}\Big((\Psi(l)+C)^2-\sum_{j=1}^{l-1}\frac{1}{j^2}\Big)\quad . \tag{51}$$

In the above equations $\Psi(x)$ represents the digamma function and $\zeta(x)$ is the Riemann zeta function.

If we introduce Eqs. 49 to 51 into Eq. 47, then we obtain the following small a/b-expansion

$$S(a,b,x)=\frac{1}{a}+\frac{1}{b}\Big(e^{x}\sum_{l=1}^{\infty}\frac{(l-1)!}{x^{l}}-\ln x-C+\frac{a}{2b}\Big(\ln^2 x+2C\ln x-\pi^2/3+C^2\ -$$
$$2e^{x}\sum_{l=2}^{\infty}\frac{(l-1)!}{x^{l}}\,(\Psi(l)+C)\Big)+\frac{a^2}{6b^2}\Big(-\ln^3 x-3C\ln^2 x+3\pi^2\ln x-3C^2\ln x-2\zeta(3)\ +$$
$$3\pi^2 C-C^3-3C\zeta(2)+3e^{x}\sum_{l=2}^{\infty}\frac{(l-1)!}{x^{l}}\Big((\Psi(l)+C)^2-\sum_{j=1}^{l-1}\frac{1}{j^2}\Big)\Big)+O(\frac{a^3}{b^3})\Big)\quad . \tag{52}$$

The divergences due to the late terms for the series in inverse powers of x in the above equation can be removed by applying Dingle's theory of terminants [7] in a similar manner as carried out in Appendix A.

For very large x, which corresponds either to the small field limit or very large values of the perpendicular wavenumber for magnetised Bose systems, Eq. 52 can be written as

$$S(a,b,x) \approx a^{-1} + (e^x/b)(1/x + 1/x^2 + 2/x^3 + \ \dots) - (ae^x/b^2)(1/x^2 \ + \\ 3/x^3 + \ \dots) + (a^2e^x/b^3)(1/x^3 + \ \dots) \ + \ \dots \quad , \tag{53}$$

where the a^{-1}-term may be dropped if a is not too small. If we were to carry out a large x-expansion of Eq. 8, then we would find that the ensuing result would agree with the large x-expansion given above. Hence, we can see that the asymptotic expansion for $\Phi(1, 1+a/b; -x)$ becomes uniform when x is very large irrespective of the value of a/b. In addition, since $S(a,b,x) = a^{-1}\Phi(a/b, a/b+1; x)$, we can see that the above expansion corresponds to the dominant contribution of the complete asymptotic expansion given on p. 60 of Slater [3] or as No. 13.5.1 in Abramowitz and Stegun [4]. For very small values of x, which corresponds to the strong field limit or to small values of the perpendicular wavenumber of magnetised Bose systems, the series expansion given in Eq. 5 is sufficient but as x increases, a cross-over into the non-uniform region eventually occurs where depending on the size of a/b, the asymptotic expansion for $\Phi(1, 1+a/b; -x)$ is determined either by Eq. 8 or Eq. 52.

In this section we have been primarily concerned with displaying the important mathematical property of non-uniformity in the asymptotic expansion for $S(a,b,x)$. In so doing, we arrived at the hitherto unknown expansion for small a/b given by Eq. 52. We will require this new expansion for our future studies of the screening potentials and modes of oscillation of the Bose plasmas in a very strong magnetic field.

IV. CONCLUSION

In this paper we have derived asymptotic expansions for the specific confluent hypergeometric function, $\Phi(1, 1+a/b; -x)$, encountered in the study of the dielectric response of both relativistic and non-relativistic charged Bose plasmas in an external magnetic field. This function, which can be expressed alternatively in terms of an incomplete gamma function, also arises in the study of the dielectric response of fermion systems. We were able to derive expansions for both large and small values of the parameter a/b, thereby displaying the function's asymptotic non-uniformity. For large a/b the asymptotic expansion was derived by first expressing the generic series, $S(a, b, x)$ given by Eq. 5, as a Laplace integral under the condition that the real parts of the parameters a and b were positive. Then we utilised the method of expanding most of the exponential as described in Dingle [7] to obtain the small b-asymptotic expansion given by Eq. 8. By analytical continuation, we were able to show that this expansion was valid when the real part of one of the parameters was negative. The expansion, however, is not valid when the ratio of the parameters is equal to a negative integer or zero, where there are simple poles. The expansion also breaks down when $a = -bx$, which corresponds to a turning point for this confluent hypergeometric function.

As a consequence of Eq. 8, we were able to give the small magnetic field expansions for the longitudinal dielectric response of both charged Bose systems mentioned above. The non-relativistic and relativistic versions appear respectively as Eqs. 39 and 41, the latter displaying considerably more intricate magnetic field behaviour. We shall study the physical properties such as the screening potential and modes of oscillation for both systems separately in the future. In addition, the transverse response functions for both systems will also be shown to be dependent on the same confluent hypergeometric functions after we evaluate them by using the gauge invariant approach adopted by WKH to evaluate the polarisation tensor for the relativistic boson pair plasma. By presenting asymptotic expansions for all response functions, we shall be able to observe the transition to the free-field behaviour of these systems in the limit as the magnetic field tends to zero. We should also mention that the series studied in this paper arises when studying the linear response of charged and neutral semions as discussed in the recent work of Chakravarty [17]. Hence, the results presented here should be relevant to these systems, and what is more, to studies of all quantum plasmas regardless of the statistics.

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VI. APPENDIX A

In this appendix we describe the methodology for determining the higher order terms in the asymptotic expansion given by Eq. 8. To do so, we need to return to Eq. 6 and write it in the following form

$$S(a,b,x) = e^x \int_0^\infty dt\ e^{-(a+bx)t} \prod_{k=2}^{\infty} \exp((-bt)^k x/k!) \quad . \tag{54}$$

To develop the asymptotic series given by Eq. (8), we utilised the method of expanding the product term in Eq. 54 into a power series so that $S(a,b,x)$ could be expressed as

$$S(a,b,x) = \exp(x) \sum_{k=0}^{\infty} c_k\ I_k(a,b,x) \quad , \tag{55}$$

where the c_k are not to be confused with the C_n of Temme and the integrals $I_k(a,b,x)$ are given by

$$I_k(a,b,x) = \int_0^\infty dt\ t^k\ e^{-(a+bx)t} = \frac{\Gamma(k+1)}{(a+bx)^{k+1}} \quad . \tag{56}$$

In developing a methodology for determining the coefficients of higher order terms in $S(a,b,x)$ we need to be able to produce a power series in t from the product in Eq. 54. Since each exponential in this product can be expressed as a power series, we shall refer to each exponential with t^k in it as a k-series. For example, $\exp(xb^2t^2/2!)$ will referred to as a 2-series while $\exp(-xb^9t^9/9!)$ will be referred to as a 9-series.

We begin our study of the coefficients by examining the terms in the product of Eq. 54 that contribute to I_2, I_3 and I_4. These are very simple since only the 2-series produces a power of t^2 and the 3-series produces a power of t^3. On the other hand, the power of t^4 is produced by contributions from the 4-series and from the 2-series. We can represent these contributions by using tree diagrams as shown in Fig. 1. The leftmost number in these diagrams represents the power of t^k in $I_k(a,b,x)$. For the case of t^2 we see that the tree diagram terminates at the ordered pair $(0,2)$. Whenever zero appears in an ordered pair such as in $(0,k)$, it will mean that the contributions along a particular path of the tree have terminated. Thus, $(0,k)$ means that the contribution to I_{k_1} has terminated with a contribution from the k-series where $k \leq k_1$. For $I_4(a,b,x)$ we see that there are two paths in our diagram, one terminating at $(0,4)$, i.e. a contribution from the 4- series and the other going through $(2,2)$ and ultimately terminating at $(0,2)$. Whenever a number other than zero appears as the left element in an ordered pair such as $(2,2)$, it will not only mean that more contributions exist to produce the required power in $I_k(a,b,x)$ but that the problem has reduced to one of finding all the contributions to the right element of the ordered pair. That is in the case of $(2,2)$ we not only have a contribution from the 2-series but also must find all the contributions yielding t^2. Thus we continue the path till it ends at $(0,2)$ since only the 2-series can contribute to $I_2(a,b,x)$. Hence, the coefficient of $I_4(a,b,x)$ consists of two paths, one coming from the 4-series and the other coming from the second order term of the 2-series. Whenever there is more than one contribution from a particular series such as the two contributions from the 2-series for $I_4(a,b,x)$, we evaluate the contribution to the coefficient from this path by taking the power of the first order contribution of the series to the number of times the series appears in the path divided by the factorial of this number. Hence the contribution to the coefficient coming from the second path for $I_4(a,b,x)$ is $(x(-b)^2/2!)^2/2!$ while that from the first path yields $(x(-b)^4/4!$.

We can see the recursive nature of the coefficients by considering higher order contributions in our asymptotic expansion for $S(a,b,x)$. The coefficient for $I_5(a,b,x)$ is determined from two paths, one coming from $(0,5)$ directly and the other consisting of a path with $(2,3)$ and $(0,3)$. Recursion is visible whenever we consider paths not terminating immediately, e.g. the path with $(2,3)$ for $I_5(a,b,x)$. In such cases all we need to do is to introduce the tree diagram for the right element to complete the tree diagram. For example, to complete the diagram for the coefficient of $I_5(a,b,x)$ we introduce the tree diagram for $k=3$ along the path with (2,3). Thus the coefficient of $I_5(a,b,x)$ is given by

$$c_5(bx) = \frac{(-b)^5 x}{5!} + \frac{(-b)^2 x}{2!}\frac{(-b)^3 x}{3!} \quad . \tag{57}$$

We are now in a position to describe the construction of the tree diagrams for the higher order coefficients of our asymptotic expansion for $S(a,b,x)$. We shall consider the contruction of the tree diagram for $k=7$. The first step is to consider all the ordered pairs (i,j) where $i \leq j$ and $i+j=7$ excluding the pair with $i=1$. Then along the path containing (2,5) we place the tree diagram for $k=5$ while we place the tree diagram for $k=4$ along the path containing (3,4) as in Fig. 2. One problem, however, arises: duplication of paths occurs. For example, the path with (2,5), (2,3) and (0,3) yields the same contribution to the coefficient for $I_k(a,b,x)$ as the path with (3,4), (2,2) and (0,2). In such cases we simply disregard the second path since all paths in the final tree diagram should yield a distinct contribution to each coefficient.

As a consequence of the above, we present the values for the coefficients in Eq. 55 up to c_{10} in the table. Higher order coefficients are perhaps better evaluated by constructing a computer code based on the above algorithm either in Pascal, C or some other language with pointers. As can be seen, the coefficients are essentially polynomials in x, whose coefficients become more complicated as k increases. This makes the evaluation of the remainder in the asymptotic expansion given by Eq. 55 a difficult exercise except in the limits of $x \ll 1$ and $x \gg 1$. For $x \ll 1$ we find that $c_k \approx (-b)^k x/k!$ while for $x \gg 1$, $c_{2k} \approx (xb^2)^k/2^k k!$ and $c_{2k+1} \approx -(xb^2/2)^{k-1} x b^3/3!(k-1)!$.

For $x \ll 1$ we can estimate the remainder of the asymptotic expansion for Eq. 55 by writing it as

$$S(a,b,x) = e^x \sum_{k=0}^{N-1} \frac{c_k\,\Gamma(k+1)}{(a+bx)^{k+1}} + R_N \quad , \tag{58}$$

where the remainder, R_N, is approximately given by

$$R_N \approx x e^x \sum_{k=N}^{\infty} \frac{(-b)^k}{(a+bx)^{k+1}} = \frac{(-b)^N x e^x}{(a+bx)^N (a+b(x+1))} \quad . \tag{59}$$

The above result can be expanded further since $x \ll 1$ and $a \gg bx$. Hence, we find that

$$R_N \approx \frac{(-1)^N x b^N}{a^{N+1}} \quad . \tag{60}$$

For the case where $x \gg 1$ we can use Dingle's theory of terminants [7] to evaluate the remainder. In utilising this theory we shall assume that N is an even integer, leaving it to the reader to consider odd integer values. Then the remainder becomes

$$R_N \approx e^x \sum_{k=p}^{\infty} \Big(\frac{xb^2}{2(a+bx)^2}\Big)^k \Big(\frac{\Gamma(2k+1)}{\Gamma(k+1)\,(a+bx)} - \frac{b\Gamma(2k+2)}{3(a+bx)^2\Gamma(k)}\Big) \quad . \tag{61}$$

Using the duplication formula for the gamma function, we can write Eq. 61 as

$$R_{N=2p} \approx f(x)\,\Big(\sum_{k=p}^{\infty} \Gamma(k+1/2) z^k - \frac{2bz}{3(a+bx)} \frac{\partial}{\partial z} \sum_{k=p}^{\infty} \Gamma(k+3/2) z^k\Big) \quad . \tag{62}$$

where $z = 2xb^2/(a+bx)^2$ and $f(x) = e^x/\sqrt{\pi}(a+bx)$. By using the technique of Borel summation Eq. 62 becomes

$$\begin{aligned} R_N \approx f(x)\Big[z^p I(p-1/2) - \frac{2b}{3(a+bx)}\Big((1-p-z^{-1})z^{p-1}\Gamma(p+1/2) + \\ (z^{p-2} - 3z^{p-1}/2)\, I(p-1/2)\Big)\Big] \quad . \end{aligned} \tag{63}$$

where the integral $I(p-1/2) = \Gamma(p+1/2)\overline{\Lambda}_{p-1/2}(-z^{-1})$ and the terminant $\overline{\Lambda}_s(-x)$ is given by

$$\overline{\Lambda}_s(-x) = \frac{1}{\Gamma(s+1)}\, P\int_0^{\infty} dt\, \frac{e^{-t}\, t^s}{1-t/x} \quad . \tag{64}$$

The fact that the remainder term involves a principal value integral is an indication that the real axis for complex values of x, i.e. ph $x = 0$, is a Stokes line. Thus if we move above or below the real axis, then the asymptotic expansion for $S(a,b,x)$ will change as a result of the Stokes phenomenon.

Dingle shows that the above integral can be expressed in terms of the incomplete gamma function on p 414 of his book [7]. He also provides asymptotic expansions as well as tables of values for $\overline{\Lambda}_s(-x)$. Hence, by terminating the asymptotic expansion at a particular p-value for known values of a, b and x, one can obtain an estimate of the remainder by using Eq. 63 which will be quite accurate for large values of p and x.

VII. APPENDIX B

In this appendix we develop the large a/b-asymptotic expansion for the series considered by Ramanujan [8], which is a generalisation of the series, $S(a,b,x)$, and is given by

$$S_s(a,b,x) = \sum_{n=0}^{\infty} \frac{x^n}{n!\,(a+bn)^s} \quad , \tag{65}$$

where Re a, Re b and Re s are initially all greater than zero. To evaluate the asymptotic expansion for $a/b \gg 1$, we utilise the following result

$$(a+bn)^s = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\; t^{s-1} e^{-(a+bn)t} \quad . \tag{66}$$

Introducing Eq. 66 into Eq. 65, we find that

$$S_s(a,b,x) = \Gamma(s)^{-1} \int_0^{\infty} dt\; t^{s-1} \exp(-at + xe^{-bt}) \quad . \tag{67}$$

We can use the method of expanding the exponential once more to obtain a similar result to Eq. 7 except that now every power of t is now multiplied by by t^{s-1}. Hence, we finally get

$$S_s(a,b,x) = \frac{e^x}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{c_k \Gamma(k+s)}{(a+bx)^{k+s}} = \frac{e^x}{(a+bx)^s\Gamma(s)} \Big(\Gamma(s) + \frac{xb^2\Gamma(s+2)}{\Gamma(3)(a+bx)^2} - \frac{xb^3\Gamma(s+3)}{\Gamma(4)(a+bx)^3} + \frac{xb^4\Gamma(s+4)}{\Gamma(5)(a+bx)^4} + \frac{x^2b^4\Gamma(s+4)}{\Gamma(3)^3(a+bx)^4} - O((\frac{b}{a+bx})^5) \Big) \quad , \tag{68}$$

where the c_k are the same as those in Appendix A. By a similar argument to that presented in Sec. 2, the above result is valid for all complex values of a and b except those where a/b equals a negative integer or zero. As expected, Eq. 68 reduces to Eq. 8 when $s = 1$.

VIII. APPENDIX C

To demonstrate the validity of our main result given by Eq. 8 and its more general counterpart given in Appendix B, we shall use the latter to develop a large q-asymptotic expansion for the Hurwitz zeta function, $\zeta(z,q)$, which we write as

$$\zeta(z,q) = q^{-z}\int_0^\infty dt\; \epsilon^{-t}\sum_{n=0}^{\infty}\frac{t^n}{(1+n/q)^z\Gamma(n+1)} = q^{-z}\int_0^\infty dt\; \epsilon^{-t}S_z(1,q^{-1},t) \quad . \tag{69}$$

where Re $z > 1$. Introducing Eq. 68 into Eq. 69 yields

$$\zeta(z,q) = \frac{q^{1-z}}{z-1} + \frac{q^{-z}}{\Gamma(3)} - q^{-(z+1)}\Big(\frac{z}{\Gamma(4)} - \frac{z}{\Gamma(3)^2}\Big) - \frac{\Gamma(z+3)q^{-(z+3)}}{\Gamma(z)} \times \Big(\frac{\Gamma(2)}{\Gamma(6)} - \frac{5\Gamma(3)}{6\Gamma(5)} + \frac{\Gamma(4)}{2\Gamma(5)} - \frac{1}{16}\Big) + O(q^{-(z+5)}) \quad . \tag{70}$$

More generally, if we let c_k^j denote the coefficient of $x^j b^k$ in the table, then the above result can be represented as

$$\zeta(z,q) = \frac{q^{1-z}}{z-1} + \frac{q^{-z}}{2} + \sum_{k=0}^{\infty}\frac{\Gamma(z+2k+1)}{\Gamma(z)\, q^{z+2k+1}}\sum_{j=0}^{2k+1} c_{j+2k+3}^{j+1}\,\Gamma(j+2k+4) \quad . \tag{71}$$

In their study of the statistical mechanics of the relativistic boson pair plasma in a magnetic field Daicic et al [18] have evaluated the large z-expansion of the Hurwitz zeta function by using Mellin transform techniques. They found

$$\zeta(z,q) = \frac{q^{1-z}}{z-1} + \frac{q^{-z}}{2} + \sum_{k=0}^{\infty}\frac{\Gamma(z+2k+1)\, B_{2k+2}}{\Gamma(z)\,\Gamma(2k+3)}\, q^{-1-z-2k} \quad , \tag{72}$$

where the B_k's are the Bernoulli numbers. Since B_2 and B_4 equal $1/6$ and $-1/30$ respectively, we find that Eq. 70 agrees with Eq. 72. Hence, by equating Eq. 71 to Eq. 72, we can see the connection between combinations of the c_k^j and the Bernoulli numbers, which is

$$B_{2k} = \Gamma(2k+1)\sum_{j=1}^{2k} c_{j+2k}^{j}\,\Gamma(j+2k+1) \quad . \tag{73}$$

TABLES

TABLE I. Coefficients for Asymptotic Expansion of $S(a, b, x)$

k	c_k
0	1
1	0
2	$xb^2/2!$
3	$-xb^3/3!$
4	$(x/4! + x^2/4 \cdot 2!)\, b^4$
5	$-(x/5! + x^2/2 \cdot 3!)\, b^5$
6	$(x/6! + 5x^2/6 \cdot 4! + x^3/2 \cdot 4!)\, b^6$
7	$-(x/7! + 4x^2/3 \cdot 5! + x^3/2 \cdot 4!)\, b^7$
8	$(x/8! + 17x^2/8 \cdot 6! + 7x^3/24 \cdot 4! + x^4/16 \cdot 4!)\, b^8$
9	$-(x/9! + 41x^2/12 \cdot 7! + 137x^3/216 \cdot 5! + x^4/12 \cdot 4!)\, b^9$
10	$(x/10! + 167x^2/30 \cdot 8! + 65x^3/48 \cdot 6! + x^4/16 \cdot 4! + x^5/32 \cdot 5!)\, b^{10}$

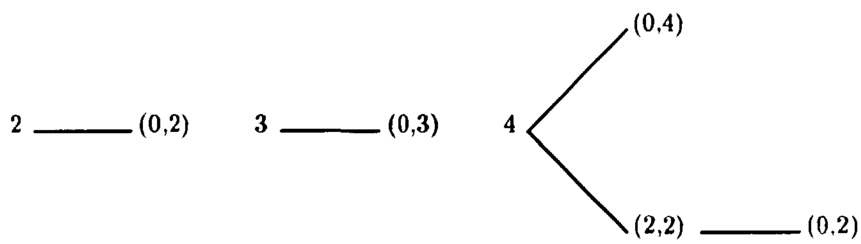

Figure 1: Tree Diagrams for c_2, c_3 and c_4

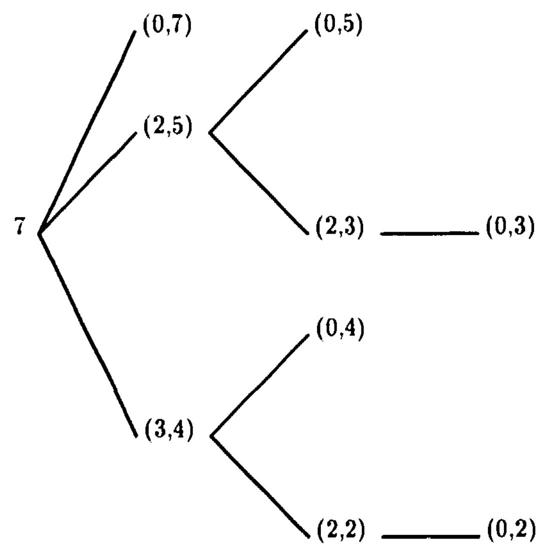

Figure 2: Construction of the tree diagram for c_7 with duplicated paths. The final result has the path ending with $(0, 2)$ omitted.