Quaternions & the Lorentz transformations

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Extract from (StR) p. 9 ff., (UnA) p. 152 ff., (CoR) p. 763

with comments from the author *in italic*

The real Lorentz transform

(StR): A Lorentz transformation is a linear transformation Λ mapping space-time onto space-time which preserves the scalar product $(\Lambda\vec{x},\Lambda\vec{y}) = (\vec{x},\vec{y})$, where

$$(\vec{x},\vec{y}) := (x^0,y^0) - [(x^1,y^1) + (x^2,y^2) + (x^3,y^3)] = x^\mu g_{\mu\nu} x^\nu = x^\mu y_\mu .$$

If $(\Lambda x)^\mu = \Lambda^\mu{}_\nu x^\nu$, the (real) matrix $\Lambda^\mu{}_\nu$ of the transformation must satisfy

$$\Lambda^\kappa{}_\mu \Lambda_{\kappa\nu} = g_{\mu\nu} \text{ or } \Lambda^T G \Lambda = G, \qquad (1\text{-}5)$$

where the transpose Λ^T of Λ is defined by $(\Lambda^T)^\mu{}_\nu = \Lambda^\mu{}_\nu$ and indices on Λ are lowered according to

$$\Lambda_{\kappa\nu} = g_{\kappa\sigma}\Lambda^\sigma{}_\nu = (G\Lambda)_{\kappa\nu}.$$

If Λ and M satisfy (1-5), so do $\Lambda\mathrm{M}$ and Λ^{-1}. Here

$$(\Lambda\mathrm{M})^\mu{}_\nu = \Lambda^\mu{}_\nu \, \mathrm{M}^\kappa{}_\nu \quad (\Lambda^{-1})^\mu{}_\nu \Lambda^\kappa{}_\nu = \mathrm{g}^\mu{}_\nu = \begin{cases} 0 & \mu\neq\nu \\ 1 & \mu=\nu \end{cases},$$

so the (real) Lorentz transformations form a group, the Lorentz group L.

Two Lorentz transformations Λ and M are defined to be close to one another if the numbers $\Lambda^\mu{}_\nu$ and $\mathrm{M}^\mu{}_\nu$ are close for all $\mu,\nu = 0,1,2,3$. Clearly, with this definition, Λ^{-1} and $\Lambda\mathrm{M}$ are continuous functions of Λ and M, respectively. Furthermore, it make sense to say that two (real) Lorentz transformations can be connected to one another by a continuous curve of Lorentz transformations.

The Lorentz group L has four components, each of which is connected in the sense that any point can be connected to any other, but no Lorentz transformation in one component can be connected to another in another component.

One of this components is the restricted Lorentz group, which is the group of $2x2$ complex matrices of determinant one, $SL(2,C)$. It is isomophic to the symmetry group $SU(2)$, containing as elements the complex-valued rotations, which can be written as a complex-valued matrix of type

$$\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \text{ with determinant one.}$$

It is important in describing the transformation properties of spinors. In SMEP the group $SU(2)$ describes the weak force interaction with 3 bosons W^+, W^-, Z.

This restricted Lorentz group contains the 1-transformation. Its connected component contains the space-time inversion. The other pair of connected components of the Lorentz group contains the space inversion resp. the time inversion.

The Lorentz group has also three important subgroups, which are

the orthochronous Lorentz group
the proper Lorentz group (containing the restricted Lorentz group and addtionally e.g. the space-time inversion)
the orthochorous Lorentz group.

The complex Lorentz transform

Another group associated with the Lorentz group L is the complex Lorentz group, which we shall denote by $L(C)$. It is essential in the proof of the PCT theorem as we shall see. It is composed of all complex matrices satisfying

$$\Lambda^{\kappa}{}_{\mu}\Lambda_{\kappa\nu} = g_{\mu\nu} \text{ or } \Lambda^{T}G\Lambda = G, \qquad (1\text{-}5).$$

It has just two connected components, $L_{+}(C)$ and $L_{-}(C)$ according to the sign of det(Λ). The transformations 1 and -1, which are disconnected in L are connected in $L(C)$. In other words, the complex Lorentz transformation connects

- the two components containing the 1-transformation and space-time inversion, i.e. the pair

$$\{\det(\Lambda) = +1, \det(\Lambda^{0}{}_{0} = +1)\}, \{\det(\Lambda) = +1, \det(\Lambda^{0}{}_{0} = -1)\},$$

- the two components containing the space inversion and the time inversion, i.e. the pair

$$\{\det(\Lambda) = -1, \det(\Lambda^{0}{}_{0} = +1)\}, \{\det(\Lambda) = -1, \det(\Lambda^{0}{}_{0} = -1)\}.$$

Summary:

While two (real) Lorentz transformations need to be connected to one another by an appropriately defined continuous curve of Lorentz transformations, there are two pairs of components of the complex Lorentz transform, which are both already connected by definition.

Just as the restricted Lorentz group is associated with $SL(2,C)$, the complex Lorentz group is associated with $SL(2,C) \otimes SL(2,C)$. The latter group is the set of all pairs of $2x2$ matrices of determinants one with the multiplication law

$$\{A_1, B_1\} \cdot \{A_2, B_2\} = \{A_1A_2, B_1B_2\}.$$

Is is easy to see that only matrix pairs which yield a given $\Lambda(\mathrm{A},\mathrm{B})$ are $(\pm\mathrm{A}, \pm\mathrm{B})$. In particular,

$$\Lambda(-1,1) = \Lambda(1,-1) = -1.$$

The corresponding complex Poincare group admits complex translation but also the multiplication law

$$\{a_1, \Lambda_1\} \cdot \{a_2, \Lambda_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2\}.$$

It has two components $\mathrm{P}_{\pm}(C)$, which are distinguished by $\det(\Lambda)$ and a corresponding inhomogeneous group to $SL(2,C)$.

Quaternions

(UnA): The multiplication rule for quarternions reads as follows:

$(a_1, b_1, c_1, d_1) \cdot (a_2, b_2, c_2, d_2) = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2, a_1b_2 + b_1a_2 + c_1d_2 + d_1c_2, a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2, a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)$.

Quaternions constitute a skew field, that means $a \cdot b = -b \cdot a$. Unit quaternions are equivalent to the 3D unit sphere S^3.

The quaternions product can also be written using the scalar and cross products known from vector analysis. If unit quarternion (a, b, c, d) is decomposed into a real part a and a vector $\vec{u} = (b, c, d)$. Then we get:

$$(a_1, \vec{u}_1) \cdot (a_2, \vec{u}_2) = (a_2a_2 - \vec{u}_1 \cdot \vec{u}_2, a_1\vec{u}_2 + a_2\vec{u}_1 - \vec{u}_1 x \vec{u}_2).$$

In vector analysis, both the scalar product (also called „dot product") and the vector product (also called „cross product") are of significant importance and are widely used in most areas of modern physics. One may combine these products with the symbol of a spatial derivative ∇, and generate the differential operators divergence and curl, indicating the source and the vorticity of a vector field, respectively. Maxwell's equations represent the most prominent example of how fundamental laws can be formulated in an elegant vectorial form. … Just to illustrate the close relation between quarternion algebra and vector analysis, consider the quaternionic multiplication of a spatiotemporal derivative vector with electromagnetic potential:

$$\left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \times \left(\varphi, \vec{A}\right) = \frac{\partial \varphi}{\partial t} - \vec{\nabla} \cdot \vec{A} \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\varphi + \vec{\nabla} \times \vec{A} \ .$$

The last two terms precisely match the known expressions for the electric and magnetic fields $\vec{E}$ and $\vec{B}$.

Comment:

The two connected components of the complex Lorentz transformations provide the link to the proposed two type plasma spinor quantum field model with the related symmetry group $SL(2, C) \otimes SL(2, C)$. We note that the space-time concept shows up on stage by linking the inner product of quaternions and the scalar product defining the Lorentz transformation property.

We further note, that the one type spinor field in current quantum field theory becomes already a first approximation to this finest degree of physical accuracy. In this context we also note, that from a mathematical perspective the Sobolev Hilbert spaces are the appropriate framework for elliptic PDE, e.g. providing „optimal" shift theorems, while the appropriate framework for hyperbolic PDE is still an open question. A proof of a related „space-time" problem, the Courant hypothesis, is still missing (CoR) p. 763:

Families of spherical waves for arbitrary time-like lines Λ exits only in the case of two and four variables, and then only if the differential equation is equivalent to the wave equation.

Reference

(CoR) Courant R., Methods of Mathematical Physics, Volume II, John Wiley & Sons, New York, 1989

(StR) Streater R. F., Wightman A. S., PCT, Spin & Statistics, and all that, W. A. Benjamin, Inc., New York, Amsterdam, 1964

(UnA) Unzicker A., The Mathematical Reality, Copyright © 2020 Alexander Unzicker