A PROOF OF THE RIEMANN HYPOTHESIS

enabled by

a new integral and series representation of the meromorphic Zeta function occuring in the symmetrical form of the Riemann functional equation

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Dedicated to my wife Vibhuta
on the occasion of her 62th birthday, August 25, 2023

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Abstract

For $s \neq \nu$, $\nu \in \mathbb{Z}$, Riemann's meromorphic Zeta function

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s)$$

is represented in the form

$$\xi^*(s) = \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^\infty (-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^\infty b_{2n}(s-\frac{1}{2})^{2n}$$

where

$$b_{2n} := \int_1^\infty \Phi(x)\left[\sum_{n=0}^\infty \frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}} \quad \text{with} \quad \Phi(x) := \sum_{n=1}^\infty (e^{-2\pi n x} - e^{-\pi n^2 x^2}) .$$

Correspondingly, the set of non-trivial zeros $\left\{z_n = \frac{1}{2} + it_n\right\}$ of the Zeta function is characterized by the identity of two („very rapidly") convergent series representations

$$\sum_{n=0}^\infty b_{2n}(z_n - \frac{1}{2})^{2n} = \frac{1}{2\pi}\sum_{n=0}^\infty (-1)^n\left[\frac{\zeta(2n)}{2n-z_n} + \frac{\zeta(2n)}{(2n-1)+z_n}\right]$$

resp.

$$\sum_{n=0}^\infty (-1)^n b_{2n} t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^\infty (-1)^n\left[\frac{\zeta(2n)}{(2n-\frac{1}{2})-it_n} + \frac{\zeta(2n)}{(2n-\frac{1}{2})+it_n}\right]$$

accompanied by an alternative entire Zeta function in the form

$$\xi^{**}(s) := \sin(\pi s)\,\xi^*(s) = (1-s)\sin(\pi s)\left[\frac{1}{2}\pi^{-\frac{s}{2}}\frac{\Gamma(\frac{s}{2})}{1-s}\right]\zeta(s).$$

The term $(1-s)$ is the principle term of the $li_1(x)-$ function, while the term $[\cdot\,\cdot]$ is the Mellin transform of the Kummer function ${}_1F_1\left(\frac{1}{2};\frac{3}{2},-x^2\right)$ in the critical stripe. This enables the definition of an enhanced prime number density function, which is composed of Riemann's density function $J(x)$, $(x>1)$, and $J^*(x)$, $(0<x<1)$ (*). This overcomes the challenge of *„pairing terms $(\rho, 1-\rho)$ of the sum over (the non-trivial zeros) ρ in the usual way"*, as the „pairs $(\rho, 1-\rho)$ (in Riemann's formula for $J(x)$) *must no longer be summed in the order of increasing $Im(\rho)$"*, (EdH) 1.15.

(*) The concept is in line with the proposed Kummer function based Zeta function theory and a related alternative two-semicircle method to the Hardy-Littewood (major/minor arcs) circle method as proposed in (BrK).

1. Notations and Main Theorem

For the notations we refer to (EdH). The baseline function for the Zeta function theory is given by $\psi(x) := \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, (EdH) 1.7. It is related to Jacobi's functional equation of the theta function ϑ enabling the symmetrical form of Riemann's functional equation in the form, (EdH) 1.7,

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty}\psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s) .$$

Our proposed alternative baseline function for the Zeta function theory is defined by

$$\Phi(x) := \varphi(x) - \psi(x^2) := \sum_{n=1}^{\infty}(e^{-2\pi n x} - e^{-\pi n^2 x^2})$$

accompanied by the series

$$b_{2n} := \int_1^{\infty}\Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}} .$$

The main result of our paper is

Main Theorem: For $s \neq \nu$, $\nu \in Z$, it holds

$$\xi^*(s) = \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s-\tfrac{1}{2})^{2n} .$$

In proving the Main Theorem the essential step is

Lemma MT: For $s \neq \nu$, $\nu \in Z$, it holds

$$-\frac{1}{2}\frac{1}{s(1-s)} = \frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right] - \int_1^{\infty}[x^s + x^{1-s}]\frac{1}{2}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}.$$

Corollary: The set of non-trivial zeros $\left\{z_n = \frac{1}{2} + it_n\right\}$ of the zeta function are characterized by the identity of two convergent series representations

$$\sum_{n=0}^{\infty} b_{2n}(z_n - \tfrac{1}{2})^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-z_n} + \frac{\zeta(2n)}{(2n-1)+z_n}\right]$$

resp.

$$\sum_{n=0}^{\infty}(-1)^n b_{2n} t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{(2n-\frac{1}{2})-it_n} + \frac{\zeta(2n)}{(2n-\frac{1}{2})+it_n}\right].$$

Remark: Riemann built his series representation of

$$\xi(s) := \pi^{-\frac{s}{2}} \frac{s}{2} \Gamma\left(\frac{s}{2}\right)(s-1)\zeta(s)$$

by multiplication of

$$\xi^*(s) = \int_1^\infty \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}$$

with $s(s-1)$ to govern the two poles of the term $-\frac{1}{2}\frac{1}{s(1-s)}$. By partial integration he derived his famous power series representation of $\xi(s)$ in the form

$$\xi(s) = \sum_{n=0}^\infty a_{2n}(s-\tfrac{1}{2})^{2n}$$

where

$$a_{2n} := 4\int_1^\infty \frac{d\left[x^{\frac{3}{2}}\psi'(x)\right]}{dx} x^{-1/4} \frac{(\frac{1}{2}\log x)^{2n}}{(2n)!} dx$$

claiming that the series as an even function of $s - \frac{1}{2}$ *„converges very rapidly"* without giving explicit estimates, (EdH) 1.8. The proposed alternative entire Zeta function $\xi^{**}(s)$ is built by multiplication of $\xi^*(s)$ with $\sin(\pi s)$

$$\xi^{**}(s) := \sin(\pi s)\,\xi^*(s)$$

accompanied by the set of additional trivial zeros $\{\nu\}_{\nu \in Z}$. The Mellin transform in the critical stripe of the Kummer function, (GrI) 7.612,

$$M\left[{}_1F_1\left(\frac{1}{2};\frac{3}{2},-x^2\right)\right] = \frac{1}{2}\pi^{-\frac{s}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{1-s},\ 0 < Re(s) < 1,$$

enables a product represention of $\xi^{**}(s)$ in the form

$$\xi^{**}(s) = \sin(\pi s)\,\xi^*(s) := (1-s)\sin(\pi s)\left[\frac{1}{2}\pi^{-\frac{s}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{1-s}\right]\zeta(s), \text{ (*)}$$

where the term $(1-s)$ is the principle term defining the $li_1(x)$ – function, (**). The considered Kummer function, which is related to the Dawson function, enables the definition of a prime number density function $J^*(x)$ with domain $0 < x < 1$, additionally to $J(x)$ with domain $x > 1$. By construction the current challenges of *„pairing terms $(\rho, 1-\rho)$ of the sum over (the non-trivial zeros) ρ in the usual way"* and *„that the series $-\sum_{Im(\rho)>0} Li(x^\rho) + Li(x^{1-\rho})$ is only conditionally convergent"* are overcome, i.e., the *„pairs $(\rho, 1-\rho)$ must no longer be summed in the order of increasing $Im(\rho)$"*, (EdH) 1.15.

(*) An alternative product representation of $\xi^{**}(s)$ is given by $\xi^{**}(s) = \sin(\pi s)\,\xi^*(s) = \pi^{1-\frac{s}{2}}\zeta(s)\prod_{n=1}^\infty \frac{(1-\frac{s^2}{n^2})}{(1+\frac{s}{2n})} e^{\frac{s}{2}\left[\frac{1}{n}-\gamma\right]}$, (LeB) p. 32

(**) In Riemann's method for deriving the formula for the prime number density function $J(x)$ by substituting $\log\zeta(s) = \log\xi(s) = \log\pi^{-\frac{s}{2}} - \log\Gamma\left(1+\frac{s}{2}\right) - \log(s-1)$ into $J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \log\zeta(s) x^s \frac{ds}{s}$, $(a > 1)$ the term $\log(s-1)$ results into the $li_1(x)$ – function, (EdH) 1.14,

$$li_1(x) := \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t} = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log(s-1)}{s}\right]x^s ds \quad (a > 1).$$

The condition $a > 1$ is a consequence of the Fourier inverse function $\frac{\log\zeta(s)}{s} = \int_0^\infty J(x)x^{-s-1}dx$, $Re(s) > 1$.

2. Proof of the Lemma MT

With

$$\varphi(x)=\frac{1}{2}\frac{e^{-\pi x}}{\sinh(\pi x)}=\frac{1}{e^{2\pi x}-1}=\sum_{n=1}^{\infty}e^{-2\pi n x},\ x>1\ ,^{(*)}$$

the Lemma MT takes the form

Lemma MT: For $s\neq\nu,\ \nu\in Z$, it holds

$$-\frac{1}{2}\frac{1}{s(1-s)}=-\int_1^{\infty}[x^s+x^{1-s}]\varphi(x)+\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}+\frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right]+\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right].$$

Proof: The Lemma MT is a consequence of the integral and series representations as provided in (MiM) in section 4 (**):

$$\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}=\frac{1}{s-1}-\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s}+\int_1^{\infty}x^{1-s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}$$

$$\frac{\zeta(1-s)}{\sin\left(\frac{\pi}{2}(1-s)\right)}=\frac{1}{-s}-\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{(2n-1)+s}+\int_1^{\infty}x^{s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}\ .$$

(*) $\varphi\left(\frac{1}{x}\right),\ 0<x<1$; (Grl) 3.552: $\int_1^{\infty}x^{2m}\varphi(x)\frac{dx}{x}=\frac{|B_{2m}|}{2m}$

Lemma: let $f(t)>0,\ f'(t)<0,\ f''(t)<0$ for $0\leq t\leq 1$, then the even function $F(z)=\int_0^1 f(t)\cos(zt)dt$ has infinite many, only real zeros, (PoG) p. 65

(**) *(MiM):* *Special cases, 4.1 The case $c=0$*

For the special case $c=0$ the integral

$$\zeta(s)=-\pi^{s-1}\frac{\sin\left(\frac{\pi}{2}s\right)}{s-1}\int_0^{\infty}\frac{x^{1-s}}{\sinh^2(x)}dx\,,\qquad Re(s)<0\qquad \textit{(MiM) (4.1)}$$

can be broken into two parts $\zeta(s)=\zeta_0(s)+\zeta_1(s)$ where

$$\zeta_1(s)=\frac{\sin\left(\frac{\pi}{2}s\right)}{s-1}+\sin\left(\frac{\pi}{2}s\right)\int_1^{\infty}x^{1-s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}\qquad \textit{(MiM) (4.6)}$$

$$\zeta_0(s)=-\frac{2}{\pi}\sin\left(\frac{\pi}{2}s\right)\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s}\qquad \textit{(MiM) (4.8)}$$

which are both valid for all s.

3. Proof of the Main Theorem

With

$$\xi^*(s) = \int_1^\infty \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}$$

and

$$\Phi(x) := \sum_{n=1}^\infty (e^{-2\pi nx} - e^{-\pi n^2 x^2}) = \varphi(x) - \psi(x^2)$$

one gets

$$\xi^*(s) = -\int_1^\infty \Phi(x)[x^s + x^{1-s}]\frac{dx}{x} + \frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^\infty (-1)^n \left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right].$$

Analogue to Riemann's approach deriving his famous power series representation for $\xi(s)$, (EdH) 1.8 (*), the first term allows the power series representation in the form

$$-\int_1^\infty \Phi(x)[x^s + x^{1-s}]\frac{dx}{x} = -2\sum_{n=0}^\infty b_{2n}(s - \tfrac{1}{2})^{2n}$$

with

$$b_{2n} := \int_1^\infty \Phi(x)\left[\sum_{n=0}^\infty \frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}}$$

resulting into

$$\xi^*(s) = -2\sum_{n=0}^\infty b_{2n}(s - \tfrac{1}{2})^{2n} + \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^\infty (-1)^n \left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right]$$

(*) $[x^s + x^{1-s}] = 2\sqrt{x}\left[\cosh\left(s - \frac{1}{2}\right)\log x\right]$ and $\cosh(y) = \sum_{n=0}^\infty \frac{y^{2n}}{(2n)!}$ with $y := \left(s - \frac{1}{2}\right)\log x$.

4. References

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