Classroom

In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

On a Definite Integral of the Fractional Part Function

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In this article, we evaluate the integral $\int_0^1 \left\{\frac{1}{x^n}\right\}$ dx as a function of n, where $\{.\}$ is the fractional part function. We also show the relationship between this integral and the Riemann zeta function, $\zeta(s)$.

1. Introduction

The Riemann zeta function which is a very important special function in the theory of numbers is defined as follows.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ .$$

For real values of s, $\zeta(s)$ converges for $s > 1$ and diverges for $s \leq 1$. Riemann in [1] analytically continued this function to all complex values of $s \neq 1$ within the half-plane of convergence, and the above series converges when the real part of s exceeds 1.

The Riemann zeta function plays an important role in the theory of numbers.

Since Euler's time to the present day, this function has been extensively studied and the famous Riemann hypothesis concerning this function still remains unsolved.

Keywords
Riemann zeta function, fractional part, Euler–Mascheroni constant.

In this article, we compute the value of the following

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integral for real values of n:

$$\int_0^1 \left\{\frac{1}{x^n}\right\} \mathrm{d}x ,$$

where $\{.\}$ is the fractional part function. From this, we arrive at the following relation:

$$\zeta\left(\frac{1}{s}\right) < \frac{1}{1-s} , \quad \text{for } 0 < s < 1. \qquad (1)$$

It should be noted however that the zeta function satisfies the following identity which is only one among many others from which the above inequality is apparent.

$$\zeta(s) = \frac{s}{s-1} - \sum_{n=1}^{\infty}(\zeta(s+n)-1)\frac{s(s+1)...(s+n-1)}{(n+1)!}, \quad \text{for } 1 < s < \infty. \qquad (2)$$

We give another proof of the same fact by evaluating the above integral. The integral has been evaluated for $n = 1$ (see [2]). For an evaluation of another similar integral and a discussion of related concepts, see [3].

2. Evaluation of the Integral

Theorem 1.

$$\int_0^1 \left\{\frac{1}{x^n}\right\} \mathrm{d}x = \begin{cases} \frac{1}{1-n} & n \le 0, \\ \frac{1}{1-n} - \zeta(\frac{1}{n}) & 0 < n < 1, \\ 1-\gamma & n = 1, \end{cases}$$

where γ is the Euler–Mascheroni constant.

Proof.

Case 1: $n < 0$

The zeta function satisfies many important identities.

This means that $0 < x^{-n} < 1$; hence $\left\{\frac{1}{x^n}\right\} = \frac{1}{x^n}$. So:

$$\int_0^1 \left\{\frac{1}{x^n}\right\} \mathrm{d}x = \left[\frac{x^{1-n}}{1-n}\right]_0^1 = \frac{1}{1-n}.$$

Case 2: $0 < n < 1$

We observe that,

$$I = \int_0^1 \left\{\frac{1}{x^n}\right\} \mathrm{d}x = -\lim_{p\to\infty} \sum_{k=1}^{p} \int_{(\frac{1}{k})^{1/n}}^{(\frac{1}{k+1})^{1/n}} \left\{\frac{1}{x^n}\right\} \mathrm{d}x.$$

Now, if

$$\left(\frac{1}{k+1}\right)^{1/n} < x < \left(\frac{1}{k}\right)^{1/n},$$

then

$$\left\{\frac{1}{x^n}\right\} = \frac{1}{x^n} - k.$$

So,

$$\int_{(\frac{1}{k})^{1/n}}^{(\frac{1}{k+1})^{1/n}} \left\{\frac{1}{x^n}\right\} \mathrm{d}x = \left[\frac{x^{1-n}}{1-n} - kx\right]_{(\frac{1}{k})^{1/n}}^{(\frac{1}{k+1})^{1/n}}.$$

Summing the above over k from 1 to ∞ we get:

$$-I = \frac{1}{1-n}\left(\zeta\left(\frac{1}{n}-1\right) - 1 - \zeta\left(\frac{1}{n}-1\right)\right) - \sum_{k=1}^{\infty} \frac{k+1-1}{(k+1)^{1/n}} + \zeta(\frac{1}{n}-1). \qquad (3)$$

Now,

$$-\sum_{k=1}^{\infty} \frac{k+1-1}{(k+1)^{1/n}} = 1 - \zeta\left(\frac{1}{n}-1\right) - 1 + \zeta\left(\frac{1}{n}\right). \qquad (4)$$

Plugging (4) into (3) gives us the following.

$$-I = \frac{-1}{1-n} + \zeta\left(\frac{1}{n}\right).$$

Hence,

$$I = \frac{1}{1-n} - \zeta\left(\frac{1}{n}\right). \qquad (5)$$

Case 3: $n = 1$

$$I = \int_0^1 \left\{\frac{1}{x}\right\} \mathrm{d}x = -\lim_{p\to\infty} \sum_{k=1}^{p} \int_{(\frac{1}{k})}^{(\frac{1}{k+1})} \left\{\frac{1}{x}\right\} \mathrm{d}x.$$

The Euler–Mascheroni constant is defined to be the limiting difference between the harmonic series and the natural logarithm.

If $\frac{1}{k+1} < x < \frac{1}{k}$ we have: $\left\{\frac{1}{x}\right\} = \frac{1}{x} - k$, so,

$$\int_{\left(\frac{1}{k}\right)}^{\left(\frac{1}{k+1}\right)} \left\{\frac{1}{x}\right\} \mathrm{d}x = [\log x - kx]_{\left(\frac{1}{k}\right)}^{\left(\frac{1}{k+1}\right)} = \log k - \log(k+1) + \frac{1}{k+1}.$$

Summing the above over k from 1 to ∞ we get:

$$-I = \lim_{p\to\infty}\left(-\log(p+1) + \sum_{k=1}^{p}\frac{1}{k+1}\right).$$

Now, the Euler–Mascheroni constant is defined to be the limiting difference between the harmonic series and the natural logarithm. That is,

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right).$$

Hence: $I = -(-1+\gamma) = 1-\gamma$.

Proof of an Inequality.

Now, from (5), we get a proof of (1) since $I > 0$:

$$\zeta\left(\frac{1}{n}\right) < \frac{1}{1-n} \quad \text{for all } 0 < n < 1.$$

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Editors Note: One of the most important unsolved problem in mathematics today is the location of the non-trivial zeros of the Riemann zeta function in the complex plane. See the book Dr. Riemanns zeros by Karl Sabbagh for a very readable introduction to this topic.

Suggested Reading

[1] G F B Riemann, *Uber die Anzahl der Primzahlen unter einer gegebenen Grosse. Monatsber. Königl.* Preuss. Akad. Wiss. Berlin, pp.671–680, November, 1859. Reprinted in *Das Kontinuum und An-dere Monographen*, Edited by H Weyl, Chelsea, New York, 1972.

[2] Eric W Weisstein, Fractional Part, From *MathWorld – A Wolfram Web Resource*, http://mathworld.wolfram.com/FractionalPart.html

[3] E C Titchmarsh, *The Theory of the Riemann Zeta Function*, pp.13–15, 2nd ed. Clarendon Press, New York, 1987.