A PROOF OF THE RIEMANN HYPOTHESIS

enabled by

a new integral and series representation of the meromorphic Zeta function occuring in the symmetrical form of the Riemann functional equation

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Dedicated to my wife Vibhuta
on the occasion of her 62th birthday, August 25, 2023

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Dedicated to all people
whose birthday or name day it is today

Abstract

For $s \neq \nu$, $\nu \in Z$, Riemann's meromorphic Zeta function

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s)$$

is represented in the form

$$\xi^*(s) = \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^\infty(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^\infty b_{2n}(s-\frac{1}{2})^{2n}$$

with

$$b_{2n} := \int_1^\infty \Phi(x)\left[\sum_{n=0}^\infty \frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}} \quad \text{and} \quad \Phi(x) := \sum_{n=1}^\infty (e^{-2\pi n x} - e^{-\pi n^2 x^2}) \,.$$

The non-trivial zeros $\left\{s_n = \frac{1}{2} + it_n\right\}$ of the Zeta function are characterized by the identity of two convergent series representations in the form

$$\sum_{n=0}^\infty(-1)^n b_{2n} t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^\infty(-1)^n\zeta(2n)\left[\frac{4n-1}{(2n-\frac{1}{2})^2 + t_n^2}\right],$$

which do not allow negative values $t_n^{2n} < 0$. In the critical stripe the corresponding alternative entire Zeta function $\xi^{**}(s) := \sin(\pi s)\,\xi^*(s)$ can be represented as a Mellin transform of a Kummer function (accompanied by the product representation ${}_1F_1\left(\frac{1}{2},\frac{3}{2},z\right) = \frac{\sqrt{\pi}}{2}e^{\frac{z}{3}}\prod(1-\frac{z}{z_n})e^{z/z_n}$) with only complex valued zeros with $\mathrm{Re}(z_n) > 1/2$ and imaginary parts lying in the horizontal stripes $(2n-1)\pi < |\mathrm{Im}(z_n)| < 2\pi n$, $n \in N$) in the form

$$\bar{\xi}(s) := \frac{2}{\pi}\frac{\xi^{**}(s)}{s(1-s)}\frac{\pi s}{\sin(\pi s)} = \pi^{-\frac{s}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{1-s}\zeta(s) = \zeta(s)M\left[{}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi x\right)\right]\left(\frac{s}{2}\right), \quad 0 < \mathrm{Re}(s) < 1 \,,$$

i.e., the representation is in line with the concept of *„a self-adjoint operator with transform $\bar{\xi}(s)$"*, as provided in (EdH) 10.3, (*).

(*) The concept is also in line with the proposed Kummer function based Zeta function theory and a related alternatively proposed two-semicircle method to the Hardy-Littewood (major/minor arcs based) circle method in (BrK).

1. Notations and Main Theorem

For the notations we refer to (EdH). The baseline function for the Zeta function theory is given by $\psi(x) := \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, (EdH) 1.7. It is related to Jacobi's functional equation of the theta function ϑ enabling the symmetrical form of Riemann's functional equation in the form, (EdH) 1.7,

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s) .$$

Riemann's related entire Zeta function is given by $\xi(s) = \frac{s}{2}(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$, (EdH) 1.8. Alternatively to $\psi(x)$ we shall apply the function

$$\Phi(x) := \varphi(x) - \psi(x^2) := \sum_{n=1}^{\infty} \Phi_n(x) := \sum_{n=1}^{\infty}(e^{-2\pi n x} - e^{-\pi n^2 x^2}) , \ x \geq 1 \ ^{(*)}.$$

The main result of our paper is an alternative $\xi^*(s)$ –function representation with three $s \leftrightarrow (1-s)$ symmetric summands in the form

$$\xi^*(s) = \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s-\tfrac{1}{2})^{2n}.$$

A corresponding alternatively defined entire Zeta function $\xi^{**}(s)$ is built by multiplication of $\xi^*(s)$ with $\sin(\pi s)$ leading to $\xi^{**}(s) := \sin(\pi s)\,\xi^*(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\sin(\pi s)\,\zeta(s)$, resp.

$$\bar{\xi}(s) := \frac{2}{\pi}\frac{\xi^{**}(s)}{s(1-s)}\frac{\pi s}{\sin(\pi s)} = \pi^{-\frac{s}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{1-s}\zeta(s) ,$$

which (in the critical stripe) can be representated as a Mellin transform in the form

$$\bar{\xi}(s) = \zeta(s)M\left[{}_1F_1\left(\frac{1}{2};\frac{3}{2}, -\pi x\right)\right]\left(\frac{s}{2}\right) \ ^{(**)\,(***)}.$$

(*) Note: $\Phi_n(x) \geq 0$ for $n \geq 2, x \geq 1$; $\Phi_1(x) < 0$ for $1 \leq x < 2$; $\Phi_1(2) = 0$; $\Phi_1(x) > 0$ for $x > 2$. Putting $\Phi_{1,2}^*(x) := -\Phi_1(x)$, $\Phi_{0,1}^*(x) := -\Phi_1\left(\frac{1}{x}\right)$, for $1 \leq x < 2$, $\Phi_{1,2}^*(x) = \Phi_{0,1}^*(x) = 0$ for $x \geq 2$, $\Phi_{2,\infty}^*(x) := \Phi(x)$ for $x \geq 2$, $\Phi_{2,\infty}^*(x) = 0$ for $x < 2$, the three terms of the sum $\Phi_{0,1}^*(x) + \Phi_{1,2}^*(x) + \Phi_{2,\infty}^*(x) > 0$ have the disjunct domains $0 < x < 1$, $1 \leq x < 2$, $2 \leq x < \infty$

(**) In the critical stripe the term $\frac{\Gamma\left(\frac{s}{2}\right)}{1-s}$ is the Mellin transform of the Kummer function ${}_1F_1\left(\frac{1}{2};\frac{3}{2}, -x\right)$; the zeros s_ν, $\nu \in Z - \{0\}$, of the function ${}_1F_1\left(\frac{1}{2},\frac{3}{2}, z\right)$ are all simple, complex valued with Re(z)>1/2, and lie in the horizontal stripes $(2n-1)\pi < |Im(z)| < 2\pi n$, $n \in N$, (SeA); we note that the latter property is strongly related to the Digamma function, (BrK). The related product representation is given by ${}_1F_1\left(\frac{1}{2},\frac{3}{2}, z\right) = \frac{\sqrt{\pi}}{2}e^{\frac{z}{3}}\prod(1 - \frac{z}{z_n})e^{z/z_n}$, (BuH) p.184; regarding Riemann's method deriving the formula for $J(x)$, EdH) 1.13, we note the series representation $\ln(\sin(\pi x) = \ln(\pi x) + \sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\frac{(2\pi)^{2n}}{(2n)!}B_{2n}x^{2n}$, $x^2 < 1$, (Grl) 1.518

(***) see also (EdH) 10.3 „A self-adjoint operator with transform $\xi(s)$", and (10.5) „$\frac{2\xi(s)}{s(s-1)}$ as a transform". The connection between $\zeta(s)$ and primes is given by Riemann's formula for $J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^s\frac{ds}{s}$, $(a > 1)$. The term $\log(s-1)$ results into the $li_1(x)$ – function $li_1(x) := \lim_{\varepsilon \to 0}\int_0^{1-\varepsilon}\frac{dt}{\log t} + \int_{1+\varepsilon}^{x}\frac{dt}{\log t} = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log(s-1)}{s}\right]x^s ds$ $(a > 1)$, (EdH) 1.14. Riemann built his famous power series representation of his entire Zeta function $\xi(s) := \pi^{-\frac{s}{2}}\frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\zeta(s)$ by multiplication of $\xi^*(s) = \int_1^{\infty}\psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}$ with $s(s-1)$ to govern the two poles of the term $-\frac{1}{2}\frac{1}{s(1-s)}$. Then, by partial integration he derived the representation of $\xi(s)$ in the form $\xi(s) = \sum_{n=0}^{\infty} a_{2n}(s-\frac{1}{2})^{2n}$ where $a_{2n} := 4\int_1^{\infty}\frac{d\left[x^{\frac{3}{2}}\psi'(x)\right]}{dx}x^{-1/4}\frac{(\frac{1}{2}\log x)^{2n}}{(2n)!}dx$. He claimed that the series as an even function of $s - \frac{1}{2}$ „*converges very rapidly*" without giving explicit estimates. …. Hadamard proved that the rapid decrease of the coefficients a_{2n} is neccessary and sufficient for the validity of the product formula $\xi(s) = \xi(0)\prod_{n=1}^{\infty}\prod_{\rho}(1 - \frac{s}{\rho})$, (EdH) 1.8.

Main Theorem: For $s \neq \nu$, $\nu \in Z$, it holds

$$\xi^*(s) = \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right)+\zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s-\tfrac{1}{2})^{2n}$$

with

$$b_{2n} := \int_1^\infty \Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}}.$$

In proving the Main Theorem the essential step (which is proven in the next section) is

Lemma MT: For $s \neq \nu$, $\nu \in Z$, it holds

$$-\frac{1}{2}\frac{1}{s(1-s)} = \frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}+\frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right] - \int_1^\infty[x^s+x^{1-s}]\frac{1}{2}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}.$$

Corollary: The set of non-trivial zeros $\left\{s_n = \frac{1}{2}+it_n\right\}$ of the zeta function are characterized by the identity of two convergent series representations

$$\sum_{n=0}^{\infty} b_{2n}(s_n-\tfrac{1}{2})^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-z_n}+\frac{\zeta(2n)}{(2n-1)+s_n}\right]$$

resp.

$$\sum_{n=0}^{\infty}(-1)^n b_{2n}t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{4n-1}{(2n-\frac{1}{2})^2+t_n^2}\right].$$

Remark: In case of existing negative values $t_n^{2n} < 0$ the two series would be no longer alternating, and, while the affected term on the left side changes its sign, the term on the corresponding right side would not.

Proof of the Main Theorem:

With $\xi^*(s) = \int_1^\infty \psi(x^2)[x^s+x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}$ and $\Phi(x) = \varphi(x) - \psi(x^2)$ one gets

$$\xi^*(s) = -\int_1^\infty \Phi(x)[x^s+x^{1-s}]\frac{dx}{x} + \frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}+\frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right].$$

Analogue to Riemann's approach deriving his famous power series representation for $\xi(s)$, (EdH) 1.8 (*), with $b_{2n} := \int_1^\infty \Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}}$ the first term allows the power series representation in the form

$$-\int_1^\infty \Phi(x)[x^s+x^{1-s}]\frac{dx}{x} = -2\sum_{n=0}^{\infty} b_{2n}(s-\tfrac{1}{2})^{2n}.$$

(*) $[x^s+x^{1-s}] = 2\sqrt{x}\left[\cosh\left(s-\frac{1}{2}\right)\log x\right]$ and $\cosh(y) = \sum_{n=0}^{\infty}\frac{y^{2n}}{(2n)!}$ with $y := \left(s-\frac{1}{2}\right)\log x$.

2. Proof of the Lemma MT

With

$$\varphi(x)=\frac{1}{2}\frac{e^{-\pi x}}{\sinh(\pi x)}=\frac{1}{e^{2\pi x}-1}=\sum_{n=1}^{\infty}e^{-2\pi n x},\ x>1\ ,^{(*)}$$

the Lemma MT takes the form

Lemma MT: For $s\neq\nu$, $\nu\in Z$, it holds

$$-\frac{1}{2}\frac{1}{s(1-s)}=-\int_1^{\infty}[x^s+x^{1-s}]\varphi(x)+\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}+\frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right]+\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right].$$

Proof:

As $\frac{1}{s-1}+\frac{1}{-s}=\frac{1}{s(s-1)}$ the Lemma MT is a consequence of the integral and series representations as provided in (MiM) in section 4 (**):

$$\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}=\frac{1}{s-1}-\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s}+\int_1^{\infty}x^{1-s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}$$

$$\frac{\zeta(1-s)}{\sin\left(\frac{\pi}{2}(1-s)\right)}=\frac{1}{-s}-\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{(2n-1)+s}+\int_1^{\infty}x^{s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}\ .$$

(*) $\int_1^{\infty}x^{2m}\varphi(x)\frac{dx}{x}=\frac{|B_{2m}|}{2m}$, (Grl) 3.552

Lemma, (PoG) p. 65: let $f(t)>0$, $f'(t)<0$, $f''(t)<0$ for $0\le t\le 1$, then the even function $F(z)=\int_0^1 f(t)\cos(zt)dt$ has infinite many, only real zeros

(**) *(MiM):* *Special cases, 4.1 The case* $c=0$

For the special case $c=0$ *the integral*

$$\zeta(s)=-\pi^{s-1}\frac{\sin\left(\frac{\pi}{2}s\right)}{s-1}\int_0^{\infty}\frac{x^{1-s}}{\sinh^2(x)}dx\ ,\qquad Re(s)<0 \qquad \textit{(MiM) (4.1)}$$

can be broken into two parts $\zeta(s)=\zeta_0(s)+\zeta_1(s)$ *where*

$$\zeta_1(s)=\frac{\sin\left(\frac{\pi}{2}s\right)}{s-1}+\sin\left(\frac{\pi}{2}s\right)\int_1^{\infty}x^{1-s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x} \qquad \textit{(MiM) (4.6)}$$

$$\zeta_0(s)=-\frac{2}{\pi}\sin\left(\frac{\pi}{2}s\right)\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s} \qquad \textit{(MiM) (4.8)}$$

which are both valid for all s.

3. References

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