Transcendental numbers and Kummer functions

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§ 1 Introduction and Notations

The considered Kummer functions are

$$ {}_1F_1(\frac{1}{2},\frac{3}{2},-z^2) \quad {}_1F_1(1,\frac{3}{2},-z^2) \ .$$

The asymptotic of the related zeros are given in [SeA]. The exponential integral function and the non-normalized (exponential) error function are given by ([AbM] 13.6, [LeN] 9.13)

$$ Ei(-x) := \int_x^\infty \frac{e^{-t}}{-t}dt = \frac{e^{-x}}{-x}\left[\sum_{k=0}^{n}\frac{k!}{x^k}+R_n\right] \quad \left(|R_n| < \frac{n!}{x^n}\right), \qquad erf(z) := \int_0^x e^{-t^2}dt = \sum_{k=0}^\infty \frac{(-1)^k}{k!}\frac{z^{2k+1}}{2k+1} = \sum_{k=0}^\infty \frac{(\frac{1}{2})_k}{(\frac{3}{2})_k}\frac{(-z^2)^k}{k!} \ .$$

The related Dawson function is defined by

$$ F(z) := e^{-z^2}\int_0^z e^{t^2}dt = \sum_{k=0}^\infty \frac{(-1)^k 2^k}{1\cdot 3\cdot ....(2k-1)}\frac{z^{2k+1}}{2k+1} $$

Their relationships to the Kummer functions above are given by ([AbM] 7.15, [LeN] 9.13)

$$ erf(z) = x\,{}_1F_1(\frac{1}{2},\frac{3}{2},-z^2) = ze^{-z^2}\,{}_1F_1(1,\frac{3}{2},z^2) $$

$$ F(z) = z\,{}_1F_1(1,\frac{3}{2},-z^2) \ .$$

The following identities are valid ([GrI 9.212], [LeN] 2.3):

i) $$ {}_1F_1(1,\frac{3}{2},z) = e^z\,{}_1F_1(\frac{1}{2},\frac{3}{2},-z) \quad , $$

$$ {}_1F_1(\frac{1}{2},\frac{3}{2},z) = e^z\,{}_1F_1(1,\frac{3}{2},-z) = \frac{1}{2}\sum_{k=0}^\infty \frac{1}{k+1/2}\frac{x^k}{k!} = \frac{1}{2}\frac{e^x}{x}\left[1+O(|x|^{-1})\right] $$

ii) $$ {}_1F_1(\frac{1}{2},\frac{3}{2},z) + 2z\,{}_1F_1'(\frac{1}{2},\frac{3}{2},z) = e^z = {}_1F_1(a,a,z) \quad , \quad F'(z)+2zF(z)=1 $$

iii) $$ li(z) = -z\,{}_1F_1(1,1,-\log z) = {}_1F_1(1,2,-\log z) $$

iv) $$ e^z - 1 = z\,{}_1F_1(1,2,z) \quad , \quad \frac{1}{2}z^2 - 2z + 1 = {}_1F_1(-2,1,z) \ .$$

Related asymptotic are given by ([OlF] chapter 3, 1.1; chapter 12 1.1, ([AbM] 7.1.23, [LeN] 2.3)

$$ erfc(z) = 1 - erf(z) \approx e^{-z^2}\sum_{k=0}^\infty (-1)^k \frac{1\cdot 3\cdot ....(2k-1)}{2^{k+1}z^{2k+1}} \quad , \quad \lim_{z\to\infty} 2zF(z) = 1 \ .$$

In [NoT] the following “irrationality” results for the considered functions are proven:

For all $x \neq 0$ in an imaginary quadratic number field K the confluent hypergeometric function

$$ {}_1F_1(1, \frac{3}{2}, x) $$

never has values in the same field K. In particular, all real zeros different from zero of the incomplete gamma function

$$ \gamma(\frac{1}{2}, x) = 2\sqrt{x}e^{-x}{}_1F_1(1, \frac{3}{2}, x) == 2\sqrt{x}{}_1F_1(\frac{1}{2}, \frac{3}{2}, -x) $$

are irrational and the complex zeros different from zero are in no imaginary quadratic extension of the rational number field Q.

In §3 below we will deal with the Bessel function given by ([AbM] 9.1.10, [LeN] 5.3)

$$ J_\nu(2\sqrt{z}) = z^{\nu/2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+k+1)}\frac{z^k}{k!} = z^{\nu/2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(\nu+k)!}\frac{z^k}{k!} \quad . $$

For the analysis of the zeros of $J_0(2\sqrt{z})$ we refer to [WaG] 15-5. The relationship to the associated Lommel polynomials is given by the Hurwitz formula ([WaG] 9-65)

$$ J_\nu(2\sqrt{z}) = \lim_{n\to\infty}\frac{g_{n,0}(z)}{n!} $$

with

$$ g_{n,0}(z) = \sum_{m=0}^{\leq \frac{n}{2}}(-1)^m\frac{(n-m)!}{(n-2m)!}\Gamma(n-m+1)\frac{z^m}{m!} \quad , $$

whereby

$$ g_{n+1,0}(z) = (n+1)g_{n,0}(z) - zg_{n-1,0}(z) \qquad g_{1,0}(z) = 1 \quad . $$

For the modified Lommel polynomials ([DiD])

$$ h_{n,1}(z) = R_{n,1}(\frac{1}{z}) = (2z)^n g_{n,0}(\frac{2}{\sqrt{z}}) $$

there is an orthogonal type recurrence relation in the form

$$ h_{n,1}(z) = 2x(n-1)h_{n-1,1}(z) - h_{n-2,1}(z) $$

leading to

$$ \frac{1}{2\pi i}\int_C z^k h_{n,1}(z)\frac{J_1(\frac{1}{x})}{J_0(\frac{1}{x})}dz = \begin{cases} 0 & k<n \\ \frac{1}{2^{n+1}(n+1)!} & k=n \end{cases} \quad . $$

The following two theorems were used to obtain so called Ω results by means of Diophantine approximations ([TiE] VIII):

Dirichlet’s theorem

Given N real numbers $a_1, a_2, ... a_N$, a positive integer q, and a positive number t_0, we can find a number t in the range $t_0 \leq t \leq t_0 q^N$ and integers $x_1, x_2, ... x_N$ such that

$$ |ta_n - x_n| \leq \frac{1}{q} \quad , \quad n = 1,2,....N \quad . $$

Kronecker's theorem

Let $a_1, a_2, ...a_N$ be linear independent real numbers, i.e. numbers such that there is no linear relation

$$\lambda_1 a_1 + ... + \lambda_N a_N = 0$$

In which the coefficients $\lambda_1, ...\lambda_N$ are integers not all zero. Let $b_1, b_2, ...b_N$ be any real numbers, and q a given positive number. Then we can find a number t and integers $x_1, x_2, ...x_N$, such that

$$\left|ta_n - b_n - x_n\right| \leq \frac{1}{q} \quad , \quad n = 1,2,....N \; .$$

In the context of the zeros of the considered Kummer functions we recall the

Eneström-Kakeya theorem ([VaR] theorem 3.2): Let

$$p_n(z) = \sum_{j=0}^{n} a_j z^n \quad , \quad n \geq 1$$

be any polynomial with $a_j > 0$ for all $j = 0,1,2,....n$. Setting

$$\alpha = \alpha[p_n] := \min_{0 \leq j < n} \left(\frac{a_i}{a_{i+1}} \right) \quad , \quad \beta = \beta[p_n] := \max_{0 \leq j < n} \left(\frac{a_i}{a_{i+1}} \right)$$

then all zeros of $p_n(z)$ lie in the annulus

$$\alpha \leq |z| \leq \beta \; .$$

The inequalities are sharp, i.e. there exist polynomials with positive coefficients having zeros on $|z| = \alpha$ or on $|z| = \beta$ ([HuA]).

§ 2 Algebraic independence of the values of Kummer functions

In [BeI] and [ShA], p.222: theorem 5, p. 229: theorem 6, transcendence and algebraic independence of the values of the Bessel and the Kummer functions are considered.

Theorem: let α, δ be algebraic numbers, linearly independently over the fields of rational numbers; let β be an algebraic number whose square differs from zero, where

$$\alpha^2 \neq -4\beta$$

then the following numbers are algebraically independent:

i) $J_0(\alpha), J_0'(\alpha)$

ii) $e^{-\alpha/2}{}_1F_1(\frac{1}{2},\frac{3}{2},\alpha), \frac{d}{d\alpha}(e^{-\alpha/2}{}_1F_1(\frac{1}{2},\frac{3}{2},\alpha)) = \frac{e^{\alpha/2}}{2\alpha}(1-(\alpha+1)e^{-\alpha}\cdot{}_1F_1(\frac{1}{2},\frac{3}{2},\alpha)$

iii) ${}_1F_1(\frac{1}{2},\frac{3}{2},\alpha), {}_1F_1'(\frac{1}{2},\frac{3}{2},\alpha) = \frac{1}{2\alpha}\left[e^{\alpha} - {}_1F_1(\frac{1}{2},\frac{3}{2},\alpha)\right]$ ([GrI] 9,213)

iv) ${}_1F_1(\frac{1}{2},\frac{3}{2},\alpha), {}_1F_1'(\frac{1}{2},\frac{3}{2},\alpha), J_0(\beta), J_0'(\beta), e^{\delta}$.

The standard technique for transcendence proofs are by contradiction (indirect), whereby appropriate conditions about algebraization of sufficiently enough values of transcendental functions and its derivatives, as well as arithmetical properties of those functions result on the one hand side in assessment estimates downwards, which will be fall below on the other hand side. Schneider ([ScT] Satz 12, p. 49, [ScT1]) proved the corresponding positive (direct) theorem:

Theorem: let $f_1(z), f_2(z)$ be entire or meromorph functions with finite growth regulatories, wherby μ denotes the maximum of both growth regulatories; the numbers $(z_k)_{k=0,1,2,...m-1}$ are pairwise different and different to the poles of $f_1(z), f_2(z)$. There is a $\eta > 0$ and there are integers n_k, m_k independently from $\eta > 0$ that

$m_k^{\tau+1} f_1^{(\tau)}(z_k), n_k^{\tau+1} f_2^{(\tau)}(z_k)$ are quite algebraic for $k = 0,1,2,....m-1, \tau \in N$

and

$$\left| f_{1,2}^{(\tau)} \right| < m_k^{\tau+1}(\tau+1)^{\eta\tau} .$$

Then, if

$m > (2\mu+1)(s(2\eta+1)-\eta+1/2)$, (s denotes the degree of the underlying field),

there must be an algebraic relationship between both functions.

§ 3 A Kummer function based characterization of the Euler constant

From [GrI] 8.211, we recall

$$Ei(-x) = \gamma + \log x + \int_0^x \frac{e^{-t} - 1}{t} dt$$

whereby

$$Ei(x) := -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^{x} \frac{e^{t}}{t} dt \quad .$$

From [BrR] (see also [GrI] 6.693) we recall

$$\int_0^\infty \frac{e^{-t/2} - J_0(t)}{t} dt = \int_0^\infty \frac{\cos(t/2) - J_0(t)}{t} dt = 0$$

With Weber's formula ([WaG] 13-24)

$$\int_0^\infty t^{\mu-\nu} \frac{J_\nu(t)}{t} dt = \frac{1}{2} \frac{2^\mu \Gamma(\frac{\mu}{2})}{2^\nu \Gamma(\nu - \frac{\mu}{2} + 1)}$$

one gets ([BrR], see also [GrI] 6.622.1)

$$\int_0^\infty \frac{e^{-t/2} - J_0(t)}{t} dt = \lim_{\mu \to 0^+} \int_0^\infty t^\mu \frac{e^{-t/2} - J_0(t)}{t} dt = \lim_{\mu \to 0^+} \left[2^\mu \Gamma(\mu) - \frac{2^{\mu-1}\Gamma(\frac{\mu}{2})}{\Gamma(1-\frac{\mu}{2})} \right] = 0$$

resp.

$$\int_0^\infty \frac{e^{-t} - J_0(2\sqrt{t})}{t} dt = \lim_{\mu \to 0^+} \int_0^\infty t^\mu \frac{e^{-t} - J_0(2\sqrt{t})}{t} dt = \lim_{\mu \to 0^+} \left[\Gamma(\mu) - \frac{\Gamma(\mu)}{\Gamma(1-\mu)} \right] = \lim_{\mu \to 0^+} \left[\frac{1}{\mu} \left\{ (1-\gamma\mu) - \frac{1-\gamma\mu}{1+\gamma\mu} + O(\mu^2) \right\} \right] = \gamma \quad .$$

In combination with the relationship

$$e^{-t} = {}_1F_1(\frac{1}{2}, \frac{3}{2}, -t) - 2t\, {}_1F_1'(\frac{1}{2}, \frac{3}{2}, -t)$$

it follows

$$\int_0^\infty \frac{{}_1F_1(\frac{1}{2}, \frac{3}{2}, -t) - 2t\, {}_1F_1'(\frac{1}{2}, \frac{3}{2}, -t) - J_0(2\sqrt{t})}{t} dt = \int_0^\infty \frac{{}_1F_1(\frac{1}{2}, \frac{3}{2}, -t) - J_0(2\sqrt{t})}{t} dt + 1 = \gamma$$

resp.

$$1 - \gamma = \int_0^\infty \frac{J_0(2\sqrt{t}) - {}_1F_1(\frac{1}{2}, \frac{3}{2}, -t)}{t} dt$$

whereby ([GrI] 7.612)

$$\int_0^\infty t^{\mu/2}\, {}_1F_1(\frac{1}{2}, \frac{3}{2}, -t) \frac{dt}{t} = \frac{\Gamma(\frac{\mu}{2})}{1-\mu} \quad .$$

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