DISSIPATIVE OPERATORS AND HYPERBOLIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

BY

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1. Introduction. The original purpose of this research was to employ the theory of semi-groups of operators to obtain a solution of the Cauchy problem for dissipative hyperbolic systems of partial differential equations with time invariant coefficients and boundary conditions. Here the term "dissipative" is used to emphasize the basic assumption imposed on the system, namely, that the *energy is nonincreasing in time.* As the work progressed, it became clear that the argument could be given an abstract formulation and that the results on hyperbolic systems should be treated as an application of the general theory of *dissipative operators*[2].

DEFINITION 1.1. *Let H_0 be a hilbert space with inner product $\langle y^0, z^0 \rangle$. A linear operator L with domain $\mathfrak{D}(L)$ is said to be dissipative if*

$$(1.1) \qquad\qquad \langle Ly^0, y^0 \rangle + \langle y^0, Ly^0 \rangle \leqq 0, \qquad\qquad y^0 \in \mathfrak{D}(L)$$

and to be maximal dissipative if it is not the proper restriction of any other dissipative operator.

The connection between dissipative operators and dissipative hyperbolic systems is almost immediately evident. In fact, the dissipative assumption on the hyperbolic system suggests that a natural setting of the problem would be a hilbert space H_0 where $\|y^0\|^2 = \langle y^0, y^0 \rangle$ is a measure of the energy of the system at a given time. A semi-group solution to the Cauchy problem (see, for instance, E. Hille and R. S. Phillips [7]) would then consist of a one-parameter family of linear bounded operators $[S(t); t \geqq 0]$ such that

    (i) $S(t_1+t_2) = S(t_1)S(t_2)$, $t_1, t_2 > 0$, $S(0) = I$;

    (ii) $S(t)$ is strongly continuous for $t \geqq 0$;

    (iii) $\|S(t)y^0\|$ is nonincreasing in t for each $y^0 \in H_0$.

Condition (i) is a consequence of the initial-value problem being well set, (ii) results from the requirement $\lim_{t \to 0+} S(t)y^0 = y^0$, and (iii) is a restatement of the dissipative assumption. Condition (iii) can also be expressed by saying that the $S(t)$ are contraction operators, that is, operators of norm $\leqq 1$. If L

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    [2] Dissipative operators are playing an increasingly important role as research on nonselfadjoint operators proceeds. Already, M. S. Livšic [9] and B. R. Mukminov [10] have obtained a remarkable expansion theorem for bounded dissipative operators having real parts of finite trace.

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is the infinitesimal generator, the differential system will be satisfied in the sense that

$$\frac{d}{dt} S(t)y^0 = L[S(t)y^0], \qquad \text{for all } y^0 \in \mathfrak{D}(L) \text{ and } t > 0,$$

the derivative being taken in the norm topology of H_0. It turns out that *an operator L is the infinitesimal generator of a strongly continuous semi-group of contraction operators on H_0 if and only if L is a maximal dissipative operator with dense domain.*

In searching for a maximal dissipative generator for our problem, one can start out with either a minimal operator L_0 or a maximal operator L_1, each representing the spatial part of the hyperbolic operator, and then determine the maximal dissipative extensions of L_0 and restrictions of L_1. These two procedures are duals of one another; the first can lead to operators which are no longer merely differential operators, whereas the second can result in differential operators whose domains need not be restricted by simple boundary conditions, but rather by "global" lateral conditions reminiscent of the lateral conditions found by W. Feller [5] in his treatise on parabolic partial differential equations. The maximal dissipative operators which are differential operators and whose domains are delimited by the usual type of boundary conditions are precisely those which are at the same time extensions of L_0 and restrictions of L_1. Chapter I is primarily concerned with giving an abstract characterization of all maximal dissipative operators L such that $L_0 \subset L \subset L_1$. In order to motivate our operator-theoretic development we shall now describe the hyperbolic system to which this theory will be applied in Chapter II.

Let Δ be a domain in the m-dimensional real euclidean space with points $\chi = (\chi^1, \chi^2, \cdots, \chi^m)$ and let $y^0(\chi, t)$ be a function of (χ, t) with values in a k-dimensional complex euclidean space. We consider the initial value problem

$$(1.2) \quad \begin{aligned} y_t^0 &= L_1 y^0 \equiv E^{-1}[(A^i y^0)_i + B y^0)], \qquad \chi \in \Delta, t > 0, \\ y^0(\chi, 0) &= f^0(\chi); \end{aligned}$$

here we use the tensor notation for summation, the subscript i denoting differentiation with respect to χ^i. The symbols E, A^i, and B represent $k \times k$ matrix-valued functions of χ alone, E being positive definite, and A^i being hermitian, and the A^i and B together satisfying the "dissipative" condition

$$(1.3) \qquad\qquad B + B^* + A^i_i \leqq \Theta, \qquad\qquad \chi \in \Delta,$$

where B^* is the adjoint of B relative to the inner product

$$(1.4) \qquad\qquad (y^0, z^0) = \eta^i \bar{\zeta}^i.$$

We further assume that in each compact subset of Δ the elements of E are

continuous, the elements of A^i are absolutely continuous with respect to χ^i, and the elements of A^i_i, and B are square integrable; no restriction is placed on the growth properties of these functions near the boundary of Δ. It will be convenient to set

$$D \equiv E^{-1}[B + B^* + A^i_i].$$

If $y^0(\chi, t)$ is a solution of the system (1.2), then at a given time the energy of the associated physical model is given by

$$(1.5) \qquad \text{Energy} = \frac{1}{2} \int_\Delta (Ey^0, y^0)d\chi.$$

Using the energy as a measure we are thus lead to the hilbert space $H_0 = L_2(\Delta; E)$ with inner product

$$(1.6) \qquad \langle y^0, z^0 \rangle = \int_\Delta (Ey^0, z^0)d\chi.$$

After an integration by parts we obtain, at least formally, the following relation for the rate of change of energy from (1.2):

$$(1.7) \qquad \begin{aligned} \langle y^0, y^0 \rangle_t &= \langle L_1 y^0, y^0 \rangle + \langle y^0, L_1 y^0 \rangle \\ &= \int_\Delta (EDy^0, y^0)d\chi + \int_\Gamma (A^i y^0, y^0)n^i d\sigma, \end{aligned}$$

where $n = (n^1, n^2, \cdots, n^m)$ is the outward normal to the boundary Γ of Δ (supposing, for the moment, that n exists). The volume integral on the right is the rate at which energy enters the system from interior sources, whereas the surface integral is the rate at which energy enters the system through the boundary. Since we shall require $\langle y^0, y^0 \rangle_t \leq 0$, in particular, for all smooth initial functions vanishing outside of compact subsets of Δ, the condition (1.3) follows and has the physical significance of requiring that there be no internal energy sources.

Solutions to (1.2) whose boundary values satisfy the condition

$$(1.8) \qquad \int_\Gamma (A^i y^0, y^0)n^i d\sigma \leq 0$$

will clearly be dissipative. These are the solutions which are commonly associated with (1.2) and they are characterized by the fact that the corresponding generators lie between the previously defined minimal operator L_0 and maximal operator L_1; we shall call such solutions *properly dissipative*. On the other hand there are dissipative solutions of (1.2) which do not satisfy (1.8). For such solutions the energy entering through the boundary is offset by the energy losses due to internal sinks, and the domain of the corresponding gen-

erator is delimited by global lateral conditions, as previously mentioned([3]).
Solutions of this kind are treated in the appendix to Chapter II.

In general, of course, the above boundary integral is not well defined.
However, in situations where it is well defined, one sees that

$$(1.9) \qquad \int_\Gamma (A^i y^0, y^0) n^i d\sigma = \langle L_1 y^0, y^0 \rangle + \langle y^0, L_1 y^0 \rangle - \int_\Delta (EDy^0, y^0) d\chi.$$

The relation (1.9) provides us with a method for defining the boundary integral whenever the terms on the right are meaningful and accordingly *we take* (1.9) *as the definition of the boundary integral for an arbitrary domain* Δ. That the so-defined relation is actually a boundary functional can be seen from the fact that it vanishes for all continuously differentiable functions $y^0(\chi)$ with compact carriers in Δ([4]).

The presence of the term $\int_\Delta (EDy^0, y^0) d\chi$ in (1.9) gives rise to some difficulty since this integral will in general not exist for all of the members of $\mathfrak{D}(L_1)$ when this operator is suitably extended to be closed. We may avoid this difficulty by treating L_1 as a transformation on $H_1 = L_2(\Delta; EF)$ to $H_2 = L_2(\Delta; EF^{-1})$, where $F(\chi)$ is the matrix-valued function

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([3]) A simple example will serve to illustrate the kind of nonproperly dissipative extensions of L_0 which we have in mind. Let y^0 be a complex-valued function of χ, $0 < \chi < 1$, and set $L_1 y^0 = y^0_x - y^0$, $\mathfrak{D}(L_1) = [y^0; y^0$ absolutely continuous with y^0 and y^0_x in $L_2(0, 1)]$. We further define $L_0 \subset L_1$ by $\mathfrak{D}(L_0) = [y^0; y^0 \in \mathfrak{D}(L_1)$ and $y^0(0) = 0 = y^0(1)]$. It is clear that L_0 is dissipative, in fact, $\langle L_1 y^0, y^0 \rangle + \langle y^0, L_1 y^0 \rangle = -2 \langle y^0, y^0 \rangle + [|y^0(1)|^2 - |y^0(0)|^2]$, which is obviously nonpositive for $y^0 \in \mathfrak{D}(L_0)$. The second term in the right member represents the boundary integral (1.8) and one sees from this that a generator of properly dissipative solutions, say L_α, must be a restriction of L_1 with domain $\mathfrak{D}(L_\alpha) = [y^0; y^0 \in \mathfrak{D}(L^1)$ and $y^0(1) = \alpha y^0(0)]$ for fixed α of absolute value ≤ 1.

On the other hand, for arbitrary $h^0 \in L_2(0, 1)$ with $\langle h^0, h^0 \rangle \leq 2$ the operator $Ly^0 = L_1 y^0 + y^0(0)h^0$, $\mathfrak{D}(L) = [y^0; y^0 \in \mathfrak{D}(L_1)$ and $y^0(1) = 0]$ is a maximal dissipative extension of L_0 (but not a restriction of L_1). In fact, it is clearly dissipative since $\langle Ly^0, y^0 \rangle + \langle y^0, Ly^0 \rangle = -|y^0(0) - \langle y^0, h^0 \rangle|^2 - [2 \langle y^0, y^0 \rangle - |\langle y^0, h^0 \rangle|^2] \leq 0$. In order to show that L (or L_α) is maximal dissipative it suffices to show, by the corollary to Theorem 1.1.1, that the range of $I - L$ (or of $I - L_\alpha$) is $L_2(0, 1)$; and this can be accomplished by the usual green's function solution of $(I - L)y^0 = f^0$. The system corresponding to L requires a mechanism for feeding energy back into the interior with density $y^0(0)h^0$.

Since $L \supset L_0$, the adjoint of L, namely M, is a restriction of the adjoint of L_0 which we denote by M_1. It is readily seen that $M_1 z^0 = -z^0_x - z^0$, $\mathfrak{D}(M_1) = [z^0; z^0$ absolutely continuous with z^0 and z^0_x in $L_2(0, 1)]$. Moreover $\langle Ly^0, z^0 \rangle - \langle y^0, M_1 z^0 \rangle = -y^0(0)[z^0(0)]^- + y^0(0) \langle h^0, z^0 \rangle$ and this vanishes for all $y^0 \in \mathfrak{D}(L)$ (and hence for all $y^0(0)$) if and only if $z^0(0) = \langle z^0, h^0 \rangle$. This condition determines the domain of $M \subset M_1$ and it follows from Theorem 1.1.2 that M is a maximal dissipative restriction of M_1. M is thus an example of a maximal dissipative restriction of a maximal operator M_1 with a domain delimited by global lateral conditions.

([4]) If $y(\chi)$ vanishes outside of a compact subset of Δ, one can replace Δ in the right member of (1.9) by a large cube Δ_0 containing Δ. If one then applies the Gauss theorem (see, for instance, H. Federer [4, pp. 313–315]), the right side of (1.9) becomes equal to the left side with Γ now replaced by the boundary Γ_0 of Δ_0, and it is clear that the integrand vanishes on Γ_0.

(1.10) $$F = I - D \geqq I, \qquad\qquad \chi \in \Delta.$$

In this case all of the volume integrals appearing in the right member of (1.9) exist even for the closure of the so-defined L_1.

With this in mind, we start with the transformation L_{21}^{00} on H_1 to H_2 defined as

(1.11)
$$L_{21}^{00} y^1 = E^{-1}[(A^i y^1)_i + B y^1],$$

$$\mathfrak{D}(L_{21}^{00}) = [y^1; y^1 \text{ continuously differentiable with compact carrier}],$$

and denote its strong closure by L_{21}^0 and its weak closure[5] by L_{21}^1. The previously mentioned operators L_0 and L_1 are the maximal restrictions of these respective transformations which have their ranges in H_0. We note that the so-defined L_1 is the operator of largest domain and of type (1.2) for which the boundary integral is meaningful in our extended sense. Consequently, any generator of properly dissipative solutions of (1.2) is necessarily a restriction of L_1. On the other hand L_0 is the largest restriction of L_1 whose domain consists only of functions which are essentially zero on the boundary. Thus the domain of an operator L such that $L_0 \subset L \subset L_1$ will be determined entirely by the boundary values assumed by its member functions. It turns out that each maximal dissipative operator of this type generates only properly dissipative solutions of (1.2) and, conversely, any generator of properly dissipative solutions of (1.2) is maximal dissipative and of this type.

Our problem, then, is to characterize all maximal dissipative operators L such that $L_0 \subset L \subset L_1$ and to this end we introduce the following.

DEFINITION 1.2. *Let $H_{12} = H_1 \times H_2$ denote the product space of elements $y = [y^1, y^2]$ where $y^1 \in H_1$ and $y^2 \in H_2$ and set*

(1.12) $$Q(y, z) = \langle y^1, z^2 \rangle + \langle y^2, z^1 \rangle - \int_\Delta (EDy^1, z^1) d\chi.$$

A linear subspace $N \subset H_{12}$ [or $P \subset H_{12}$] is said to be negative [positive] if

$$Q(y, y) \leqq 0 \text{ for all } y \in N \qquad [Q(z, z) \geqq 0 \text{ for all } z \in P];$$

it is called maximal negative [maximal positive] if it is not the proper subspace of a negative [positive] subspace of H_{12}. The Q-orthogonal complement of a set S is defined as

$$[z; Q(y, z) = 0 \text{ for all } y \in S].$$

Following J. W. Calkin [1] (cf. M. I. Višik [13] and Lars Hörmander

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[8]), we represent the boundary data assumed by functions in $\mathfrak{D}(L_{21}^1)$ by the cosets of the quotient space

$$(1.13) \qquad\qquad H = \mathfrak{G}(L_{21}^1)/\mathfrak{G}(L_{21}^0),$$

where $\mathfrak{G}(L_{21}^i)$ denotes the graph of L_{21}^i, $i = 0, 1$. As we shall see, the bilinear form $Q(y, z)$ depends only on the cosets to which y and z belong and hence can be used to define the form $Q(y^0, z^0)$ where $y^0, z^0 \in H$ are the images of y and z, respectively, under the natural homomorphism. Finally, there is a one-to-one correspondence between the maximal negative subspaces N of H (relative to Q) and the maximal dissipative operators L such that $L_0 \subset L \subset L_1$; this correspondence is given by

$$(1.14) \qquad\qquad \mathfrak{D}(L) = [y^0; [y^0, L_1 y^0] \to N].$$

This, then, is the sense in which we have been able to characterize all properly dissipative solutions of (1.2).

We note that the formal adjoint of L_1 is of the form

$$(1.15) \qquad\qquad M_1 z^0 = E^{-1}[-(A^i z^0)_i + (B^* + A_i^i)z^0],$$

and it is readily verified that

$$(1.16) \qquad\qquad z_t^0 = M_1 z^0$$

is again a dissipative hyperbolic system. If we denote the adjoint of L by M, then we find that M is again maximal dissipative, $M_0 \subset M \subset M_1$, and that M generates properly disipative solutions of (1.16). Moreover, the domain of M is determined by the Q-orthogonal complement of N in H.

Boundary conditions of the elastic type are not included in the above development. In order to treat such boundary conditions we have coupled the system (1.2) at the boundary to a relatively simple "dissipative" system. As we shall see, the general theory developed in Chapter I applies as well to the coupled system.

The present work is an outgrowth of an earlier paper by the author [12] which treated the Cauchy problem (1.2) for the case of one spatial variable, in essence, by a green's function method. Much of the motivation for the ideas and concepts introduced here is to be found in [12]. However, the operator-theoretic approach is new and we feel that it has enabled us to treat (1.2) in the general case ($m \geq 1$) with much less work than was required by the previous method for the case $m = 1$. On the other hand, we have not been able to obtain the wealth of detail which is to be found in [12]. K. Yosida [14] has also written a paper treating the wave equation from the semi-group point of view in which he establishes the existence of a single solution, assuming Δ to be the entire euclidean space; moreover Yosida has imposed rather severe regularity conditions on the coefficients of the wave equation.

Chapter I. Theory of dissipative operators

1.1. Basic properties of dissipative operators. The central problem treated in this chapter is the extension of dissipative operators to maximal dissipative operators. The existence of such a maximal dissipative extension is a simple consequence of the maximal principle; however, mere existence is by no means sufficient for our purposes. The present section treats of certain basic properties of dissipative operators obtained, for the most part, by means of a variant of the von Neumann-Cayley transform theory (see, for instance, [11; §VI.3]). This transform theory furnishes some insight into the extension problem by providing a construction for all possible maximal dissipative extensions of a given dissipative operator with dense domain in terms of corresponding extensions of an associated contraction operator.

Definition 1.1.1. *If L is a transformation on H_0 to itself, we denote the range of L by $\Re(L)$ and the graph by $\mathfrak{G}(L)$;*

$$\mathfrak{G}(L) = [[y^0, Ly^0]; y^0 \in \mathfrak{D}(L)] \subset H_0 \times H_0.$$

Lemma 1.1.1. *Let L be a dissipative operator and suppose $\lambda > 0$. Then for $f^0 = \lambda y^0 - Ly^0$, $y^0 \in \mathfrak{D}(L)$, we have*

$$(1.1.1) \qquad\qquad \lambda \| y^0 \| \leq \| f^0 \|.$$

Further, the map $[y^0, Ly^0] \to f^0$ of $\mathfrak{G}(L)$ onto $\Re(\lambda I - L)$ is one-to-one and bicontinuous.

Proof. It is clear that

$$2\lambda\langle y^0, y^0 \rangle \leq 2\lambda\langle y^0, y^0 \rangle - [\langle Ly^0, y^0 \rangle + \langle y^0, Ly^0 \rangle]$$
$$= \langle f^0, y^0 \rangle + \langle y^0, f^0 \rangle \leq 2\| y^0 \| \| f^0 \|$$

from which (1.1.1) follows directly. Further, setting $c'(\lambda) = \max(1, \lambda)$, we have

$$\| f^0 \| \leq c'(\lambda)[\| y^0 \| + \| Ly^0 \|] \leq c'(\lambda)[\| y^0 \| + (\lambda\| y^0 \| + \| f^0 \|)]$$
$$\leq c'(\lambda)[2 + \lambda^{-1}]\| f^0 \|,$$

and this proves the last assertion of the lemma.

We note that (1.1.1) implies that $\lambda I - L$ is one-to-one when $\lambda > 0$ so that if $\Re(\lambda I - L) = H_0$, then L is necessarily maximal dissipative. It also follows from the lemma that L is closed if and only if $\Re(\lambda I - L)$ is closed.

We now define the operator

$$(1.1.2) \qquad J = (I + L)(I - L)^{-1} \quad \text{with} \quad \mathfrak{D}(J) = \Re(I - L).$$

It is easy to see that J is a contraction operator. For, setting

$$(1.1.3) \qquad u^0 = y^0 - Ly^0 \quad \text{and} \quad Ju^0 = y^0 + Ly^0, \qquad\qquad y^0 \in \mathfrak{D}(L),$$

we obtain

$$\|Ju^0\|^2 = \|y^0\|^2 + \|Ly^0\|^2 + [\langle Ly^0, y^0 \rangle + \langle y^0, Ly^0 \rangle]$$
$$\leq \|y^0\|^2 + \|Ly^0\|^2 - [\langle Ly^0, y^0 \rangle + \langle y^0, Ly^0 \rangle] = \|u^0\|^2;$$

here we have made use of the fact that L is dissipative. Further solving the Equations (1.1.3) for y^0 and Ly^0 we get

$$(1.1.4) \qquad y^0 = \frac{1}{2}\left[Ju^0 + u^0\right] \quad \text{and} \quad Ly^0 = \frac{1}{2}\left[Ju^0 - u^0\right], \qquad u^0 \in \mathfrak{D}(J),$$

from which it follows that $I+J$ is one-to-one and that $\mathfrak{R}(I+J) = \mathfrak{D}(L)$. Since J is bounded, it will be closed if and only if $\mathfrak{D}(J) = \mathfrak{R}(I-L)$ is closed and hence if and only if L is closed.

Conversely, suppose J is a given contraction operator such that $I+J$ is one-to-one. Then the operator L defined by means of (1.1.4) is linear (and hence single-valued) with $\mathfrak{D}(L) = \mathfrak{R}(I+J)$. Moreover, for $y^0 \in \mathfrak{D}(L)$ we have

$$\langle Ly^0, y^0 \rangle + \langle y^0, Ly^0 \rangle = \frac{1}{4}\left[\langle (Ju^0 - u^0), (Ju^0 + u^0) \rangle + \langle (Ju^0 + u^0), (Ju^0 - u^0) \rangle\right]$$

$$= \frac{1}{2}\left[\|Ju^0\|^2 - \|u^0\|^2\right] \leq 0,$$

so that L is a dissipative operator.

Finally we show that $I+J$ will be one-to-one if $\mathfrak{R}(I+J)$ is dense in H_0. To this end, suppose that $Ju^0 + u^0 = \theta$, let v^0 be any element of $\mathfrak{D}(J)$, and set $g^0 = v^0 + Jv^0$. Then

$$\|J(v^0 - \alpha u^0)\|^2 \leq \|v^0 - \alpha u^0\|^2$$

which reduces to

$$\alpha\langle u^0, g^0 \rangle + \bar{\alpha}\langle g^0, u^0 \rangle \leq \|v^0\|^2 - \|Jv^0\|^2.$$

Since α is arbitrary, it follows that $\langle u^0, g^0 \rangle = 0$ and by assumption this must hold for a dense set of g^0. Consequently $u^0 = \theta$ and this proves that $I+J$ is one-to-one.

We note that the Banach-Steinhaus theorem provides any contraction operator J with a unique extension of the same kind having the closure of $\mathfrak{D}(J)$ as its domain. On the other hand, a contraction operator J with a closed domain can always be extended in a trivial fashion to a contraction operator J_1 on H_0 by setting $J_1 y^0 = \theta$ for all $y^1 \in \mathfrak{D}(J)^{\perp}$. We summarize these results in

THEOREM 1.1.1. *If L_0 is a dissipative operator with dense domain, then the operator J_0 defined as in (1.1.3) is a contraction operator; L_0 and J_0 are closed together. Conversely, if J_0 is a contraction operator with $\mathfrak{R}(I+J_0)$ dense in H_0, then L_0 defined as in (1.1.4) is a dissipative operator with dense domain and J_0 is again given by (1.1.3). The relations (1.1.3) and (1.1.4) establish a one-to-one inclusion preserving correspondence between all dissipative extensions L of*

L_0 and all contraction extensions J of J_0. In particular, the maximal dissipative extensions of L_0 correspond to the (closed) contraction extensions of J_0 with domains equal to H_0.

COROLLARY. *Let* $\lambda > 0$ *and suppose that* L *is a dissipative operator with dense domain. Then* L *is maximal dissipative if and only if* $\Re(\lambda I - L) = H_0$.

Proof. It is clear that L and $\lambda^{-1}L$ are dissipative and maximal dissipative together. It follows from the previous theorem that $\lambda^{-1}L$ is maximal dissipative if and only if $\Re(I - \lambda^{-1}L) = \Re(\lambda I - L)$ fills out H_0.

The nontrivial contraction extensions of J_0 are governed by the following.

LEMMA 1.1.2. *Let* J *be a linear operator defined on the two-dimensional subspace generated by* y^0 *and* z^0 *where* $\langle y^0, z^0 \rangle = 0$. *A necessary and sufficient condition that* J *be a contraction operator is that*

$$(1.1.5) \qquad |\langle Jy^0, Jz^0 \rangle|^2 \leqq \delta y^0 \delta z^0,$$

where $\delta u^0 = \|u^0\|^2 - \|Ju^0\|^2$.

Proof. The operator J will be a contraction operator if and only if

$$\|J(\alpha y^0 + \gamma z^0)\|^2 \leqq \|\alpha y^0 + \gamma z^0\|^2 = |\alpha|^2 \|y^0\|^2 + |\gamma|^2 \|z^0\|^2$$

for all α, γ. This is equivalent with

$$|\alpha|^2 \delta y^0 + |\gamma|^2 \delta z^0 - \alpha\bar{\gamma}\langle Jy^0, Jz^0 \rangle - \bar{\alpha}\gamma\langle Jz^0, Jy^0 \rangle \geqq 0.$$

For fixed $|\alpha|$ and $|\gamma|$ the minimum of the left member is attained when $\arg(\alpha\bar{\gamma}) = \arg(\langle Jz^0, Jy^0 \rangle)$ in which case

$$|\alpha|^2 \delta y^0 + |\gamma|^2 \delta z^0 - 2|\alpha||\gamma||\langle Jy^0, Jz^0 \rangle| \geqq 0$$

and this is easily seen to be true for all $|\alpha|, |\gamma|$ if and only if (1.1.5) is satisfied.

It follows from Theorem 1.1.1 that a maximal dissipative operator with dense domain is necessarily closed([6]). We next prove a converse to this statement.

LEMMA 1.1.3. *If* L *is maximal dissipative and closed, then it has a dense domain.*

([6]) The following is an example of a maximal dissipative operator L which is not closed. In view of the above correspondence theory, it is clear that for the construction of such an L it suffices to define a contraction operator J with $I+J$ one-to-one which has no proper extension with these same properties, and yet such that $\mathfrak{D}(J)$ is not closed. To this end suppose that $\{\phi_n^0\}$ is a complete orthonormal system for H_0 and define J_1 with domain H_0 so that $J_1\phi_1^0 = -\phi_1^0$ and $J_1\phi_n^0 = \phi_n^0$ for all $n > 1$. Then $(I+J_1)y^0 = \theta$ if and only if $y^0 = \alpha\phi_1^0$. Thus the above requirements for J will be satisfied by any restriction of J_1 for which $\mathfrak{D}(J)$ is dense in H_0 and for which $\mathfrak{D}(J)$ is maximal with respect to the property of not including ϕ_1^0. Now $\mathfrak{D}_0 \equiv$ linear extension of $[\sum_{n=1}^{\infty} 2^{-n}\phi_n^0, \phi_2^0, \phi_3^0, \cdots]$ is dense in H_0 and does not include ϕ_1^0. Applying the maximal principle to the subspaces of H_0 containing $\mathfrak{D}_0$ yet not containing ϕ_1^0, yields a domain with the desired properties.

Proof. We first show that $\Re(I-L) = H_0$. To this end we define J as in (1.1.3). Then J is a contraction operator with $\mathfrak{D}(J) = \Re(I-L)$. Since L is a closed operator, Lemma 1.1.1 implies that $\Re(I-L)$ is a closed subspace. Consequently if $\Re(I-L)$ is a proper subspace of H_0 there exists a nonzero vector $u^0 \in \mathfrak{D}(J)^\perp$. We can therefore extend J to a contraction operator J_1 by setting $J_1(y^0 + \alpha u^0) = Jy^0$ for all $y^0 \in \mathfrak{D}(J)$. Now $I+J_1$ is one-to-one. For suppose $J_1(y^0 + \alpha u^0) = -(y^0 + \alpha u^0)$. Since $I+J$ is one-to-one we see that $\alpha \neq 0$ and hence that

$$\left\| y^0 + \alpha u^0 \right\| = \left\| J_1(y^0 + \alpha u^0) \right\| = \left\| Jy^0 \right\| \leqq \left\| y^0 \right\| < \left\| y^0 + \alpha u^0 \right\|,$$

which is impossible. It follows that $I+J_1$ is one-to-one and hence that J_1 defines a proper dissipative extension of L as in (1.1.4). Since L was assumed to be maximal dissipative, we conclude that $\Re(I-L) = \mathfrak{D}(J) = H_0$. Next, suppose that $\mathfrak{D}(L) = \Re(I+J)$ is not dense in H_0. Then the adjoint of $I+J$ takes a nonzero vector, say v^0, into zero; that is, $J^*v^0 = -v^0$. Now J^* is obviously a contraction operator along with J. Applying Lemma 1.1.2 to J^* (with $\delta v^0 = 0$), we see that $\langle J^*y^0, v^0 \rangle = -\langle J^*y^0, J^*v^0 \rangle = 0$ whenever y^0 is orthogonal to v^0. It follows from this that $Jv^0 = -v^0$, contrary to $I+J$ being one-to-one. We therefore conclude that $\mathfrak{D}(L) = \Re(I+J)$ is dense in H_0.

THEOREM 1.1.2. *If L is a maximal dissipative operator with dense domain, then so is L^*, and if L corresponds as in Theorem 1.1.1 to the contraction operator J, then L^* corresponds to the contraction operator J^*.*

Proof. Since L is maximal dissipative with dense domain, the corresponding contraction operator J will have H_0 for its domain and $\Re(I+J)$ will be dense. Hence J^* is well-defined and it is obviously a contraction operator. Moreover, $\Re(I+J^*)$ is also dense in H_0. If this were not so, then its adjoint, $I+J$, would take a nonzero vector into θ. However, as we have seen in the proof of Theorem 1.1.1, this is contrary to $\Re(I+J)$ being dense. Consequently $\Re(I+J^*)$ is dense and J^* corresponds as in Theorem 1.1.1 to a maximal dissipative operator, say M, with dense domain. Further $\langle Ju^0, v^0 \rangle = \langle u^0, J^*v^0 \rangle$ implies

$$\langle Ju^0 - u^0, J^*v^0 + v^0 \rangle = \langle Ju^0 + u^0, J^*v^0 - v^0 \rangle$$

and hence $\langle Ly^0, z^0 \rangle = \langle y^0, Mz^0 \rangle$ for all $y^0 \in \mathfrak{D}(L)$, $z^0 \in \mathfrak{D}(M)$. As a consequence $M \subset L^*$. We further note that $\Re(I-M) = \mathfrak{D}(J^*) = H_0$. Thus if L^* were a proper extension of M, then $I-L^*$ would take a nonzero vector into θ and this is impossible when the range of its adjoint, $I-L$, equals H_0.

COROLLARY. *Let L_0 be a dissipative operator with dense domain and set $M_1 = L_0^*$. Then there exist maximal dissipative operators L and M, adjoints of each other and both having dense domains, such that*

$$L \supset L_0 \quad and \quad M \subset M_1.$$

Proof. Let L be any maximal dissipative operator extending L_0 and set $M = L^*$. According to the above theorem M is again maximal dissipative with dense domain. Finally, since $L_0 \subset L$, it follows that $M = L^* \subset L_0^* = M_1$.

The next theorem exhibits the importance of maximal dissipative operators in the theory of semi-groups.

THEOREM 1.1.3. *A necessary and sufficient condition for an operator L to generate a strongly continuous semi-group of contraction operators on H_0 to itself is that L be a maximal dissipative operator with dense domain.*

Proof. If L is the infinitesimal generator of a strongly continuous semi-group of contraction operators, say $[S(t); t \geqq 0]$, then $\|S(t)y^0\| \leqq \|y^0\|$ and for $y^0 \in \mathfrak{D}(L)$

$$\langle y^0, Ly^0 \rangle + \langle Ly^0, y^0 \rangle = \frac{d}{dt} \langle S(t)y^0, S(t)y^0 \rangle \Big|_{t=0} \leqq 0.$$

It follows that L is dissipative. According to the general theory of semigroups (see, for instance, Hille and Phillips [7]), $\mathfrak{D}(L)$ is dense in H_0 and the resolvent set of L contains all real $\lambda > 0$. This implies, in particular, that $\Re(\lambda I - L) = H_0$ for $\lambda > 0$ so that L is maximal dissipative by the corollary to Theorem 1.1.1. Conversely, if L is maximal dissipative with dense domain and $\lambda > 0$, then the same corollary implies that $\Re(\lambda I - L) = H_0$. This together with the inequality (1.1.1) shows that the resolvent of L, namely $\Re(\lambda; L) = (\lambda I - L)^{-1}$, exists and satisfies the inequality $\|\lambda \Re(\lambda; L)\|_0 \leqq 1$, $\lambda > 0$. Thus the hypothesis of the Hille-Yosida theorem (see [7]) is satisfied, and it follows that L generates a strongly continuous semi-group of contraction operators.

Semi-groups of isometries form a particularly interesting subclass of the semi-groups of contraction operators since they correspond to energy conserving solutions of the Cauchy problem. They were first studied by J. L. B. Cooper [2; 3] who gave a penetrating characterization of them. As we shall see, the generators of such semi-groups satisfy the following property.

DEFINITION 1.1.2. *An operator L is called conservative if*

$$(1.1.6) \qquad \langle Ly^0, y^0 \rangle + \langle y^0, Ly^0 \rangle = 0, \qquad\qquad y^0 \in \mathfrak{D}(L).$$

LEMMA 1.1.4. *The operator L is conservative if and only if iL is symmetric. If L is conservative and maximal dissipative with dense domain, then iL is maximal symmetric. Conversely, if iL is maximal symmetric with dense domain, then either L or $-L$ is conservative and maximal dissipative. Finally, L and L^* are conservative with dense domains if and only if iL is self-adjoint.*

Proof. If L is conservative, then replacing y^0 in (1.1.6) by $u^0 + \epsilon v^0$, $\epsilon = \pm 1$, $\pm i$, in turn and combining the resulting relations in the usual way, one obtains

$$(1.1.7) \qquad \langle Lu^0, v^0 \rangle + \langle u^0, Lv^0 \rangle = 0, \qquad\qquad u^0, v^0 \in \mathfrak{D}(L).$$

It follows that iL is symmetric whenever L is conservative; the converse assertion is obvious. On the other hand if L is conservative and maximal dissipative with dense domain, then $\Re(iI - iL) = \Re(I - L) = H_0$ by the corollary to Theorem 1.1.1. This shows that one of the deficiency indices of iL is zero and hence that iL is maximal symmetric. Conversely if iL is maximal symmetric with dense domain then $\Re(I \pm L) = \Re(iI \pm iL) = H_0$ for one of the indicated signs and since L and $-L$ are conservative together it follows from the above cited corollary that one of these operators is maximal dissipative. Finally if L and L^* are conservative with dense domains, then iL and $(iL)^* = -iL^*$ are symmetric and hence iL is self-adjoint; again the converse is obvious.

LEMMA 1.1.5. *Suppose L_0 is a closed conservative operator with dense domain and set $L_1 = -L_0^*$. Then each maximal dissipative extension of L_0 is a restriction of L_1.*

Proof. Since iL_0 is symmetric, it is clear that $L_0 \subset L_1$. Suppose L is a maximal dissipative extension of L_0 and let J_0 and J be the contraction operators corresponding to L_0 and L, respectively, as in Theorem 1.1.1. Then J is an extension of J_0 with $\mathfrak{D}(J) = H_0$, and it is readily seen that J_0 is an isometry with closed domain. Hence by Lemma 1.1.2, $v^0 = Ju^0 \in \Re(J_0)^\perp$ whenever $u^0 \in \mathfrak{D}(J_0)^\perp$. In other words, if

$$\langle y^0 - L_0 y^0, u^0 \rangle = 0, \qquad\qquad y^0 \in \mathfrak{D}(L_0),$$

then

$$\langle y^0 + L_0 y^0, v^0 \rangle = 0, \qquad\qquad y^0 \in \mathfrak{D}(L_0).$$

In this case $L_1 u^0 = -L_0^* u^0 = -u^0$ and $L_1 v^0 = -L_0^* v^0 = v^0$ so that by (1.1.4)

$$L(v^0 + u^0) = v^0 - u^0 = L_1(v^0 + u^0).$$

Since $\mathfrak{D}(L)$ is the linear extension of $\mathfrak{D}(L_0)$ and vectors of the kind $(v^0 + u^0)$, it follows that $L \subset L_1$.

The next theorem is due to Cooper [3], however the argument is new.

THEOREM 1.1.4. *A necessary and sufficient condition for an operator L to generate a strongly continuous semi-group of isometries is that L be conservative and maximal dissipative with dense domain.*

Proof. If $[S(t)]$ is a semi-group of isometries, then $\|S(t)y^0\| = \|y^0\|$ for all $t \geq 0$. Thus for $y^0 \in \mathfrak{D}(L)$

$$\langle Ly^0, y^0 \rangle + \langle y^0, Ly^0 \rangle = \frac{d}{dt} \langle S(t)y^0, S(t)y^0 \rangle \bigg|_{t=0} = 0$$

and we see that L is conservative. In addition, it follows from Theorem 1.1.3 that L is also maximal dissipative with dense domain. Conversely, suppose

L is known to be conservative and maximal dissipative with dense domain. Then Theorem 1.1.3 also asserts that L generates a strongly continuous semi-group of contraction operators, say $[S(t)]$. For $y^0 \in \mathfrak{D}(L)$, $S(t)y^0$ lies in $\mathfrak{D}(L)$ for all $t > 0$ and, making use of the fact that L is conservative, we get

$$\frac{d}{dt} \langle S(t)y^0, S(t)y^0 \rangle = \langle LS(t)y^0, S(t)y^0 \rangle + \langle S(t)y^0, LS(t)y^0 \rangle = 0.$$

Consequently $\|S(t)y^0\| = \|y^0\|$, $t > 0$, for each $y^0 \in \mathfrak{D}(L)$ and since $\mathfrak{D}(L)$ is dense in H_0, this relation holds for all $y^0 \in H_0$; thus $S(t)$ is an isometry. This completes the proof.

THEOREM 1.1.5. *A necessary and sufficient condition for a conservative maximal dissipative operator with dense domain to generate a strongly continuous group is that iL be self-adjoint. In this case the group consists of unitary operators.*

The argument can be taken verbatim from [12, Theorem 8.2].

1.2. **Dual hilbert spaces.** The theory which we are about to present requires two auxiliary dual hilbert spaces in addition to the basic hilbert space H_0. These are defined by means of a positive definite self-adjoint operator F with (dense) domain $\mathfrak{D}(F)$ satisfying the condition

$$(1.2.1) \qquad\qquad \langle Fy^0, y^0 \rangle \geqq \langle y^0, y^0 \rangle, \qquad\qquad y^0 \in \mathfrak{D}(F).$$

It is clear that F^{-1} exists and is a positive definite operator with domain H_0 and norm $\leqq 1$.

We now define two new inner products

$$(1.2.2) \qquad\qquad \langle y^0, z^0 \rangle_1 \equiv \langle Fy^0, z^0 \rangle, \qquad\qquad y^0, z^0 \in \mathfrak{D}(F),$$

and

$$(1.2.3) \qquad\qquad \langle y^0, z^0 \rangle_2 \equiv \langle F^{-1}y^0, z^0 \rangle, \qquad\qquad y^0, z^0 \in H_0.$$

Finally H_1 and H_2 are the respective completions of the two pre-hilbert spaces defined by these inner products. A Cauchy sequence consisting of repetitions of a fixed element y^0 of $\mathfrak{D}(F)$ will, of course, define an element, say y^1, of H_1 and for the sake of clarity we will write $y^0 = I_{01}y^1$. Likewise a Cauchy sequence consisting of a single element y^0 of H_0 will define an element y^2 in H_2 and we shall write $y^2 = I_{20}y^0$. Both of these maps are one-to-one.

LEMMA 1.2.1. *The map I_{01} can be uniquely extended to be one-to-one on H_1 into H_0 and of norm $\leqq 1$. Similarly the map I_{20} is one-to-one and of norm $\leqq 1$.*

Proof. The elements of H_1 are defined in the usual manner as classes of Cauchy sequences in $\mathfrak{D}(F)$. Suppose that $\{y_n^0\}$ is such a sequence defining an element y^1 of H_1. Then

$$\left\| \overset{0}{y_n} - \overset{0}{y_m} \right\| \leqq \left\| \overset{0}{y_n} - \overset{0}{y_m} \right\|_1 \to 0$$

as n, $m \to \infty$. Thus $\{ \overset{0}{y_n} \}$ is also a Cauchy sequence in H_0 and converges to some element, say $y^0 \in H_0$. We define $I_{01} y^1 = y^0$. It is clear that this extends our previously defined I_{01} and that the so-defined I_{01} is linear and of norm $\leqq 1$. To show that the extended I_{01} is one-to-one, we suppose that the above $y^0 = \theta$. Then for arbitrary $z^0 \in \mathfrak{D}(F)$ with $z^1 = I_{01}^{-1} z^0$, we have

$$\langle \overset{1}{y}, \overset{1}{z} \rangle_1 = \lim_{n \to \infty} \langle F \overset{0}{y_n}, \overset{0}{z} \rangle = \lim_{n \to \infty} \langle \overset{0}{y_n}, F \overset{0}{z} \rangle = \langle \overset{0}{y}, F \overset{0}{z} \rangle = 0.$$

Thus $\langle y^1, z^1 \rangle_1 = 0$ for a set of z^1 dense in H_1 and it follows that $y^1 = \theta$. The uniqueness of this extension is a consequence of the fact that $I_{01}^{-1}[\mathfrak{D}(F)]$ is dense in H_1. The assertions about I_{20} follow directly from the properties of F^{-1}.

LEMMA 1.2.2. *There is a unitary map of H_1 onto H_2 which takes the Cauchy sequence $\{ \overset{0}{y_n} \} \subset \mathfrak{D}(F)$, defining an element of H_1, into the Cauchy sequence $\{ F \overset{0}{y_n} \}$, defining an element of H_2. This map can be thought of as an extension of F and we shall denote it by F_{21}, with $F_{12} = F_{21}^{-1}$. For $y^1 \in H_1$ and $z^2 \in H_2$ we define*

(1.2.4) $$\langle y^1, z^2 \rangle \equiv \langle y^1, F_{12} z^2 \rangle_1 = \langle F_{21} y^1, z^2 \rangle_2.$$

For Cauchy sequences $\{ \overset{0}{y_n} \} \subset \mathfrak{D}(F)$ defining $y^1 \in H_1$ and $\{ \overset{0}{z_k} \} \subset H_0$ defining $z^2 \in H_2$ we have

(1.2.5) $$\langle \overset{1}{y}, \overset{2}{z} \rangle = \lim_{n,k \to \infty} \langle \overset{0}{y_n}, \overset{0}{z_k} \rangle.$$

In particular if $y^1 = I_{01}^{-1} y^0$ and $z^2 = I_{20} z^0$, then $\langle y^1, z^2 \rangle = \langle y^0, z^0 \rangle$.

Proof. The unitary character of F_{12} follows immediately from two relations

$$\langle F \overset{0}{y_n}, F \overset{0}{y_n} \rangle_2 = \langle F^{-1}(F \overset{0}{y_n}), F \overset{0}{y_n} \rangle = \langle F \overset{0}{y_n}, \overset{0}{y_n} \rangle = \langle \overset{0}{y_n}, \overset{0}{y_n} \rangle_1$$

and

$$\langle \overset{0}{z_n}, \overset{0}{z_n} \rangle_2 = \langle F^{-1} \overset{0}{z_n}, \overset{0}{z_n} \rangle = \langle F(F^{-1} \overset{0}{z_n}), F^{-1} \overset{0}{z_n} e \rangle = \langle F^{-1} \overset{0}{z_n}, F_1^{-1} \overset{0}{z_n} \rangle_1.$$

Further, since $\{ F^{-1} \overset{0}{z_k} \}$ defines the element $F_{12} z^2$ in H_1, we have

$$\langle \overset{1}{y}, \overset{2}{z} \rangle \equiv \langle \overset{1}{y}, F_{12} \overset{2}{z} \rangle_1 = \lim_{n,k \to \infty} \langle F \overset{0}{y_n}, F^{-1} \overset{0}{z_k} \rangle = \lim_{n,k \to \infty} \langle \overset{0}{y_n}, \overset{0}{z_k} \rangle.$$

By choosing $\overset{0}{z_k} = z^0$ for all k, we see that $z^2 = I_{20} z^0$ and $y^0 \equiv \lim_n \overset{0}{y_n} = I_{01} y^1$, which proves the last assertion of the lemma.

One sees from the unitary nature of F_{12} that

$$\left| \langle y^1, z^2 \rangle \right| = \left| \langle y^1, F_{12} z^2 \rangle_1 \right| \leqq \| y^1 \|_1 \| F_{12} z^2 \|_1 = \| y^1 \|_1 \| z^2 \|_2.$$

DEFINITION 1.2.1. *Let C_{21} be a bounded linear transformation on H_1 to H_2. Then C_{21} will be called symmetric if $\langle C_{21}y^1, z^1 \rangle = \langle y^1, C_{21}z^1 \rangle$ for all $y^1, z^1 \in H_1$, and it will be called positive if $\langle C_{21}y^1, y^1 \rangle \geq 0$ for all $y^1 \in H_1$, in symbols, $C_{21} \geq \Theta$.*

LEMMA 1.2.3. *Let C be a linear operator on H_0 with $\mathfrak{D}(C) = \mathfrak{D}(F)$ and such that*

$$(1.2.6) \qquad\qquad \langle F^{-1}Cy^0, Cy^0 \rangle \leq k \langle Fy^0, y^0 \rangle, \qquad\qquad y^0 \in \mathfrak{D}(F).$$

Then C takes each Cauchy sequence $\{y_n^0\} \subset \mathfrak{D}(F)$, defining an element of H_1, into a Cauchy sequence $\{Cy_n^0\}$, defining an element of H_2, and in this way defines a bounded linear transformation C_{21} on H_1 to H_2 of norm $\leq k^{1/2}$. If C is symmetric (or positive), then C_{21} will likewise be symmetric (or positive).

Proof. The condition (1.2.6) implies that $\{Cy_n^0\}$ is a Cauchy sequence defining an element of H_2 and hence that

$$\langle C_{21}y^1, C_{21}y^1 \rangle_2 = \lim_{n \to \infty} \langle F^{-1}Cy_n^0, Cy_n^0 \rangle \leq k \lim_{n \to \infty} \langle Fy_n^0, y_n^0 \rangle \leq k \langle y^1, y^1 \rangle_1.$$

It follows from this that C_{21} is a linear transformation on H_1 to H_2 of bound $\leq k^{1/2}$. Let y^1 and z^1 be defined by the Cauchy sequences $\{y_n^0\}$ and $\{z_n^0\} \subset \mathfrak{D}(F)$, respectively. Then if C is symmetric

$$\langle C_{21}y^1, z^1 \rangle = \lim_{n \to \infty} \langle Cy_n^0, z_n^0 \rangle = \lim_{n \to \infty} \langle y_n^0, Cz_n^0 \rangle = \langle y^1, C_{21}z^1 \rangle,$$

whereas if C is positive

$$\langle C_{21}y^1, y^1 \rangle = \lim_{n \to \infty} \langle Cy_n^0, y_n^0 \rangle \geq 0.$$

COROLLARY. *The operators I, $D \equiv I - F$, and $F(\lambda) \equiv \lambda I - D$, $\lambda > 0$, each with domain $\mathfrak{D}(F)$, have bounded linear extensions on H_1 to H_2. If we denote these extensions by I_{21}, D_{21} and $F(\lambda)_{21}$, respectively, then $I_{21} = I_{20}I_{01}$, $D_{21} = I_{21} - F_{21}$, and $F(\lambda)_{21} = \lambda I_{21} - D_{21}$. Further I_{21}, $-D_{21}$ and $F(\lambda)_{21}$ are positive symmetric.*

Proof. Let C denote I, $-D$, and $F(\lambda)$ in turn. In each case C is positive symmetric and there exists a k such that

$$0 \leq \langle Cy^0, y^0 \rangle \leq k^{1/2} \langle Fy^0, y^0 \rangle, \qquad\qquad y^0 \in \mathfrak{D}(F).$$

Applying the Schwarz inequality with respect to the inner product $\langle Cz^0, z^0 \rangle$, we obtain

$$[\langle F^{-1}(Cy^0), Cy^0 \rangle]^2 \leq \langle Cy^0, y^0 \rangle \langle CF^{-1}Cy^0, F^{-1}Cy^0 \rangle$$
$$\leq k \langle Fy^0, y^0 \rangle \langle FF^{-1}Cy^0, F^{-1}Cy^0 \rangle,$$

and this implies (1.2.6). Thus the operators have bounded linear extensions on H_1 to H_2 and the asserted relations follow if we merely consider the Cauchy sequences used in defining these extensions. We note, incidently, for both I_{21} and $-D_{21}$ that $k \leq 1$.

1.3. **Negative subspaces.** Next let $H_{12} = H_1 \times H_2$ denote the product space of elements $y = [y^1, y^2]$, $y^1 \in H_1$ and $y^2 \in H_2$ with inner product

$$(1.3.1) \qquad\qquad \langle y, z \rangle = \langle y^1, z^1 \rangle_1 + \langle y^2, z^2 \rangle_2.$$

When $F = I$ we note that $H_1 = H_0 = H_2$ and H_{12} becomes $H_0 \times H_0$.

DEFINITION 1.3.1. *If L_{21} is a transformation on H_1 to H_2 with domain $\mathfrak{D}(L_{21})$, then its graph is a subspace of H_{12} defined as*

$$\mathfrak{G}(L_{21}) = [[y^1, L_{21}y^1]; y^1 \in \mathfrak{D}(L_{21})].$$

As we shall see, the graphs of the dissipative operators in which we are interested correspond to negative subspaces of H_{12} relative to the hermitian symmetric bilinear form

$$(1.3.2) \qquad\qquad Q(y, z) \equiv \langle y^1, z^2 \rangle + \langle y^2, z^1 \rangle - \langle D_{21}y^1, z^1 \rangle.$$

It is at any rate clear in the case $F = I$ (and hence $D_{21} = \Theta$) that the graph of a dissipative operator is negative.

It is easy to see that Q is a continuous form; in fact

$$(1.3.3) \qquad | Q(y, z) | \leq \|y^1\|_1 \|z^2\|_2 + \|y^2\|_2 \|z^1\|_1 + \|y^1\|_1 \|z^1\|_1 \leq 2\|y\|\|z\|.$$

The continuity of Q shows that the closure of a negative (or positive) subspace is again negative (or positive). It follows from the maximal principle that each negative (positive) subspace is contained in a maximal negative (maximal positive) subspace, which is necessarily closed since otherwise its closure would again be negative (positive). Moreover the continuity of Q also requires that the Q-orthogonal complement of a set be a closed linear subspace.

It is convenient at this point to introduce the following operators on H_{12} to itself:

$$
\begin{aligned}
& U[y^1, y^2] = [y^1, -y^2 + D_{21}y^1], \\
(1.3.4) \quad & V[y^1, y^2] = [F_{12}y^2, -F_{21}y^1], \\
& W[y^1, y^2] \equiv UV[y^1, y^2] = [F_{12}y^2, F_{21}y^1 + D_{21}F_{12}y^2].
\end{aligned}
$$

The corollary to Lemma 1.2.3 shows that U, V, and W are all linear bounded operators. Moreover

$$(1.3.5) \qquad\qquad U^2 = I \quad \text{and} \quad V^2 = -I,$$

so that the inverses of these operators are likewise bounded operators on H_{12}. A simple calculation gives

$$Q(Uy,\ Uz) = -\ Q(y,\ z),$$

(1.3.6) $Q(y,\ Uz) = -\langle y^1,\ z^2\rangle + \langle y^2,\ z^1\rangle,$

$$Q(Wy,\ z) = Q(y,\ Wz) = \langle F_{21}y^1,\ z^1\rangle + \langle F_{12}y^2, z^2\rangle = \langle y,\ z\rangle.$$

The first of the identities (1.3.6) shows that the mapping $y\to Uy$ of H_{12} on itself defines a one-to-one correspondence between the negative subspaces and the positive subspaces and it is clear that this correspondence preserves inclusion and Q-orthogonality. As a consequence, a statement about negative subspaces will, in general, imply a dual statement about positive subspaces.

DEFINITION 1.3.2. *Given a subset $S\subset H_{12}$, we define its domain $\mathfrak{D}(S)$ and range $\mathfrak{R}(S)$ by*

$$\mathfrak{D}(S) = [y^1;\ [y^1,\ y^2] \in S],$$
$$\mathfrak{R}(S) = [y^2;\ [y^1,\ y^2] \in S].$$

We further set

$$\mathfrak{R}_\lambda(S) = [\lambda I_{21}y^1 - y^2;\ [y^1,\ y^2] \in S].$$

A linear subspace S of H_{12} is the graph of a linear transformation on H_1 to H_2 if and only if S contains no element $y = [y^1,\ y^2]$ of the form $y^1 = \theta$, $y^2 \neq \theta$. It follows from this that S is a graph if and only if $U[S]$ is a graph. We note that $\mathfrak{D}(S) = \mathfrak{D}(U[S])$. The graph of a closed linear transformation is by definition a closed subspace. It should be observed that the graph of a linear transformation on H_1 to H_2 may be negative and the transformation may even be maximal in this respect, yet the graph itself may be the proper subset of some other negative subspace. As the following lemma shows, this situation will not occur if the domain of the transformation is dense in H_1.

LEMMA 1.3.1. *A negative (or positive) subspace with dense domain is the graph of a linear transformation on H_1 to H_2.*

Proof. From the above remarks, it is clear that it suffices to consider only the case of a negative subspace N with dense domain. Suppose there were a $u = [\theta,\ u^2]\in N$, $u^2\neq\theta$; we note that $Q(u,\ u) = 0$. Since $\mathfrak{D}(N)$ is dense in H_1, there is a $y = [y^1,\ y^2]\in N$ such that $\langle y^1,\ u^2\rangle = \langle y^1,\ F_{12}u^2\rangle_1 \neq 0$. Now $y + \alpha u\in N$ and hence

$$Q(y + \alpha u,\ y + \alpha u) = Q(y,\ y) + \alpha Q(u,\ y) + \bar\alpha Q(y,\ u) \leqq 0.$$

As this holds for all α, we conclude that $Q(y,\ u) = \langle y^1,\ u^2\rangle$ must equal zero, contrary to our choice of y.

COROLLARY. *If the negative (positive) subspace N is the graph of a linear transformation on H_1 to H_2 with dense domain, then the same is true of each maximal negative (maximal positive) subspace containing N.*

Next we obtain some properties of negative subspaces.

Lemma 1.3.2. *Let N be a negative subspace and suppose $\lambda > 0$. Then for* $f^2 = \lambda I_{21} y^1 - y^2$, $[y^1, y^2] \in N$, *we have*

(1.3.7) $$c(\lambda) \|y^1\|_1 \leqq \|f^2\|_2,$$

where $c(\lambda) = \min(1/2, \lambda)$.

Proof. We obtain directly from the definition of f^2 that $\lambda \langle I_{21} y^1, y^1 \rangle - \langle y^2, y^1 \rangle = \langle f^2, y^1 \rangle$ and adding this to its complex conjugate we get

(1.3.8) $2\lambda \langle I_{21} y^1, y^1 \rangle - \langle D_{21} y^1, y^1 \rangle - Q(y, y) = \langle y^1, f^2 \rangle + \langle f^2, y^1 \rangle \leqq 2\|y^1\|_1 \|f^2\|_2.$

According to the corollary to Lemma 1.2.3 both I_{21} and $-D_{21}$ are positive symmetric and $F_{21} = I_{21} - D_{21}$; this together with the fact that $Q(y, y) \leqq 0$ for $y \in N$ implies

$$c(\lambda) \|y^1\|_1^2 = c(\lambda) \langle F_{21} y^1, y^1 \rangle \leqq \|y^1\|_1 \|f^2\|_2,$$

so that (1.3.7) follows.

Lemma 1.3.3. *Let $\lambda > 0$ and suppose N is a negative subspace. Then the mapping*

$$y = [y^1, y^2] \to f^2 \equiv \lambda I_{21} y^1 - y^2$$

of N onto $\Re_\lambda(N)$ is one-to-one and bicontinuous.

Proof. Making use of the inequality (1.3.7) we have

$$\|y^1\|_1 \leqq [c(\lambda)]^{-1} \|f^2\|_2,$$
$$\|y^2\|_2 \leqq \|f^2\|_2 + \lambda \|I_{21} y^1\|_2 \leqq \|f^2\|_2 + \lambda \|y^1\|_1 \leqq [1 + \lambda/c(\lambda)] \|f^2\|_2,$$

and

$$\|f^2\|_2 \leqq \lambda \|I_{21} y^1\|_2 + \|y^2\|_2 \leqq \lambda \|y^1\|_1 + \|y^2\|_2 \leqq c'(\lambda)[\|y^1\|_1 + \|y^2\|_2],$$

where $c'(\lambda) = \max(1, \lambda)$. The result now follows directly.

Corollary. *Let $\lambda > 0$. Then a negative subspace N is closed if and only if $\Re_\lambda(N)$ is a closed subspace of H_2.*

Lemma 1.3.4. *Let $\lambda > 0$ and suppose N is a negative subspace. Then N is maximal negative if and only if $\Re_\lambda(N) = H_2$.*

Proof. Suppose first that N is maximal negative but that $\Re_\lambda(N)$ is a proper subspace of H_2. Since N is closed, we see from the previous lemma that $\Re_\lambda(N)$ is also closed. Thus there is a $u^2 \neq \theta$ such that $\langle F_{12} u^2, f^2 \rangle = 0$ for all $f^2 \in \Re_\lambda(N)$. Now we set

$$z^1 = F_{12} u^2, \qquad z^2 = -F(\lambda)_{21} F_{12} u^2, \qquad z = [z^1, z^2],$$

where $F(\lambda)_{21} = \lambda I_{21} - D_{21}$ as in Lemma 1.2.3. Clearly $z \in H_{12}$. For any $y \in H_{12}$ we have

$$Q(y, z) = - \langle y^1, F(\lambda)_{21} F_{12} u^2 \rangle + \langle y^2, F_{12} u^2 \rangle - \langle D_{21} y^1, F_{12} u^2 \rangle$$
$$= - \langle \lambda I_{21} y^1 - y^2, F_{12} u^2 \rangle.$$

Hence if $y \in N$, then $Q(y, z) = 0$. On the other hand

$$Q(z, z) = - 2\lambda \langle I_{21} F_{12} u^2, F_{12} u^2 \rangle + \langle D_{21} F_{12} u^2, F_{12} u^2 \rangle$$
$$\leqq - 2c(\lambda) \langle F_{12} u^2, u^2 \rangle < 0.$$

Thus $z \notin N$ and consequently $N_1 \equiv [y + \alpha z; y \in N]$ contains N properly. Moreover

$$Q(y + \alpha z, y + \alpha z) = Q(y, y) + |\alpha|^2 Q(z, z) \leqq 0,$$

and this shows that N_1 is a negative subspace, contrary to N being maximal negative.

Conversely, suppose N is a negative subspace with $\mathfrak{R}_\lambda(N) = H_2$. Lemma 1.3.3 asserts that the mapping $y \to f^2$ of N onto $\mathfrak{R}_\lambda(N)$ is one-to-one and that this remains true for any negative subspace containing N. Since N already maps onto H_2, it follows that N is maximal negative.

Specializing the above result to the case $F = I$ and dissipative operators we readily obtain another proof of the corollary to Theorem 1.1.1. In fact, we have only to note that if L is dissipative then its graph is negative, and if $\mathfrak{D}(L)$ is dense then any negative subspace containing $\mathfrak{G}(L)$ is the graph of a dissipative extension of L.

The foregoing lemmas can be used to connect maximal negative subspaces of H_{12} with maximal dissipative operators on H_0. To this end we introduce the following

DEFINITION 1.3.3. *A transformation L_{21} on H_1 to H_2 will be said to engender the operator L on H_0 to itself if*

(1.3.9)
$$\mathfrak{D}(L) = [I_{01} y^1; y^1 \in \mathfrak{D}(L_{21}) \text{ and } L_{21} y^1 \in I_{20}[H_0]],$$
$$L y^0 = I_{20}^{-1} L_{21} I_{01}^{-1} y^0.$$

THEOREM 1.3.1. *Let L_{21} be a linear transformation on H_1 to H_2 whose graph is a maximal negative subspace of H_{12} and let L denote the operator on H_0 engendered by L_{21}. Then L is a closed maximal dissipative operator with dense domain, and the graph of $I_{20} L I_{01}$ is dense in the graph of L_{21}.*

Proof. According to Lemma 1.3.4, the set $[\lambda I_{21} y^1 - L_{21} y^1; y^1 \in \mathfrak{D}(L_{21})]$ fills out H_2. Further one sees from Definition 1.3.3 that $\lambda I_{21} y^1 - L_{21} y^1 \in I_{20}[H_0]$ if and only if $I_{01} y^1 \in \mathfrak{D}(L)$. Consequently,

$$\mathfrak{R}(\lambda I - L) = I_{20}^{-1} \{ \lambda I_{21} - I_{21} \} I_{01}^{-1} [\mathfrak{D}(L)] = H_0.$$

On the other hand, setting $y = [y^1, L_{21} y^1]$ and $y^0 = I_{01} y^1$, $y^0 \in \mathfrak{D}(L)$, we see that

$$\langle y^0, Ly^0 \rangle + \langle Ly^0, y^0 \rangle = \langle y^1, L_{21}y^1 \rangle + \langle L_{21}y^1, y^1 \rangle$$
$$= Q(y, y) + \langle D_{21}y^1, y^1 \rangle \leqq 0,$$

so that L is a dissipative operator. Lemma 1.1.1 now implies that L is closed and maximal dissipative and Lemma 1.1.3 asserts that $\mathfrak{D}(L)$ is dense in H_0. Next suppose that $y = [y^1, y^2] \in \mathfrak{G}(L_{21})$, in which case $y^2 = L_{21}y^1$. Then $f^2 = \lambda I_{21}y^1 - y^2 \in H_2$ is determined by a Cauchy sequence $\{f_n^0\} \subset H_0$ such that $f_n^2 \equiv I_{20}f_n^0 \to f^2$. Since $\Re(\lambda I - L) = H_0$, there exists a sequence $\{u_n^2\} \in \mathfrak{D}(L)$ such that $f_n^0 = \lambda u_n^0 - Lu_n^0$ for all n. Setting $u_n^1 = I_{01}^{-1}u_n^0$, we see that $f_n^2 = \lambda I_{21}u_n^1 - L_{21}u_n^1$; hence the relation (1.3.7) implies that the sequence $\{u_n^1\}$ converges in H_1 to say u^1. Thus $[u_n^1, L_{21}u_n^1 = \lambda I_{21}u_n^1 - f_n^2]$ converges in H_{12} to $[u^1, \lambda I_{21}u^1 - f^2]$ and since L_{21} is closed it follows that $u^1 \in \mathfrak{D}(L_{21})$ and $L_{21}u^1 = \lambda I_{21}u^1 - f^2$. Consequently $w^1 = u^1 - y^1$ is a solution of $\lambda I_{21}w^1 - L_{21}w^1 = \theta$ and (1.3.7) requires that $w^1 = \theta$. In other words $[u_n^1, L_{21}u_n^1] \to [y^1, L_{21}y^1]$ in H_{12}, and since

$$\{u_n^1\} \subset I_{01}^{-1}[\mathfrak{D}(L)]$$

and $L_{21}u_n^1 = I_{20}LI_{01}u_n^1$, the last assertion of the theorem has now been verified.

1.4. **Duality theory.** The present section contains a duality theory for maximal negative and maximal positive subspaces of H_{12}. As we shall see, this theory is intimately related to an adjoint theory for maximal dissipative operators. Our central result is

THEOREM 1.4.1. *Let N be a maximal negative subspace and let P denote its Q-orthogonal complement. Then P is maximal positive and N is the Q-orthogonal complement of P.*

We shall prove this theorem with the help of the following two lemmas.

LEMMA 1.4.1. *Let N be a closed subspace and let P denote its Q-orthogonal complement. Then N is again the Q-orthogonal complement of P.*

Proof. The relation (1.3.6) shows that

(1.4.1) $Q(y, z) = \langle W^{-1}y, z \rangle.$

Consequently P is the ordinary orthogonal complement of $W^{-1}[N]$. Since W is a continuous operator $W^{-1}[N]$ is a closed subspace and hence $W^{-1}[N]$ is again the orthogonal complement of P. This being so, the relation (1.4.1) now implies that N is the Q-orthogonal complement of P.

LEMMA 1.4.2. *If N is a negative and P a positive subspace of H_{12}, and if N and P are Q-orthogonal complements of one another, then both are maximal.*

Proof. It is clear that N and P are both closed linear subspaces of H_{12}. Moreover, because of the U isomorphism between positive and negative subspaces, it suffices to show that N is maximal negative. If this were not so, there would be a closed negative subspace, say N_1, which properly contains

N. Choose $u \in N_1 \cap N^\perp$, $u \neq \theta$. Then $Q(u, u) \leq 0$ and $\langle y, u \rangle = 0$ for all $y \in N$. According to (1.3.6), $\langle y, u \rangle = Q(y, Wu)$ so that $Wu \in P$ and therefore $Q(Wu, Wu) \geq 0$. We now have

$$(1.4.2) \quad \begin{aligned} \langle u^1, u^2 \rangle + \langle u^2, u^1 \rangle - \langle D_{21}u^1, u^1 \rangle &= Q(u, u) \leq 0, \\ \langle u^1, u^2 \rangle + \langle u^2, u^1 \rangle + \langle D_{21}F_{12}u^2, F_{12}u^2 \rangle &= Q(Wu, Wu) \geq 0. \end{aligned}$$

As a consequence

$$- \langle D_{21}u^1, u^1 \rangle - \langle D_{21}F_{12}u^2, F_{12}u^2 \rangle \leq 0$$

and since $D_{21} \leq \Theta$, we obtain

$$\langle D_{21}u^1, u^1 \rangle = 0 = \langle D_{21}F_{12}u^2, F_{12}u^2 \rangle.$$

Combining this with the inequalities (1.4.2) yields

$$Q(u, u) = 0 = Q(Wu, Wu).$$

Now Q is a nonpositive quadratic form on N_1 and hence the Schwarz inequality (relative to $-Q$ on N_1) implies that $Q(y, u) = 0$ for all $y \in N_1$. It follows that u belongs to the Q-orthogonal complement of N, namely P. Similarly, P positive, $Wu \in P$, and $Q(Wu, Wu) = 0$ imply that $Q(Wu, z) = 0$ for all $z \in P$; thus $Wu \in N$. Consequently u and Wu are Q-orthogonal so that we obtain from (1.3.6) the result $\langle u, u \rangle = Q(Wu, u) = 0$. This being contrary to our choice of u, we conclude that N is maximal negative.

Proof of Theorem 1.4.1. We are given that N is maximal negative and that P is its Q-orthogonal complement. We first show that P is a positive subspace. If this were not so, there would exist a $z \in P$ such that $Q(z, z) < 0$. Since $Q(y, z) = 0$ for all $y \in N$, we see that z does not belong to N and further that

$$Q(y + \alpha z, y + \alpha z) = Q(y, y) + |\alpha|^2 Q(z, z) \leq 0, \qquad y \in N.$$

It follows that $N_1 \equiv [y + \alpha z; y \in N]$ is a negative subspace which contains N properly. This being impossible, we conclude that P is indeed positive. By Lemma 1.4.1, N and P are Q-orthogonal complements of one another so that it follows from Lemma 1.4.2 that P is a maximal positive subspace.

LEMMA 1.4.3. *If N has a dense domain then its Q-orthogonal complement is a graph and, conversely, if N is closed graph then its Q-orthogonal complement has a dense domain.*

Proof. Let P denote the Q-orthogonal complement of N. If P has an element of the form $z = [\theta, z^2]$, then $Q(y, z) = \langle y^1, z^2 \rangle = 0$ for all $y \in N$ and, assuming $\mathfrak{D}(N)$ to be dense, this implies $z^2 = \theta$. It follows that P is a graph if $\mathfrak{D}(N)$ is dense. Suppose next that N is a closed graph but that $\mathfrak{D}(P)$ is not dense in H_1. Then there is a $u^2 \in H_2$, $u^2 \neq \theta$, such that $\langle u^2, z^1 \rangle = 0$ for all $z^1 \in \mathfrak{D}(P)$. We see as above that $u = [\theta, u^2]$ is Q-orthogonal to P and hence by Lemma 1.4.1

that $u \in N$, contrary to N being a graph. This establishes the second half of the lemma.

A somewhat deeper result in this direction is given by

LEMMA 1.4.4. *Suppose N is both a maximal negative (maximal positive) subspace and a graph. Then $\mathfrak{D}(N)$ is dense in H_1 and the Q-orthogonal complement of N is also a graph with dense domain.*

Proof. Let P denote the Q-orthogonal complement of N. Since N is maximal negative, P will be positive by Theorem 1.4.1 and since N is a closed graph, $\mathfrak{D}(P)$ will be dense by the previous lemma. As a consequence, Lemma 1.3.1 asserts that P is also a graph. Applying Lemma 1.4.3 once more we see that $\mathfrak{D}(N)$ is dense.

Incidental to the above development is a new proof of Theorem 1.1.2.

THEOREM 1.4.2. *If L is a maximal dissipative operator with dense domain, then so is its adjoint operator.*

Proof. In this case we suppose $F = I$ so that $H_1 = H_0 = H_2$. The graph of L, say N, is by hypothesis negative with dense domain and since L is maximal dissipative, it follows from Lemma 1.3.1 that N is maximal negative. Denoting the Q-orthogonal complement of N in $H_0 \times H_0$ by P, we see that P is a maximal positive subspace by Theorem 1.4.1. and a graph with dense domain by Lemma 1.4.3. Consequently $U[P]$ is a maximal negative subspace and a graph with dense domain. Now $z \in U[P]$ if and only if

$$0 = Q(y, Uz) = - \langle y^1, z^2 \rangle + \langle y^2, z^1 \rangle, \qquad y^2 = Ly^1,$$

for all $y^1 \in \mathfrak{D}(L)$, that is, if and only if $z^2 = L^* z^1$. Thus $U[P] = \mathfrak{G}(L^*)$ and L^* is a maximal dissipative operator with dense domain.

Most essential for our purposes is the following

THEOREM 1.4.3. *Let L_{21} be a linear transformation on H_1 to H_2 whose graph N is a maximal negative subspace of H_{12} and let P be the Q-orthogonal complement of N. In this case $U[P]$ is also the graph of a linear transformation, say M_{21}. Next let L and M denote the operators on H_0 engendered by L_{21} and M_{21}, respectively. Then L and M are maximal dissipative operators, with dense domains, and adjoints of each other.*

Proof. Since N is maximal negative and a graph, it follows from Theorem 1.4.1 and Lemma 1.4.4 that P is maximal positive and a graph. Consequently $U[P]$ is maximal negative and a graph. Theorem 1.3.1 therefore implies that L and M are both closed maximal dissipative operators with dense domains. Employing the previous theorem we see that L^* is maximal dissipative with dense domain. Now for $y^0 \in \mathfrak{D}(L)$, $z^0 \in \mathfrak{D}(M)$, $y^1 = I_{01}^{-1} y^0$, and $z^1 = I_{01}^{-1} z^0$, it is clear that $y = [y^1, L_{21} y^1] \in N$ and $z = [z^1, M_{21} z^1] \in U[P]$. Thus $Uz \in P$ and

$$\langle y^0, Mz^0 \rangle - \langle Ly^0, z^0 \rangle = \langle y^1, M_{21} z^1 \rangle - \langle L_{21} y^1, z^1 \rangle = - Q(y, Uz) = 0.$$

Consequently $M \subset L^*$ and since M is itself maximal dissipative, it follows that $M = L^*$.

1.5. The boundary space. As we have already mentioned in the introduction, the boundary data assumed by functions in the domain of our differential operator will be represented by a certain space of cosets which we will call the boundary space. We now give an abstract formulation of such a space.

DEFINITION 1.5.1. *A closed subspace N_0 of H_{12} will be called a null space if*

$$(1.5.1) \qquad\qquad Q(y, u) = 0 \qquad\qquad \text{for all } y, u \in N_0.$$

Suppose N_0 is a null space and let N_1 denote its Q-orthogonal complement; clearly $N_0 \subset N_1$. The quotient space

$$(1.5.2) \qquad\qquad H = N_1/N_0$$

will be called a boundary space.

It is easy to see that the boundary space H can be made into a hilbert space. In fact, if β denotes the projection operator $N_1 \to N_1 \cap N_0^\perp$, then $\beta y = \beta u$ if and only if $y - u \in N_0$. Thus H is algebraically isomorphic with $N_1 \cap N_0^\perp$ and the inner product defined on $N_1 \cap N_0^\perp$ can be used to implement an inner product in H. We shall denote the points of H by $y, z, \cdots$, and we shall indicate that $y \in N_1$ belongs to the coset $y \in H$ by writing $\beta y = y$. It will be clear from the context whether βy is to be thought of as an element of H or as an element of $N_1 \cap N_0^\perp$.

The following lemma gives a simple characterization of $N_1 \cap N_0^\perp$.

LEMMA 1.5.1. *Let N_0 be a null space and let N_1 denote its Q-orthogonal complement. Then $y \in N_1 \cap N_0^\perp$ if and only if both y and Wy belong to N_1.*

Proof. By definition a necessary and sufficient condition for $y \in N_1 \cap N_0^\perp$ is that y belong to N_1 and be orthogonal to N_0. By (1.3.6) the last half of this condition is equivalent with Wy being Q-orthogonal to N_0, in other words, it is equivalent with Wy belonging to N_1.

LEMMA 1.5.2. *Let N_0 be a null space and let N_1 denote its Q-orthogonal complement. If $y, u \in N_1$, then*

$$(1.5.3) \qquad\qquad Q(y, u) = Q(\beta y, \beta u),$$
$$(1.5.4) \qquad\qquad |Q(\beta y, \beta u)| \leq 2\|\beta y\|\|\beta u\|,$$

and if $Q(\beta y, \beta u) = 0$ for all $\beta u \in N_1 \cap N_0^\perp$, then $\beta y = \theta$.

Proof. It follows from the definition of N_1 that $Q(y, u) = 0$ for all $y \in N_1$ and $u \in N_0$. Since $y - \beta y$ and $u - \beta u$ both belong to N_0, we have

$$Q(y, u) = Q(\{\beta y + [y - \beta y]\}, \{\beta u + [u - \beta u]\}) = Q(\beta y, \beta u).$$

The relation (1.5.4) is an immediate consequence of (1.3.3). Finally if $Q(\beta y, \beta u) = 0$ for all βu, then by (1.5.3), $Q(\beta y, u) = 0$ for all $u \in N_1$. This means

that βy belongs to the Q-orthogonal complement of N_1 and hence, by Lemma 1.4.1, that βy belongs to N_0. Thus βy is orthogonal to itself and therefore $\beta y = \theta$.

It follows that the value of $Q(y, u)$ depends only on the cosets to which y and u belong. This suggests that we introduce the

DEFINITION 1.5.2. *We define the form $\boldsymbol{Q}$ on $\boldsymbol{H}$ as*

$$\boldsymbol{Q}(\beta y, \beta u) = Q(\beta y, \beta u).$$

It is clear that $\boldsymbol{Q}$ is a continuous bilinear form and it follows from the preceding lemma that it is nonsingular in the sense that $\boldsymbol{Q}(y, u) = 0$ for all $u \in \boldsymbol{H}$ implies that $y = \boldsymbol{\theta}$.

LEMMA 1.5.3. *Let N_0 be a null space and let N_1 denote its Q-orthogonal complement. The mapping $\boldsymbol{N} = \beta N$ defines a one-to-one correspondence between subspaces of N_1 which contain N_0 and subspaces of $\boldsymbol{H}$. This correspondence preserves negativity, positivity, inclusion, and Q-orthogonal subspaces correspond to $\boldsymbol{Q}$-orthogonal subspaces. In particular, subspaces of N_1 which are maximal negative (maximal positive) relative to the subspaces of N_1 contain N_0 and correspond to maximal negative (maximal positive) subspaces of $\boldsymbol{H}$.*

Proof. All the assertions of the lemma save the last, follow directly from properties of the homomorphism β established in Lemma 1.5.2. Actually even the last assertion is evident, once it has been shown that a subspace N which is maximal negative, say, relative to the subspaces of N_1 necessarily contains N_0. However, if N is negative, so is $\boldsymbol{N} = \beta N$ and $\beta^{-1}\boldsymbol{N}$. Since $\beta^{-1}\boldsymbol{N}$ is a negative subspace of N_1 containing N and N_0, it follows from the maximal property that N contains N_0.

LEMMA 1.5.4. *Suppose N_0 is a null space and let N_1 denote its Q-orthogonal complement. Then any negative (positive) subspace N of N_1 which is maximal negative (maximal positive) relative to the subspaces of N_1 is also maximal negative (maximal positive) relative to the subspaces of H_{12}. Further if P is the Q-orthogonal complement relative to N_1 of such an N, then P is also Q-orthogonal to N relative to H_{12} and maximal positive (maximal negative) relative to the subspaces of H_{12} (and a fortiori maximal relative to the subspaces of N_1).*

Proof. Suppose first that $N \subset N_1$ is a maximal negative subspace relative to the subspaces of N_1. It follows from the previous lemma that N consists of cosets of N_0 and, in particular that N contains N_0. On the other hand if N' is any negative subspace of H_{12} containing N_0, then since Q is negative on N' we obtain as a simple consequence of the Schwarz inequality that each element of N_0 is Q-orthogonal to every element in N'; in other words N' is contained in the Q-orthogonal complement of N_0, namely N_1. It follows that any negative extension in H_{12} of the given N necessarily lies in N_1 so that N is also maximal negative relative to the subspaces of H_{12}. Likewise we

note that the Q-orthogonal complement of N relative to H_{12} is necessarily a subspace of N_1 since N_1 is the Q-orthogonal complement of a subspace of N, namely N_0. Thus P is the Q-orthogonal complement of a maximal negative subspace and therefore is itself maximal positive by Theorem 1.4.1. Finally, if N had been maximal positive (instead of maximal negative) relative to the subspaces of N_1, then $U[N]$ would be maximal negative relative to the subspaces of $U[N_1]$. Since $U[N_1]$ is the Q-orthogonal complement of the null space $U[N_0]$, the above argument applies and we see that $U[N]$ is maximal negative in H_{12} and hence that N is maximal positive in H_{12}. It then follows as above that P is maximal negative in H_{12}.

REMARK. If in the above lemma, the null space N_0 had possessed a dense domain, then the relations

$$N_0 \subset N, P \subset N_1$$

imply that $\mathfrak{D}(N)$, $\mathfrak{D}(P)$, and $\mathfrak{D}(N_1)$ are also dense, and hence by Lemmas 1.3.1 and 1.4.3 that N, P, and N_1 are graphs.

LEMMA 1.5.5. *Let $\mathbf{N}$ be a maximal negative (maximal positive) subspace of $\mathbf{H}$ and let $\mathbf{P}$ be its $\mathbf{Q}$-orthogonal complement. Then $\mathbf{P}$ is maximal positive (maximal negative) and $\mathbf{N}$ is the $\mathbf{Q}$-orthogonal complement of $\mathbf{P}$. In addition, $N = \beta^{-1}\mathbf{N}$ and $P = \beta^{-1}\mathbf{P}$ are subspaces of N_1, maximal with respect to H_{12}, and Q-orthogonal complements.*

Proof. Suppose $\mathbf{N}$ is, say, maximal negative in $\mathbf{H}$. Then according to Lemma 1.5.3, $N = \beta^{-1}\mathbf{N}$ will be maximal negative relative to the subspaces of N_1 and hence, by Lemma 1.5.4, maximal negative relative to the subspaces of H_{12}. Lemma 1.5.4 also asserts that the Q-orthogonal complement of N, say P, is maximal positive and a subset of N_1. It now follows from Lemma 1.5.3 that $\mathbf{P} = \beta P$ is maximal positive as well as the $\mathbf{Q}$-orthogonal complement of $\mathbf{N}$. Likewise, since N is the Q-orthogonal complement of P by Theorem 1.4.1, we see that $\mathbf{N}$ will be the $\mathbf{Q}$-orthogonal complement of $\mathbf{P}$.

We now state one of our principal results.

THEOREM 1.5.1. *Let N_0 be a null space with dense domain, let N_1 be its Q-orthogonal complement, let $\mathbf{H} = N_1/N_0$. In this case N_0 and N_1 will be graphs of linear transformations, say, L_{21}^0 and L_{21}^1, respectively, which engender the operators L_0 and L_1, respectively. Suppose, in addition, that $\mathfrak{D}(I_{20}L_0I_{01})$ is dense in H_1. Then there is a one-to-one correspondence between the maximal negative subspaces $[\mathbf{N}]$ of $\mathbf{H}$, taken with respect to the form $\mathbf{Q}$, and the maximal dissipative operators $[L]$ such that $L_0 \subset L \subset L_1$, this correspondence being defined by*

$$(1.5.6) \qquad \mathfrak{D}(L) = [y^0; \beta[I_{01}^{-1}y^0, I_{20}L_1y^0] \in \mathbf{N}],$$

which is dense in H_0.

The adjoint operator $M = L^*$ is again maximal dissipative with dense domain

and can be described as follows: $U[N_0]$ *and* $U[N_1]$ *are graphs of linear trans-*
formations, say, M_{21}^0 *and* M_{21}^1, *respectively, which engender the operators* M_0 *and*
M_1, *respectively. Let* P *be the* Q-*orthogonal complement of* N. *Then* P *is maximal*
positive, $M_0 \subset M \subset M_1$, *and*

$$(1.5.7) \qquad \mathfrak{D}(M) = [z^0; \beta U[I_{01}^{-1} z^0, I_{20} M_1 z^0] \in P].$$

Proof. According to Lemma 1.5.3 there is a one-to-one correspondence be-
tween the maximal negative subspaces $[N]$ of H and the subspaces of N_1
which are maximal negative relative to the subspaces of N_1, this correspond-
ence being given by $N = \beta^{-1} N$. By Lemma 1.5.4 each such N is maximal nega-
tive in H_{12} and a graph, say of L_{21}. Theorem 1.3.1 now asserts that L_{21}
engenders a maximal dissipative operator L with dense domain and that the
graph of $I_{20} L I_{01}$ is dense in N. It follows that each L obtained in this way
can correspond to only one maximal negative subspace in N_1 and hence to
only one maximal negative subspace of H. Since $L_{21}^0 \subset L_{21} \subset L_{21}^1$, it is further
clear that $L_0 \subset L \subset L_1$. Finally, L is a restriction of L_1 and hence is completely
determined once its domain is given. By Definition 1.3.3

$$\mathfrak{D}(L) = [y^0; [I_{01}^{-1} y^0, I_{20} L_1 y^0] \in N],$$

or, equivalently, $\mathfrak{D}(L)$ can be given by (1.5.6).

The adjoint of L is described in Theorem 1.4.3. If P is the Q-orthogonal
complement of N and if M_{21} is the transformation whose graph is $U[P]$,
then $M = L^*$ is engendered by M_{21}. It also follows from Theorem 1.4.3 that
M is maximal dissipative and has a dense domain. According to Lemma 1.5.4,
$N_0 \subset P \subset N_1$ so that $M_{21}^0 \subset M_{21} \subset M_{21}^1$ and hence $M_0 \subset M \subset M_1$. In particular,
then M is a restriction of M_1 with domain

$$\mathfrak{D}(M) = [z^0; U[I_{01}^{-1} z^0, I_{20} M_1 z^0] \in P].$$

With P denoting the Q-orthogonal complement of N, we see by Lemma 1.5.5
that P is maximal positive and that $P = \beta^{-1} P$. Consequently, $\mathfrak{D}(M)$ may also
be described as in (1.5.7).

It remains to show that any maximal dissipative operator L', such that
$L_0 \subset L' \subset L_1$, corresponds as above to some maximal negative subspace N of
H. Suppose, therefore, that L' is such an operator and set $L_{21}' = I_{20} L' I_{01}$ and
$N' = \mathfrak{G}(L_{21}')$. By assumption $L_{21}' \supset I_{20} L_0 I_{01}$. We now show that N' is a negative
subspace of H_{12}. In the first place, since L' is dissipative,

$$\langle L_{21}' y^1, y^1 \rangle + \langle y^1, L_{21}' y^1 \rangle = \langle L' I_{01} y^1, I_{01} y^1 \rangle + \langle I_{01} y^1, L' I_{01} y^1 \rangle \leq 0$$

for all $y^1 \in \mathfrak{D}(L_{21}')$. Suppose then that $Q(y, y) > 0$ for some $y \in N'$. Choosing
a sequence $\{u_n^1\}$ in $\mathfrak{D}(I_{20} L_0 I_{01})$ with the property that $u_n^1 \to y^1$ and setting

$u_n = [u_n^1, L_{21}' u_n^1]$, we see that $u_n \in N_0$ and hence that

$$0 < Q(y, y) = Q(y - u_n, y - u_n)$$

$$= \langle (y^1 - u_n^1), L_{21}'(y^1 - u_n^1) \rangle + \langle L_{21}'(y^1 - u_n^1), (y^1 - u_n^1) \rangle$$

$$- \langle D_{21}(y^1 - u_n^1), (y^1 - u_n^1) \rangle \leqq - \langle D_{21}(y^1 - u_n^1), (y^1 - u_n^1) \rangle$$

$$\leqq \langle F_{21}(y^1 - u_n^1), (y^1 - u_n^1) \rangle = \| y^1 - u_n^1 \|_1^2.$$

However, this is impossible since the last term tends to zero as $n \to \infty$. It follows that N' is a negative subspace, obviously contained in N_1. There therefore exists a subspace $N \subset N_1$, maximal negative with respect to all subspaces of N_1 which contains N'. Now N is the graph of a transformation L_{21} and as in the first part of the proof, L_{21} engenders a maximal dissipative operator $L \subset L_1$ which corresponds to the maximal negative subspace $N = \beta N$ as in (1.5.6). It is clear from the construction that $L' \subset L$ and since L' is maximal dissipative we conclude that $L' = L$. In other words, L' corresponds as above to the maximal negative subspace N.

COROLLARY. *Theorem* 1.5.1 *characterizes all maximal and properly dissipative restrictions of L_1, that is, all maximal restrictions such that $Q(y, y) \leqq 0$ for each $y = [I_{01}^{-1} y^0, I_{20} L y^0]$ with $y^0 \in \mathfrak{D}(L)$.*

Proof. It is clear that if L is characterized as in (1.5.6), then $Q(y, y) \leqq 0$ for all $y = [I_{01}^{-1} y^0, I_{20} L y^0]$ with $y^0 \in \mathfrak{D}(L)$. Conversely, suppose that L is a maximal and properly dissipative restriction of L_1. Then $\beta [\mathfrak{G}(I_{20} L I_{01}^{-1})]$ is a negative subspace of H and hence contained in some maximal negative subspace, say N'. Let $L' \subset L_1$ be defined by (1.5.6) with N replaced by N'. Then clearly $L \subset L'$ and since both L and L' are maximal dissipative it follows that $L = L'$, which was to be proved.

The following remarks are of interest in connection with the boundary space for L_1. According to Lemma 1.5.1, the elements of $N_1 \cap N_0^{\perp}$ are just the elements y of $\mathfrak{G}(L_{21}^1)$ for which $W y \in \mathfrak{G}(L_{21}^1)$, in other words, the $[y^1, L_{21}^1 y^1]$ such that $F_{12} L_{21}^1 y^1$ as well as y^1 belong to $\mathfrak{D}(L_{21}^1)$ and for which

$$L_{21}^1 [F_{12} L_{21}^1 y^1] = F_{21} y^1 + D_{21} F_{12} L_{21}^1 y^1.$$

Since $\mathfrak{D}(M_{21}^1) = \mathfrak{D}(L_{21}^1)$ and $M_{21}^1 = -L_{21}^1 + D_{21}$, the above relation can be rewritten as

$$(1.5.8) \qquad\qquad M_{21}^1 F_{12} L_{21}^1 y^1 = - F_{21} y^1.$$

Thus we can represent H as the subspace of $\mathfrak{G}(L_{21}^1)$ which corresponds to the solution space of (1.5.8). We shall not pursue this matter any further; nevertheless it should be remarked that this is clearly a starting point for an investigation of the boundary space of L_1.

The next two lemmas deal with properties of L_0 and L_1. For one spatial variable it was shown in [12] that both L_0 and L_1 are closed; in the many variable case L_0 is closed, but it seems doubtful that L_1 is closed. Our best result in this direction is the following.

LEMMA 1.5.6. *Let L_{21} be a closed transformation on H_1 to H_2 whose graph is negative and let L denote the operator engendered by L_{21}. Then L is closed.*

Proof. Suppose $[y_n^0, Ly_n^0] \to [y^0, g^0]$ in $H_0 \times H_0$ and set $y_n^1 = I_{01}^{-1} y_n^0$ and $f_n^2 = I_{21} y_n^1 - L_{21} y_n^1$. Then $I_{21} y_n^1 = I_{20} y_n^0 \to I_{20} y^0$, $L_{21} y_n^1 = I_{20} L y_n^0 \to I_{20} g^0$, and $f_n^2 \to I_{20} y^0 - I_{20} g^0$. Consequently Lemma 1.3.2 applies and we see that $\{y_n^1\}$ is a Cauchy sequence in H_1 converging to some element, say y^1. Since I_{01} is continuous it follows that $I_{01} y^1 = y^0$. Thus $[y_n^1, L_{21} y_n^1] \to [y^1, I_{20} g^0]$ in H_{12} and, L_{21} being closed, we see that $y^1 \in \mathfrak{D}(L_{21})$ and that $L_{21} y^1 = I_{20} g^0$. According to Definition 1.3.3, $y^0 = I_{01} y^1 \in \mathfrak{D}(L)$ and $Ly^0 = I_{20}^{-1} L_{21} y^1 = g^0$; the operator L is therefore closed.

LEMMA 1.5.7. *In the notation of Theorem 1.5.1, $\mathfrak{D}(L_0)$ is dense in H_0 and $M_1 \subset L_0^*$.*

Proof. According to Lemma 1.2.1, I_{01} is a continuous map of H_1 into H_0 and it is clear from this construction that $\mathfrak{R}(I_{01})$ contains $\mathfrak{D}(F)$. Since $\mathfrak{D}(I_{20} L_0 I_{01})$ is dense in H_1, it follows that $\mathfrak{D}(L_0) = I_{01} [\mathfrak{D}(I_{20} L_0 I_{01})]$ is dense in $\mathfrak{D}(F)^- = H_0$. On the other hand for $y^0 \in \mathfrak{D}(L_0)$, $z^0 \in \mathfrak{D}(M_1)$, $y^1 = I_{01}^{-1} y^0$, and $z^1 = I_{01}^{-1} z^0$, we have

$$\langle L_0 y^0, z^0 \rangle - \langle y^0, M_1 z^0 \rangle = \langle L_{21}^0 y^1, z^1 \rangle - \langle y^1, M_{21}^1 z^1 \rangle$$
$$= Q([y^1, L_{21}^0 y^1], [z^1, L_{21}^1 z^1]) = 0$$

so that $M_1 \subset L_0^*$.

LEMMA 1.5.8. *Let N_0 be a null space and the graph of a linear transformation, say L_{21}^0, which engenders the operator L_0. Suppose, in addition, that $\mathfrak{D}(I_{20} L_0 I_{01})$ is a dense in H_1. Then L_0 is conservative if and only if $I = F$.*

Proof. For $y^0 \in \mathfrak{D}(L_0)$ and $y^1 = I_{01}^{-1} y^0$ we have

$$(1.5.9) \qquad \langle L_0 y^0, y^0 \rangle + \langle y^0, L_0 y^0 \rangle = \langle D_{21} y^1, y^1 \rangle.$$

If L_0 is conservative, then $\langle D_{21} y^1, y^1 \rangle = 0$ for all $y^1 \in \mathfrak{D}(I_{20} L_0 I_{01})$, which is assumed to be dense in H_1. Thus this relation holds for all $y^1 \in H_1$; replacing y^1 by $u^1 + \epsilon v^1$, $\epsilon = 1, i$, in turn and adding, we obtain $\langle D_{12} u^1, v^1 \rangle = 0$ for all $u^1, v^1 \in H_1$, from which it follows that $D_{21} = \Theta$. Now for $y^0 \in \mathfrak{D}(F)$, it is readily seen that

$$(I - F) y^0 = I_{20}^{-1} D_{21} I_{01}^{-1} y^0 = \theta$$

so that $F=I$. Conversely, if $F=I$ then the relation (1.5.9) shows that L_0 is conservative.

Now if $F=I$ we have $H_1=H_0=H_2$ and there is no longer any distinction between an operator L_{21} and the operator L which it engenders. Thus, in this case, the operators L_0 and L_1 of Theorem 1.5.1 are closed and it follows from $Q(y, z) = \langle y^1, z^2 \rangle + \langle y^2, z^1 \rangle$ that $L_1 = -L_0^*$. If L is a conservative extension of L_0, then

$$L_1 = -L_0^* \supset -L^* \supset L;$$

whereas if iL is a self-adjoint restriction of iL_1, then

$$L_0 = -L_1^* \subset -L^* = L.$$

In either case $L_0 \subset L \subset L_1$. Conditions that $L \supset L_0$ be a conservative maximal dissipative operator have been given by J. W. Calkin [1, Theorems 2.9 and 4.1]. For the sake of completeness we now give another version of this result which is slightly more general in that it includes all maximal dissipative extensions of L_0. In view of Lemma 1.5.8, we may, without loss of generality, assume that $F=I$.

THEOREM 1.5.2. *Suppose $F=I$ and let L_0 be a closed conservative operator with dense domain. Set $L_1 = -L_0^*$ and $H = \mathfrak{G}(L_0)/\mathfrak{G}(L_1)$. Then there is a one-to-one correspondence between the maximal negative subspaces $[N]$ of H, taken with respect to the form Q, and the maximal dissipative operators $[L]$ such that $L_0 \subset L$, this correspondence being defined by the fact that L is necessarily a restriction of L_1 and*

$$\mathfrak{D}(L) = [y^0; \beta[y^0, L_1 y^0] \in N].$$

Moreover, L will be conservative if and only if N is a null subspace of H, whereas iL will be self-adjoint if and only if $N=P$, where P is the Q-orthogonal complement of the maximal negative subspace N.

Proof. Since $F=I$ it is easy to see that $N_0 = \mathfrak{G}(L_0)$ is a null space with dense domain and that $N_1 = \mathfrak{G}(L_1)$ is its Q-orthogonal complement. According to Lemma 1.1.5 each maximal dissipative extension of L_0 is also a restriction of L_1. Consequently the first assertion of the theorem is a special case of Theorem 1.5.1. For $y = [y^0, L_1 y^0]$

$$(1.5.10) \qquad \langle L_1 y^0, y^0 \rangle + \langle y^0, L_1 y^0 \rangle = Q(y, y) = Q(\beta y, \beta y).$$

Therefore, if L is conservative, then $Q(y, y) = 0$ for all $y \in N$. Replacing y by $u + \epsilon v$, $\epsilon = 1, i$, in turn, and adding, we obtain $Q(u, v) = 0$ for all $u, v \in N$; hence N is a null space. Conversely, if N is a null space, then (1.5.10) shows that L is conservative. On the other hand, Lemma 1.1.4 asserts that iL is self-adjoint if and only if L and $M = L^*$ are conservative. The above argument shows that if L is conservative that N is a null space. Further, if P denotes the

Q-orthogonally complement of $\mathfrak{G}(L)$, then $\mathfrak{G}(M) = U[P]$, as in the proof of Theorem 1.5.1. For $z = [z^0, Mz^0] \in U[P]$ we have

$$(1.5.11) \qquad \langle Mz^0, z^0 \rangle + \langle z^0, Mz^0 \rangle = Q(Uz, Uz) = \mathbf{Q}(\beta Uz, \beta Uz).$$

Since U maps $U[P]$ in a one-to-one fashion on P, we see as above that M being conservative is equivalent with P and hence $\mathbf{P} = \beta P$ being null. Finally if $\mathbf{N}$ and $\mathbf{P}$ are null, then each is contained in its $\mathbf{Q}$-orthogonal complement, namely $\mathbf{P}$ and $\mathbf{N}$, respectively; consequently $\mathbf{N} = \mathbf{P}$. Conversely, if $\mathbf{N} = \mathbf{P}$ then this subspace is both positive and negative, and hence it is a null space. The relations (1.5.10) and (1.5.11) then imply that L and M are conservative, and thus that iL is self-adjoint.

It will be observed that there is a lack of symmetry between the expression (1.5.6) for $\mathfrak{D}(L)$ and (1.5.7) for $\mathfrak{D}(M)$. This is a consequence of the fact that both domains are delimited by means of the boundary space for L, whereas symmetry requires that $\mathfrak{D}(M)$ be delimited in terms of the boundary space for M. The purpose of the next section is to rectify this situation.

1.6. The dual boundary space. Throughout this section N_0 will be a fixed null space with Q-orthogonal complement N_1 and $\mathbf{H} = N_1/N_0$ will denote the associated boundary space. It is clear that $N_0' = U[N_0]$ is also a null space and the relation $Q(Uy, Uz) = -Q(y, z)$ shows that $N_1' = U[N_1]$ is the Q-orthogonal complement of N_0'. Thus

$$(1.6.1) \qquad \mathbf{H}' = N_1'/N_0'$$

is again a boundary space, and we refer to this space as the *dual boundary space*. We proceed to study the relation between these two boundary spaces.

As before $\mathbf{H}'$ is isomorphic with $N_1' \cap N_0'^{\perp}$ and we shall denote the Q-induced bilinear form on $\mathbf{H}'$ by $\mathbf{Q}'$. Likewise, we shall let β' denote both the natural homomorphism of N_1' onto $\mathbf{H}'$ and the projection operator of N_1' onto $N_1' \cap N_0'^{\perp}$. Given an operator T on H_{12} we shall symbolize its restriction on $N_1 \cap N_0^{\perp}$ by T_0 and its restriction on $N_1' \cap N_0'^{\perp}$ by T_0'.

In view of the way in which $\mathbf{H}'$ has been defined, one should expect the operator U to provide the principal link between $\mathbf{H}$ and $\mathbf{H}'$. The following lemma shows that this is the case.

LEMMA 1.6.1. *The transformation $\beta' U_0$ is one-to-one and bicontinuous on $N_1 \cap N_0^{\perp}$ to $N_1' \cap N_0'^{\perp}$ with inverse $\beta U_0'$. Moreover*

$$(1.6.2) \qquad Q(\beta' U_0 y, \beta' U_0 z) = - Q(y, z), \qquad y, z \in N_1 \cap N_0^{\perp}.$$

Proof. Given $z \in N_1' \cap N_0'^{\perp} \subset U[N_1]$, there exists a $y \in N_1 \cap N_0^{\perp}$ and a $u \in N_0$ such that $z = Uy + Uu$. Since $Uu \in N_0'$, we see that $z = \beta' z = \beta' Uy$; this shows that $\beta' U_0$ is onto $N_1' \cap N_0'^{\perp}$. On the other hand, given $y \in N_1 \cap N_0^{\perp}$, set $z = \beta' U_0 y$ and $v = z - Uy$. Then $\beta' v = z - z = \theta$ from which it follows that $v \in N_0'$. Consequently $Uv = U_0' z - y \in N_0$ and $y = \beta y = \beta U_0' z$; in other words $(\beta U_0')(\beta' U_0)$

$= I_0$. Since $\beta' U_0$ was onto, we infer that $\beta U_0' = (\beta' U_0)^{-1}$ and hence that $\beta' U_0$ is one-to-one and bicontinuous. Finally the relation (1.6.2) follows from (1.3.6) and (1.5.3).

Lemma 1.6.2. V_0 *is a unitary map on* $N_1 \cap N_0^{\perp}$ *to* $N_1' \cap N_0'^{\perp}$ *with* $V_0^{-1} = -V_0'$.

Proof. It is clear from (1.3.4) that V is a unitary map on H_{12} to itself with $V^{-1} = -V$. It therefore suffices to show that V_0 maps $N_1 \cap N_0^{\perp}$ onto $N_1' \cap N_0'^{\perp}$. To this end let $y \in N_1 \cap N_0^{\perp}$ and suppose u is an arbitrary element of N_0. Then by (1.3.6)

$$0 = \langle y, u \rangle = Q(UVy, u) = -Q(Vy, Uu);$$

since $N_0' = U[N_0]$ it follows that $V_0 y$ belongs to the Q-orthogonal complement of N_0', that is to N_1'. Further

$$\langle Vy, Uu \rangle = Q(WVy, Uu) = Q(UVVy, Uu) = -Q(VVy, u) = Q(y, u) = 0$$

so that $V_0 y$ is orthogonal to N_0'. Together these results imply that $V_0 y \in N_1' \cap N_0'^{\perp}$. Similarly, it can be shown that V_0' maps $N_1' \cap N_0'^{\perp}$ into $N_1 \cap N_0^{\perp}$. On the other hand $V^2 = -I$ implies $V_0 V_0' = -I_0'$, and this shows that the map is onto.

Lemma 1.6.3. βW_0 *is symmetric, one-to-one, and bicontinuous on* $N_1 \cap N_0^{\perp}$ *to itself with* $(\beta W_0)^{-1} = -V_0' \beta' U_0 = \beta_1 (W^{-1})_0$. *Similarly* $\beta' W_0'$ *is symmetric, one-to-one, bicontinuous on* $N_1' \cap N_0'^{\perp}$ *to itself with* $(\beta' W_0')^{-1} = -V_0 \beta U_0' = \beta_1' (W^{-1})_0'$. *Here* β_1 *and* β_1' *are projections on* H_{12} *to* $N_1 \cap N_0^{\perp}$ *and* $N_1' \cap N_0'^{\perp}$, *respectively.*

Proof. Since $W = UV$, it is an immediate consequence of Lemmas 1.6.1 and 1.6.2 that $\beta W_0 = (\beta U_0') V_0$ is one-to-one and bicontinuous on $N_1 \cap N_0^{\perp}$ to itself. Further $\beta W_0 (V_0' \beta' U_0) = \beta U_0' V_0 V_0' \beta' U_0 = -I_0$ so that $(\beta W_0)^{-1} = -V_0' \beta' U_0$. Now for $y \in N_1 \cap N_0^{\perp}$, $(I - \beta') U_0 y \in N_0'$ and since V is unitary and maps $N_1' \cap N_0'^{\perp}$ onto $N_1 \cap N_0^{\perp}$, we see that $V(I - \beta') U_0 y \in (N_1 \cap N_0^{\perp})^{\perp}$. Hence $\beta_1 V(I - \beta') U_0 y = \theta$ and $-\beta_1 (W^{-1})_0 = \beta_1 V U_0 = \beta_1 V \beta' U_0 = V_0' \beta' U_0$. Finally we see from (1.3.6) that for $y, z \in N_1 \cap N_0^{\perp}$ we have

$$\langle \beta W_0 y, z \rangle = \langle Wy, z \rangle = Q(W^2 y, z) = Q(y, W^2 z) = \langle y, Wz \rangle = \langle y, \beta W_0 z \rangle$$

and hence βW_0 is symmetric. The results for $\beta' W_0'$ follow in a similar fashion.

Corollary. *For* $y, z \in N_1 \cap N_0^{\perp}$ *we have*

(1.6.3) $$\langle (\beta W_0)^{-1} y, z \rangle = Q(y, z) = \langle y, (\beta W_0)^{-1} z \rangle.$$

Proof. It suffices to note that for $y, z \in N_1 \cap N_0^{\perp}$

$$Q(y, z) = \langle W^{-1} y, z \rangle = \langle \beta_1 (W^{-1})_0 y, z \rangle = \langle (\beta W_0)^{-1} y, z \rangle.$$

It is clear that the transformations $\beta' U_0$, $\beta U_0'$, βW_0, and $\beta' W_0'$ induce corresponding transformations on $\boldsymbol{H}$ and $\boldsymbol{H'}$ which we denote by $\boldsymbol{U}$, $\boldsymbol{U'}$, $\boldsymbol{W}$,

and W', respectively. These transformations are all one-to-one and bicontinuous, $U' = U^{-1}$, $Q'(Uy, Uz) = -Q(y, z)$, and W and W' are symmetric with

$$(1.6.4) \quad \begin{aligned} Q(Wy, z) &= \langle y, z \rangle = Q(y, Wz), \\ Q'(W'y', z') &= \langle y', z' \rangle = Q'(y', W'z'). \end{aligned}$$

It readily follows that negative (maximal negative, positive, maximal positive) subspaces of H correspond under the U mapping to positive (maximal positive, negative, maximal negative) subspaces of H'.

Returning to Theorem 1.5.1, we can now define $\mathfrak{D}(M)$ in a manner completely analogous with $\mathfrak{D}(L)$.

THEOREM 1.6.1. *In the notation of Theorem* 1.5.1., *we have*

$$(1.6.5) \qquad \mathfrak{D}(M) = [z^0; \beta'[I_{01}^{-1}z^0, I_{20}M_1z^0] \in U[P]],$$

and $U[P]$ *is a maximal negative subspace of* H'.

Proof. As was noted above, P being maximal positive in H implies that $U[P]$ is maximal negative in H'. Further, each $z \in N_1'$ has the representation $z = \beta'z + v$ where $v \in N_0'$. Thus $Uz = U_0'\beta'z + Uv$ and since $Uv \in N_0$, we have $\beta Uz = \beta U_0'\beta'z$. Consequently $\beta Uz \in P$ if and only if $\beta U_0'\beta'z \in P$ and hence by Lemma 1.6.1, if and only if $\beta'z = \beta'U_0(\beta U_0'\beta'z)$ belongs to $\beta'U_0[P] = U[P]$. The relation (1.6.5) now follows directly from (1.5.7).

The previous result can also be used to characterize the negative and maximal negative subspaces of H. In fact, W^{-1} being symmetric, we see that H splits into two orthogonal manifolds H^+ and H^-, $H = H^+ \oplus H^-$; these are respectively the positive and negative eigenspaces of W^{-1}. If we define

$$\begin{aligned} \langle y, z \rangle_+ &= \langle W^{-1}y, z \rangle && \text{for } y, z \in H^+, \\ \langle y, z \rangle_- &= \langle W^{-1}y, z \rangle && \text{for } y, z \in H^-, \end{aligned}$$

then H^+ and H^- are again hilbert spaces. In view of the fact that W and its inverse are bounded operators, the new topologies are equivalent with the original H topologies for each subspace. Suppose next that N is a negative subspace of H and for $y \in N$ set $y = y^+ + y^-$, where $y^+ \in H^+$ and $y^- \in H^-$. Then

$$Q(y, y) = \langle W^{-1}y, y \rangle = \|y^+\|_+^2 - \|y^-\|_-^2 \leq 0.$$

As a consequence of this inequality $Jy^- \equiv y^+$ defines a contraction transformation on H^- to H^+ with

$$N = [y^- + Jy^-; y^- \in \mathfrak{D}(J)].$$

Conversely if J is a contraction transformation of this kind, then the set of vectors $[y^- + Jy^-; y^- \in \mathfrak{D}(J)]$ is a negative subspace of H. We have therefore defined an inclusion preserving correspondence between negative sub-

spaces of H and contraction transformations on H^- to H^+. It is clear that N will be maximal negative if and only if the corresponding contraction transformation is defined on all of H^-.

CHAPTER II. DISSIPATIVE HYPERBOLIC SYSTEMS

2.1. Properly dissipative generators of the basic system. We return now to the dissipative hyperbolic system described in the introduction. By applying the foregoing theory we shall obtain a characterization of the properly dissipative generators first of the system (1.2) and later of this system coupled at the boundary to an auxiliary dissipative system. In the appendix to this chapter we shall construct a class of maximal dissipative extensions of our minimal operator and the dual class of maximal dissipative restrictions of our maximal operator, neither class being generators of properly dissipative solutions.

As remarked in the introduction, energy considerations require that the basic hilbert space be $H_0 = L_2(\Delta; E)$ and the desire to work with a finite boundary integral motivates our use of the auxiliary hilbert spaces H_1 and H_2. We construct H_1 and H_2 as in §1.2 by means of the self-adjoint operator F defined as

$$(2.1.1) \qquad\qquad Fy^0 = F(\cdot)y^0(\cdot)$$

where $F(\chi) = I - D(\chi)$ and $D(\chi) \leqq \Theta$ for all $\chi \in \Delta$;

$$(2.1.2) \qquad \mathfrak{D}(F) = [y^0; \ y^0(\cdot) \quad \text{and} \quad F(\cdot)y^0(\cdot) \in L_2(\Delta; E)].$$

It is readily verified that F is self-adjoint and satisfies the condition (1.2.1). Since the functions in $\mathfrak{D}(F)$ are dense in $L_2(\Delta; EF)$ and those in $\mathfrak{R}(F)$ $= L_2(\Delta; E)$ are dense in $L_2(\Delta; EF^{-1})$, we see at once that H_1 and H_2 are unitarily equivalent with $L_2(\Delta; EF)$ and $L_2(\Delta; EF^{-1})$, respectively. In what follows we shall not distinguish between the abstractly defined H_1 and H_2 and their realizations $L_2(\Delta; EF)$ and $L_2(\Delta; EF^{-1})$. Thus for $y^1 \in L_2(\Delta; EF)$ and $z^2 \in L_2(\Delta; EF^{-1})$ we have

$$\langle y^1, z^2 \rangle = \int_\Delta (Ey^1, z^2)d\chi;$$

and the bilinear form Q is defined equivalently by (1.3.2) and (1.12).

Next we define the transformation L_{21}^{00} as in (1.11). Then $\mathfrak{D}(L_{21}^{00})$ is dense in H_1 and an integration by parts shows that $Q(y, z) = 0$ for all $y, z \in \mathfrak{G}(L_{21}^{00})$. Thus $\mathfrak{G}(L_{21}^{00})$ is a negative subspace of H_{12} with dense domain and hence, by Lemma 1.3.1, its closure, which we denote by N_0, is also a graph, say of the transformation L_{21}^0. It follows from the continuity of Q that N_0 is a null space in the sense of Definition 1.5.1. In the present development, the operator L_0 engendered (see Definition 1.3.3) by L_{21}^0 will be called the *minimal operator* of the spatial part of the hyperbolic operator (1.2). Lemma 1.5.6 shows that

L_0 is closed. Further, it is clear that L_0 is an extension of

$(2.1.3)$
$$L_{00}y^0 = E^{-1}[(A^iy^0)_i + By^0],$$
$$\mathfrak{D}(L_{00}) = [y^0; y^0 \text{ continuously differentiable with compact carrier in } \Delta],$$

as can be seen from the fact that $L_{00} = I_{20}^{-1}L_{21}^{00}I_{01}^{-1}$. This latter relation also shows that $\mathfrak{D}(I_{20}L_0I_{01}) \supset \mathfrak{D}(L_{21}^{00})$ and hence that $\mathfrak{D}(I_{21}L_0I_{01})$ is dense in H_1. Incidently, we conjecture that L_0 is not the smallest closed extension of L_{00}[7]. However, from our point of view this is all to the good since we want L_0 to be the largest operator on H_0 (and L_{21}^0 to be the largest transformation on H_1 to H_2) of the type $E^{-1}[(A^iy^i)_i + By^i]$, $j=0$ (and 1), for which the domain functions have "essentially" zero boundary data. Finally we note that $\mathfrak{D}(L_{00})$ is dense in H_0 and consequently so is $\mathfrak{D}(L_0)$.

We also define the formal adjoints of these operators in an entirely analogous manner and it is clear that corresponding operators will possess the same properties. For instance, we define

$(2.1.4)$
$$M_{21}^{00}z^1 = E^{-1}[-(A^i z^1)_i + (B^* + A^i_i)z^1]$$

with $\mathfrak{D}(M_{21}^{00}) = \mathfrak{D}(L_{21}^{00})$. It is readily verified that $\mathfrak{G}(M_{21}^{00}) = U[\mathfrak{G}(L_{21}^{00})]$ and, since U is bicontinuous, this relation continues to hold for the closures of these graphs. Thus if M_{21}^0 denotes the smallest closed linear extension of M_{21}^{00}, then $\mathfrak{G}(M_{21}^0) = U[\mathfrak{G}(L_{21}^0)]$ and it follows that $\mathfrak{G}(M_{21}^0)$ is also a null space along with $\mathfrak{G}(L_{21}^0)$. It is also evident from the definition of U that $\mathfrak{D}(M_{21}^0) = \mathfrak{D}(L_{21}^0)$. However, the domain of the operator M_0 engendered by M_{21}^0 will in general not coincide with the domain of L_0, engendered by L_{21}^0. We note that $\mathfrak{D}(M_0)$ is dense in H_0 and that $\mathfrak{D}(I_{20}M_0I_{01})$ is dense in H_1.

We now define N_1 to be the Q-orthogonal complement of N_0. Since $\mathfrak{D}(N_0)$ is dense in H_1, it follows from Lemma 1.4.3 that N_1 is also a graph, say of the transformation L_{21}^1. Further, N_0 being a null space, we conclude that $N_1 \supset N_0$ and hence, that $L_{21}^1 \supset L_{21}^0$. For $y = [y^1, y^2]$ and $z = [z^1, z^2]$ we see that $y \in N_1$ if and only if

$(2.1.5)$
$$0 = Q(y, z) = \langle y^1, L_{21}^0z^1 \rangle + \langle y^2, z^1 \rangle - \langle D_{21}y^1, z^1 \rangle$$
$$= - \langle y^1, M_{21}^0z^1 \rangle + \langle y^2, z^1 \rangle, \qquad z^1 \in \mathfrak{D}(M_{21}^0).$$

Thus L_{21}^1 can be thought of as the "adjoint" of M_{21}^0 relative to this mixed inner product; in the usual notation for adjoint we have $L_{21}^1 = F_{21}(M_{21}^0)^*F_{21}$

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[7] Assuming the above conjecture to be correct, our use of the term "minimal operator" differs from the customary usage (cf. L. Hörmander [8]), which applies the term to the smallest closure of L_{00}, say L_0'. On the other hand, L_0' would be a more appropriate minimal operator than our L_0 if we wished to determine *all* maximal dissipative extensions of a minimal operator; likewise, $L_1' = (M_0')^*$ would be a more appropriate maximal operator than our L_1 if we wished to determine all maximal dissipative restrictions of a maximal operator.

since M_{21}^0 is the smallest closed extension of M_{21}^{00}, it is clear that y will belong to $\mathfrak{G}(L_{21}^1)$ if and only if (2.1.5) merely holds for all $z^1 \in \mathfrak{D}(M_{21}^{00})$. Now if $y^1 \in L_2(\Delta; EF)$ is continuously differentiable on Δ and if

$$y^2 = E^{-1}[(A^i y^1)_i + B y^1] \in L_2(\Delta; EF^{-1}),$$

then one sees by direct computation that (2.1.5) will be satisfied for all $z^1 \in \mathfrak{D}(M_{21}^{00})$; thus $y = [y^1, y^2]$ belongs to $\mathfrak{G}(L_{21}^1)$ and so does the closure of this set of function pairs. A complete description of L_{21}^1 is given in the following.

LEMMA 2.1.1. *Suppose that the elements of E, the A^i, and B are continuously differentiable in Δ. Then $[y^1, y^2] \in H_{12}$ belongs to $\mathfrak{G}(L_{21}^1)$ if and only if there is a sequence $\{y_n^1\} \subset \mathfrak{D}(L_{21}^{00})$ such that*

$$\left[\overset{1}{y_n}, \overset{00}{L_{21}} \overset{1}{y_n} \right] \rightarrow \left[\overset{1}{y}, \overset{2}{y} \right] \text{ in } L_2(\Lambda; EF) \times L_2(\Lambda; EF^{-1})$$

for each compact $\Lambda \subset \Delta$.

Proof. The sufficiency argument is trivial in view of the fact that (2.1.5) need only be verified for $z^1 \in \mathfrak{D}(M_{21}^{00})$ and each such z^1 has a compact support in Δ. The necessity can be established by a method developed by K. O. Friedrichs [6, pp. 365–373]. The reduction of the present problem to that treated by Friedrichs is readily accomplished by means of the following unitary transformations:

$$S: \quad y^1 \rightarrow Y = (EF)^{1/2} y^1, \qquad L_2(\Delta; EF) \quad \text{onto } L_2(\Delta; I),$$
$$T: \quad z^2 \rightarrow z = (EF)^{-1/2} E z^2, \qquad L_2(\Delta: EF^{-1}) \text{ onto } L_2(\Delta; I).$$

These are multiplicative transformations and since the factors are continuously differentiable, it is clear that continuously differentiable functions map onto functions of the same kind and likewise functions with compact support map onto functions having the same support. Setting

$$\overset{1}{\mathfrak{L}} = T L_{21}^1 S^{-1},$$
$$\overset{00}{\mathfrak{L}} Y = T \overset{00}{L_{21}} S^{-1} Y = (\overset{i}{\mathfrak{a}} Y)_i + \mathfrak{B} Y,$$
$$\overset{00}{\mathfrak{M}} z = T \overset{00}{M_{21}} S^{-1} z = -(\overset{i}{\mathfrak{a}} z)_i + (\mathfrak{B}^* + \overset{i}{\mathfrak{a}_i}) z,$$

where

$$\mathfrak{a}^i = (EF)^{-1/2} A^i (EF)^{-1/2} \text{ and}$$
$$\mathfrak{B} = (EF)^{-1/2} B (EF)^{-1/2} - [(EF)^{-1/2}]_i A^i (EF)^{-1/2},$$

it is clear that

$$\mathfrak{D}(\mathfrak{M}^{00}) = \mathfrak{D}(\mathfrak{L}^{00}) = [Y; Y \text{ continuously differentiable with compact support in } \Delta].$$

Moreover, $[Y, f] \in \mathfrak{G}(\mathfrak{L}^1)$ if and only if

$$(2.1.6) \qquad \int_\Delta (y, \mathfrak{M}^{00}z)d\chi = \int_\Delta (f, z)d\chi \qquad \text{for all } z \in \mathfrak{D}(\mathfrak{M}^{00}).$$

The main steps in Friedrichs' argument are as follows. He begins with a function $j(\chi^1)$ possessing derivatives of all orders and such that $j(\chi^1) \geqq 0$, $j(\chi^1) = 0$ for $|\chi^1| \geqq 1$, and $\int_{-1}^1 j(\chi^1)d\chi^1 = 1$. Setting

$$j_\epsilon(\chi) = \epsilon^{-m} \prod_{i=1}^m j(\epsilon^{-1}\chi^i),$$

he defines the mollifier operator

$$[\mathcal{J}_\epsilon y](\chi) = \int_\Delta j_\epsilon(\chi - \bar\chi)y(\bar\chi)d\bar\chi.$$

It is easy to show that both $\mathcal{J}_\epsilon$ and $\mathcal{J}_\epsilon^*$ approximate the identity in $L_2(\Delta; I)$ as $\epsilon \to 0+$. For $z \in \mathfrak{D}(\mathfrak{M}^{00})$ and ϵ sufficiently small, $\mathcal{J}_\epsilon^* z \in \mathfrak{D}(\mathfrak{M}^{00})$ so that (2.1.6) becomes

$$(2.1.7) \qquad \langle y, \mathfrak{M}^{00}\mathcal{J}_\epsilon^* z \rangle = \langle f, \mathcal{J}_\epsilon^* z \rangle = \langle \mathcal{J}_\epsilon f, z \rangle.$$

Now $\mathfrak{M}^{00}\mathcal{J}_\epsilon^*$ and $\mathfrak{L}^{00}\mathcal{J}_\epsilon$ are integral operators with kernels $k_\epsilon^1(\chi, \bar\chi)$ and $k_\epsilon^2(\chi, \bar\chi)$, respectively. Setting

$$(2.1.8) \qquad [\mathcal{K}_\epsilon y](\chi) = \int_\Delta [k_\epsilon^1(\bar\chi, \chi)^* - k_\epsilon^2(\chi, \bar\chi)]y(\bar\chi)d\bar\chi$$

and considering $\mathcal{K}_\epsilon$ as a transformation on $L_2(\Delta; I)$ to $L_2(\Lambda; I)$, Friedrichs shows that $\mathcal{K}_\epsilon y \to \theta$ as $\epsilon \to 0$ for each compact $\Lambda \subset \Delta$. To complete the proof, we choose a sequence of domains $\{\Delta_n\}$ such that $\bar\Delta_n \subset \Delta_{n+1} \subset \Delta$ and $\Delta = \bigcup_n \Delta_n$. Suppose that Δ_n is a distance greater than $\delta_n > 0$ from the complement of Δ and suppose further that $\delta_n \to 0$. For $\epsilon_n = \delta_n/4$, we define

$$y_n(\chi) = [\mathcal{J}_{\epsilon_n} y](\chi), \qquad\qquad\qquad \chi \in \Delta_n,$$
$$= 0, \qquad\qquad\qquad\qquad\qquad \chi \notin \Delta_{n+1},$$

and elsewhere $y_n(\chi)$ is defined so as to be continuously differentiable in Δ. Then $\{y_n\} \subset \mathfrak{D}(\mathfrak{L}^{00})$ and $y_n \to y$ in $L_2(\Lambda; I)$ for each compact $\Lambda \subset \Delta$. Moreover, according to (2.1.7)

$$\langle (\mathcal{J}_{\epsilon_n} f - \mathfrak{L}^{00}y_n), z \rangle = \langle \mathcal{K}_{\epsilon_n} y, z \rangle$$

for each $z \in \mathfrak{D}(\mathfrak{M}^{00})$ with support contained in Δ_n. It follows that $\mathfrak{L}^{00}y_n = \mathcal{J}_{\epsilon_n} f - \mathcal{K}_{\epsilon_n} y$ for all $\chi \in \Delta_n$ and hence that $\mathfrak{L}^{00}y_n \to f$ in $L_2(\Lambda; I)$, again for each compact $\Lambda \subset \Delta$. It readily follows that the sequence $\{S^{-1}y_n\} \subset \mathfrak{D}(L_{21}^{00})$ satisfies the assertion of the lemma.

   Returning to the main thread of our argument, $\dot M_{21}^1$ is defined in a manner analogous to L_{21}^1 as the transformation whose graph is the Q-orthogonal

complement of $\mathfrak{G}(M_{21}^0)$. It is clear from the properties of U that $\mathfrak{G}(M_{21}^1)$ $= U[\mathfrak{G}(L_{21}^1)]$ and it follows from this that the analogue of Lemma 2.1.1 holds for M_{21}^1. Finally let L_1 and M_1 be the operators engendered by L_{21}^1 and M_{21}^1, respectively. According to Lemma 1.5.7, $L_1 \subset M_0^*$ and $M_1 \subset L_0^*$ and we suspect that in general these inclusions are proper. We shall call L_1 the *maximal operator* of the spatial part of the hyperbolic operator (1.2). It follows from the way in which L_1 was defined, that among the differential operators of this type it possesses the largest domain to which the boundary integral (1.9) can be meaningfully extended. Consequently, any generator of properly dissipative solutions of (1.2) must necessarily be a restriction of L_1. An analogous assertion holds for M_1.

Next we introduce the boundary spaces for L_1 and M_1, namely the quotient spaces

$$H = \mathfrak{G}(L_{21}^1)/\mathfrak{G}(L_{21}^0) \quad \text{and} \quad H' = \mathfrak{G}(M_{21}^1)/\mathfrak{G}(M_{21}^0),$$

respectively. These cosets correspond to sets of functions in $\mathfrak{D}(L_{21}^1)$ and $\mathfrak{D}(M_{21}^1)$, respectively, which exhibit the same boundary behavior. Now H can be represented by $\mathfrak{G}(L_{21}^1) \cap \mathfrak{G}(L_{21}^0)^\perp$ and under this representation the homomorphism is simply the projection β of $\mathfrak{G}(L_{21}^1)$ onto $\mathfrak{G}(L_{21}^1) \cap \mathfrak{G}(L_{21}^0)^\perp$. Similarly H' has the representation $\mathfrak{G}(M_{21}^1) \cap \mathfrak{G}(M_{21}^0)^\perp$ and under this representation the homomorphism takes the form of the projection β' of $\mathfrak{G}(M_{21}^1)$ onto $\mathfrak{G}(M_{21}^1) \cap \mathfrak{G}(M_{21}^0)^\perp$.

As shown in §1.5, the bilinear form Q induces the bilinear forms $\mathbf{Q}$ and $\mathbf{Q}'$ on H and H', respectively. Finally we recall that the transformation $\mathbf{U}$ (see §1.6) on H to H' is one-to-one, bicontinuous, and takes maximal negative (maximal positive) subspaces of H into maximal positive (maximal negative) subspaces of H' in a one-to-one fashion. With all this in mind, we now state our principal result.

THEOREM 2.1.1. *There is a one-to-one correspondence between the maximal negative subspaces $[\mathbf{N}]$ of H, taken with respect to $\mathbf{Q}$, and the closed maximal dissipative operators $[L]$ on H_0 such that $L_0 \subset L \subset L_1$, the correspondence being defined by*

$$(2.1.9) \qquad \mathfrak{D}(L) = [y^0; y^0 \in \mathfrak{D}(L_1) \text{ and } \beta[I_{01}^{-1} y^0, I_{20} L_1 y^0] \in \mathbf{N}],$$

which is dense in H_0. The adjoint operator $M = L^*$ is again maximal dissipative with dense domain, $M_0 \subset M \subset M_1$, and

$$(2.1.10) \quad \mathfrak{D}(M) = [z^0; z^0 \in \mathfrak{D}(M_1) \text{ and } \beta'[I_{01}^{-1} z^0, I_{20} M_1 z^0] \in \mathbf{U}[P]],$$

where $\mathbf{P}$ is the $\mathbf{Q}$-orthogonal complement of $\mathbf{N}$ and $\mathbf{U}[\mathbf{P}]$ is a maximal negative subspace of H'.

This theorem merely paraphrases Theorems 1.5.1 and 1.6.1. It also char-

acterizes all maximal and properly dissipative restrictions of L_1 as can be seen from the corollary to Theorem 1.5.1.

According to Lemma 1.5.8 we can obtain a conservative extension of L_0 only if $F = I$, that is, only if $D(\chi) = \Theta$ for all $\chi \in \Delta$. In this case L_0 is the smallest closed extension of L_{00} and $L_1 = -L_0^*$ Theorem 1.5.2 then gives a characterization of the conservative maximal dissipative extensions of L_0 as well as the self-adjoint extensions of iL_0.

2.2. **Coupled systems.** In order to treat dissipative hyperbolic systems with boundary conditions of the "elastic" type by the above method, it is necessary to take into account the interchange of energy between the main system and the boundary system. Because of this energy interchange the central system by itself is no longer dissipative. Nevertheless the coupled system is dissipative and hence one is lead to treat the entire coupled system as a unit. This is our starting point; the reader will find further motivation for the material in this section in [12, §7].

Much work remains to be done before one can be sure that the problems treated in this chapter and especially those treated in this section have been correctly formulated. In view of this fact we have been content to study a rather simple boundary system.

The uncoupled components of our system will consist of the central system considered in the previous section, acting in the space H_0, and a boundary system acting in a hilbert space $\dot{H}_0$ and governed by the equation

$$(2.2.1) \qquad\qquad \overset{0}{\dot{y}_t} = B\overset{0}{\dot{y}}, \qquad\qquad t > 0,$$

where $\dot{B}$ is a bounded linear operator.

To the right member of (2.2.1) we now add a coupling term of the form Cy^i where C is a closed linear operator acting on the boundary value assumed at the time in the central system, namely

$$(2.2.2) \qquad\qquad y^i = \beta[I_{01}^{-1}\overset{0}{y}, I_{20}L_1\overset{0}{y}].$$

Thus the coupled system acts in the space $\mathfrak{H}_0 \equiv H_0 \times \dot{H}_0$ with elements $\mathfrak{y}^0 = [y^0, \dot{y}^0]$ and inner product

$$\langle \mathfrak{y}^0, \mathfrak{z}^0 \rangle = \langle y^0, z^0 \rangle + \langle \dot{y}^0, \dot{z}^0 \rangle,$$

and it is governed by the equation

$$(2.2.3) \qquad\qquad \overset{0}{\mathfrak{y}_t} = \mathfrak{L}_1\overset{0}{\mathfrak{y}}, \qquad\qquad t > 0,$$

where

$$(2.2.4) \qquad\qquad \mathfrak{L}_1\mathfrak{y}^0 = [L_1 y^0, \dot{B}\dot{y}^0 + Cy^i]$$

and, roughly speaking, $\mathfrak{D}(\mathfrak{L}_1)$ consists of those elements $\mathfrak{y}^0$ for which $y^0 \in \mathfrak{D}(L_1)$, $y^i \in \mathfrak{D}(C)$, and $\dot{y}^0 \in \dot{H}_0$. A precise description of $\mathfrak{D}(\mathfrak{L}_1)$ will appear in the

course of the subsequent development.

We shall impose the following conditions on C:

(i) *There exists a subset S of $\mathfrak{D}(CWC^*)$ with $[[\dot{y}^0, WC^*\dot{y}^0]; \dot{y}^0 \in S]$ dense in the graph of WC^* such that $WC^*[S] \subset \beta[\mathfrak{G}(I_{20}L_1 I_{01})]$;*

(ii) $\dot{B} + \dot{B}^* - CWC^* \leqq \Theta$ on $\mathfrak{D}(CWC^*)$.

When $F = I$ Condition (i) is not very restrictive. For in this case $\beta[\mathfrak{G}(I_{20}L_1 I_{01})] = H$ and the condition simply requires that the set $[[\dot{y}^0, WC^*\dot{y}^0];$ $\dot{y}^0 \in \mathfrak{D}(CWC^*)]$ be dense in the graph of WC^*. However if $\mathfrak{D}(CWC^*)$ is merely dense in $\dot{H}_0$, then we can take a restriction G_0 of WC^* with domain $\mathfrak{D}(CWC^*)$ and redefine C as $G_0^* W^{-1}$. It is readily verified that the so-defined C is a closed extension of the original C and satisfied (i) in its entirety.

Condition (ii) is a dissipative condition similar to (1.3). In fact, let us imagine the central system as extending beyond the boundary of Δ. Then at the boundary of Δ there will be a discontinuity in y^0 due to the back reaction of the coupling. Thus to a given y^0 there is associated an *interior boundary value y^i* defined by (2.2.2) and an *exterior boundary value y^e* given by

$$(2.2.5) \qquad\qquad y^e = y^i + G\dot{y}^0,$$

where G can be thought of as a back coupling operator describing the effect of the boundary system on the central system.

Next we assume that C and G are so related that no energy is lost in the coupling itself. In order to see what this means, let $\mathfrak{y}^0(t)$ be a solution of (2.2.3). Then the rate of increase of energy in the system is readily computed as

$$(2.2.6) \qquad \langle \mathfrak{y}^0, \mathfrak{y}^0 \rangle_t = \int_\Delta (EDy^0, y^0)d\chi + Q(y^i, y^i) + \langle (\dot{B} + \dot{B}^*)\dot{y}^0, \dot{y}^0 \rangle$$
$$+ \langle Cy^i, \dot{y}^0 \rangle + \langle \dot{y}^0, Cy^i \rangle.$$

Replacing $Q(y^i, y^i)$ by its equivalent,

$$Q(y^e, y^e) - Q(G\dot{y}^0, G\dot{y}^0) - Q(y^i, \mathfrak{G}\dot{y}^0) - Q(G\dot{y}^0, y^i),$$

we see that $\langle \mathfrak{y}^0, \mathfrak{y}^0 \rangle_t$ is expressible in terms of the rate of increase of energy in the interior of the central and boundary systems plus the flow of energy through the exterior boundary provided that the cross product terms nullify each other. Thus our assumption amounts to setting

$$(2.2.7) \qquad Q(y^i, G\dot{y}^0) + Q(G\dot{y}^0, y^i) = \langle Cy^i, \dot{y}^0 \rangle + \langle \dot{y}^0, Cy^i \rangle$$

for all $y^i \in \mathfrak{D}(C)$ and $\dot{y}^0 \in \mathfrak{D}(G)$. We shall see below that this requires that $G = WC^*$ and hence the rate of increase of energy due to sources interior to the boundary system is

$$\langle (B + B^*)\dot{y}^0, \dot{y}^0 \rangle - Q(G\dot{y}^0, G\dot{y}^0) = \langle (B + \dot{B}^* - CWC^*)\dot{y}^0, \dot{y}^0 \rangle.$$

Finally if we assume that the boundary system is dissipative, we arrive at Condition (ii).

LEMMA 2.2.1. *The assumption* (2.2.7) *is equivalent with*

$$(2.2.8) \qquad\qquad C = G^*W^{-1} \quad and \quad G = WC^*.$$

Proof. Since (2.2.7) holds for both $\dot{y}^0$, y^i and $i\dot{y}^0$, y^i, it is equivalent with $\mathcal{Q}(y^i, G\dot{y}^0) = \langle Cy^i, \dot{y}^0 \rangle$ and hence by (1.6.4) with

$$(2.2.9) \qquad\qquad \langle W^{-1}y^i, G\dot{y}^0 \rangle = \langle Cy^i, \dot{y}^0 \rangle$$

for all $y^i \in \mathfrak{D}(C)$ and $\dot{y}^0 \in \mathfrak{D}(G)$. However the relation (2.2.9) is valid if and only if $C = G^*W^{-1}$, or, what amounts to the same thing, if and only if $G = WC^*$.

After this digression, we return to the main line of argument. The operator

$$D_0 = \dot{B} + \dot{B}^* - CWC^*$$

with domain $\mathfrak{D}(CWC^*)$ is clearly symmetric and by assumption negative. We can therefore apply the Friedrichs' extension theorem to obtain a self-adjoint extension $\dot{D} \leq \Theta$ (see, for instance, [11, §VI.2]). Then, proceeding as in §1.2, we construct the dual hilbert spaces $\dot{H}_1$ and $\dot{H}_2$ from the positive definite operator $\dot{F} = I - \dot{D}$ and define the transformations $\dot{F}_{21}$, $\dot{F}_{12}$, $\dot{D}_{21}$, $\dot{I}_{01}$, $\dot{I}_{20}$, and $\dot{I}_{21}$. It follows from the manner in which the Friedrichs' extension is obtained that $\mathfrak{D}(CWC^*) \subset \dot{I}_{01}[\dot{H}_1]$ and further that $\dot{I}_{01}^{-1}[\mathfrak{D}(CWC^*)]$ is dense in $\dot{H}_1$. Moreover $\mathfrak{D}(G) \subset \dot{I}_{01}[\dot{H}_1]$. In fact according to Condition (i), for each $\dot{y}^0 \in \mathfrak{D}(G)$ there exists a sequence $\{\dot{y}^0_n\} \in \mathfrak{D}(CWC^*)$ such that $[\dot{y}^0_n, G\dot{y}^0_n] \rightarrow [\dot{y}^0, G\dot{y}^0]$. Thus

$$(2.2.10) \qquad \begin{aligned} \left\| \dot{y}^0_n - \dot{y}^0_m \right\|_1^2 &= \langle (I - \dot{B} - \dot{B}^*)(\dot{y}^0_n - \dot{y}^0_m), (\dot{y}^0_n - \dot{y}^0_m) \rangle \\ &\quad - \langle W^{-1}G(\dot{y}^0_n - \dot{y}^0_m), G(\dot{y}^0_n - \dot{y}^0_m) \rangle \rightarrow 0 \end{aligned}$$

so that $\{\dot{y}^0_n\}$ defines an element of $\dot{H}_1$ whose $\dot{I}_{01}$ image is just $\dot{y}^0$.

Setting

$$(2.2.11) \qquad\qquad B_{21} = \dot{I}_{20}\dot{B}\dot{I}_{01} \quad and \quad \dot{B}^*_{21} = I_{20}\dot{B}^*\dot{I}_{01},$$

it readily follows that $\dot{B}_{21}$ and $\dot{B}^*_{21}$ are bounded operators satisfying the relation

$$(2.2.12) \qquad\qquad \langle \dot{B}_{21}\dot{y}^1, \dot{z}^1 \rangle = \langle \dot{y}^1, \dot{B}^*_{21}\dot{z}^1 \rangle, \qquad\qquad \dot{y}^1, \dot{z}^1 \in \dot{H}_1.$$

Further it is clear that $\dot{B}_{21}$ and $\dot{B}^*_{21}$ engender, respectively, restrictions of $\dot{B}$ and $\dot{B}^*$ with domain $\dot{I}_{01}[\dot{H}_1]$. We also note that the operator

$$(2.2.13) \qquad\qquad [CG]_{21} \equiv \dot{F}_{21} - I_{21} + \dot{B}_{21} + \dot{B}^*_{21}$$

is bounded and symmetric on $\dot{H}_1$ to $\dot{H}_2$ and extends $\dot{I}_{20}CWC^*\dot{I}_{01}^{-1}$ whose domain, as we have already remarked, is dense in $\dot{H}_1$. Finally we set

$$(2.2.14) \qquad\qquad D_{21} = B_{21} + \overset{*}{\dot{B}}_{21} - [CG]_{21}$$

and define the quadratic form $\dot{Q}$ on $\dot{H}_{12} = \dot{H}_1 \times \dot{H}_2$ as

$$(2.2.15) \qquad \dot{Q}(\dot{y}, \dot{z}) = \langle \dot{y}^1, \dot{z}^2 \rangle + \langle \dot{y}^2, \dot{z}^1 \rangle - \langle \dot{D}_{21}\dot{y}^1, \dot{z}^1 \rangle.$$

We now define

$$(2.2.16) \qquad\qquad G_1 = G\dot{I}_{01}.$$

Then G is engendered by G_1 (since $\mathfrak{D}(G) \subset \dot{I}_{01}[\dot{H}_1]$), $\mathfrak{D}(G_1)$ is dense in $\dot{H}_1$ with $\dot{I}_{01}^{-1}[\mathfrak{D}(CWC^*)]$, and G_1 is closed with G. Next we set

$$(2.2.17) \qquad\qquad C_1 = \dot{F}_{21}\overset{*}{G}_1 W^{-1}.$$

Then C_1 is also closed by virtue of the fact that W^{-1} is bounded, G_1^* is closed, and $\dot{F}_{21}$ is unitary. It is clear that $G_1^* \supset \dot{I}_{01}^* G^*$. On the other hand $\dot{F}_{21}\dot{I}_{01}^* = \dot{I}_{20}$ as can be seen from

$$\langle \dot{I}_{20}\overset{0}{\dot{y}}, \overset{1}{\dot{z}} \rangle = \langle \overset{0}{\dot{y}}, \dot{I}_{01}\overset{1}{\dot{z}} \rangle = \langle \dot{I}_{01}^* \overset{0}{\dot{y}}, \overset{1}{\dot{z}} \rangle_1 = \langle \dot{F}_{21}\dot{I}_{01}^* \overset{0}{\dot{y}}, \overset{1}{\dot{z}} \rangle;$$

and therefore $C_1 \supset \dot{I}_{20}C$. Hereafter we shall denote the operator engendered by C_1 by the symbol C_0. It follows that $C_0 \supset C$ and also that $C_1 G_1 \supset \dot{I}_{20}CG_1 = \dot{I}_{20}CG\dot{I}_{01}$. Finally we show that

$$(2.2.18) \qquad\qquad \mathfrak{D}(C_1) \supset \mathfrak{R}(G_1).$$

In fact, we conclude from Condition (i) that each element in the graph of G_1 can be approximated by a sequence $[\dot{y}_n^1, G_1\dot{y}_n^1]$ where $\{\dot{y}_n^1\} \subset \dot{I}_{01}^{-1}[\mathfrak{D}(CG)]$ and since $\dot{I}_{20}CG_1$ has the bounded extension $[CG]_{21}$, we see that $\lim C_1 G_1\dot{y}_n^1$ also exists and hence that $\lim G_1\dot{y}_n^1 \in \mathfrak{D}(C_1)$. It also follows that $C_1 G_1 \subset [CG]_{21}$.

With these preliminaries out of the way, we now proceed to apply the theory set forth in Chapter I. The operator $\mathfrak{F}$ on $\mathfrak{H}_0$ is defined as

$$(2.2.19) \qquad\qquad \mathfrak{F} = \begin{pmatrix} F & \Theta \\ \Theta & \dot{F} \end{pmatrix}$$

where $\mathfrak{D}(\mathfrak{F}) = \mathfrak{D}(F) \times \mathfrak{D}(\dot{F})$. The dual spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are constructed as in §1.2 and it is readily verified that $\mathfrak{H}_1 = H_1 \times \dot{H}_1$ and $\mathfrak{H}_2 = H_2 \times \dot{H}_2$. We denote the elements of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ by $\mathfrak{y}^1 = [y^1, \dot{y}^1]$ and $\mathfrak{y}^2 = [y^2, \dot{y}^2]$, respectively. We also obtain the transformations $\mathfrak{F}_{21}, \mathfrak{F}_{12}, \mathfrak{D}_{21}, \mathfrak{I}_{01}, \mathfrak{I}_{20}, \mathfrak{I}_{21}$, which are all of the form

$$\mathfrak{T}_{ij} = \begin{pmatrix} T_{ij} & \Theta \\ \Theta & \dot{T}_{ij} \end{pmatrix}.$$

Finally we define the quadratic form on $\mathfrak{H}_{12} = \mathfrak{H}_1 \times \mathfrak{H}_2$ as

$$(2.2.20) \quad \mathfrak{Q}(\mathfrak{y}, \mathfrak{z}) = \langle \mathfrak{y}^1, \mathfrak{z}^2 \rangle + \langle \mathfrak{y}^2, \mathfrak{z}^1 \rangle - \langle \mathfrak{D}_{21}\mathfrak{y}^1, \mathfrak{z}^1 \rangle = Q(y, z) + Q(\dot{y}, \dot{z})$$

and the operator

(2.2.21) $$\mathfrak{U}[\mathfrak{y}^1, \mathfrak{y}^2] = [\mathfrak{y}^1, -\mathfrak{y}^2 + \mathfrak{D}_{21}\mathfrak{y}^1].$$

Next we define the transformation

(2.2.22) $$\mathfrak{L}_{21}^{0}\mathfrak{y}^1 = [L_{21}^{1}y^1, B_{21}\dot{y}^1 + C_1y^i]$$

where $y^i = \beta[y^1, L_{21}^1 y^1]$ and

(2.2.23) $$\mathfrak{D}(\mathfrak{L}_{21}^0) = [[y^1, \dot{y}^1]; y^1 \in \mathfrak{D}(L_{21}^1), \dot{y}^1 \in \mathfrak{D}(G_1), y^i = -G_1\dot{y}^1].$$

We set $\mathfrak{N}_0 = \mathfrak{G}(\mathfrak{L}_{21}^0)$ and let $\mathfrak{L}_0$ denote the transformation engendered by $\mathfrak{L}_{21}^0$. It follows directly that

(2.2.24) $$\mathfrak{L}_0\mathfrak{y}^0 = [L_1y^0, B\dot{y}^0 + C_0y^i]$$

where y^i is defined as in (2.2.2) and

(2.2.25) $$\mathfrak{D}(\mathfrak{L}_0) = [[y^0, \dot{y}^0]; y^0 \in \mathfrak{D}(L_1), \dot{y}^0 \in \mathfrak{D}(G), y^i = -\mathfrak{G}\dot{y}^0 \in \mathfrak{D}(C_0)].$$

LEMMA 2.2.2. $\mathfrak{N}_0$ *is a closed null space.*

Proof. Suppose $\{\mathfrak{y}_n^1\} \subset \mathfrak{D}(\mathfrak{L}_{21}^0)$, $\mathfrak{y}_n^1 \to \mathfrak{y}^1$, and $\mathfrak{L}_{21}^0\mathfrak{y}_n^1 \to \mathfrak{g}^2 = [g^2, \dot{g}^2]$. Then $y_n^1 \to y^1$ and $L_{21}^1 y_n^1 \to g^2$, and since L_{21}^1 is a closed transformation it follows that $y^1 \in \mathfrak{D}(L_{21}^1)$, and $L_{21}^1 y^1 = g^2$, and $y_n^i \to y^i$. Thus $\dot{y}_n^1 \to \dot{y}^1$ and $G_1\dot{y}_n^1 = -y_n^i \to -y^i$, and since G_1 is closed $\dot{y}^1 \in \mathfrak{D}(G_1)$ and $G_1\dot{y}^1 = -y^i$. Finally $C_1y_n^i = [\mathfrak{L}_{21}^0\mathfrak{y}_n^1]^2 - \dot{B}_{21}\dot{y}_n^1$ also converges and since C_1 is closed the limit is C_1y^i. Consequently $\mathfrak{y}^1 \in \mathfrak{D}(\mathfrak{L}_{21}^0)$ and $\mathfrak{L}_{21}^0\mathfrak{y}^1 = \mathfrak{g}^2$. This proves that $\mathfrak{N}_0$ is closed.

In order to show that $\mathfrak{N}_0$ is a null space, let $\mathfrak{y}, \mathfrak{z} \in \mathfrak{N}_0$. Then

(2.2.26) $$\mathfrak{Q}(\mathfrak{y}, \mathfrak{z}) = Q(y, z) + \langle \dot{y}^1, C_1z^i \rangle + \langle C_1y^i, \dot{z}^1 \rangle + \langle \dot{y}^1, [CG]_{21}\dot{z}^1 \rangle.$$

Now

$$Q(y, z) = \mathbf{Q}(y^i, z^i) = \langle G_1\dot{y}^1, W^{-1}G_1z^1 \rangle = \langle \dot{y}^1, C_1G_1\dot{z}^1 \rangle = \langle \dot{y}^1, [CG]_{21}\dot{z}^1 \rangle.$$

Further

$$\langle \dot{y}^1, C_1z^i \rangle = -\langle \dot{y}^1, C_1G_1\dot{z}^1 \rangle = -\langle \dot{y}^1, [CG]_{21}\dot{z}^1 \rangle,$$

$$\langle C_1y^i, \dot{z}^1 \rangle = -\langle C_1G_1\dot{y}^1, \dot{z}^1 \rangle = -\langle [CG]_{21}\dot{y}^1, \dot{z}^1 \rangle = -\langle \dot{y}^1, [CG]_{21}\dot{z}^1 \rangle.$$

Combining these relations with (2.2.26) we see that $\mathfrak{Q}(\mathfrak{y}, \mathfrak{z}) = 0$, which was to be proved.

LEMMA 2.2.3. $\mathfrak{D}(\mathfrak{J}_{20}\mathfrak{L}_0\mathfrak{J}_{01})$ *and* $\mathfrak{D}(\mathfrak{N}_0)$ *are dense in* $\mathfrak{H}_1$.

Proof. Since $\mathfrak{D}(\mathfrak{J}_{20}\mathfrak{L}_0\mathfrak{J}_{01}) \subset \mathfrak{D}(\mathfrak{N}_0)$, it suffices to show that $\mathfrak{D}(\mathfrak{J}_{20}\mathfrak{L}_0\mathfrak{J}_{01})$ is dense in $\mathfrak{H}_1$. To this end let $\mathfrak{g}^1 = [g^1, \dot{g}^1]$ be an arbitrary element of $\mathfrak{H}_1$ and let $\epsilon > 0$ be given. As we have already noted, there is a $\dot{u}^1 \in \mathfrak{D}(CG_1)$ such that $\|\dot{g}^1 - \dot{u}^1\|_1 < \epsilon$. According to Condition (i) there will exist a $\dot{y}^1 \in \mathfrak{D}(CG_1)$ with $G_1\dot{y}^1 \in \beta[\mathfrak{G}(I_{20}L_1I_{01})] \cap \mathfrak{D}(C)$ such that $\|\dot{I}_{01}\dot{u}^1 - \dot{I}_{01}\dot{y}^1\| < \epsilon$ and $\|G_1\dot{u}^1 - G_1\dot{y}^1\| < \epsilon$. It therefore follows as in (2.2.10) that $\|\dot{y}^1 - \dot{u}^1\|_1^2 \leq (1 + 2\|\dot{B}\|)\epsilon^2 + \|W^{-1}\|\epsilon^2$. For

such a $\dot{y}^1$ there is a $v^1 \in \mathfrak{D}(I_{20}L_1I_{01})$ such that $v^i = -G_1\dot{y}^1$. We now approximate $(g^1 - v^1)$ in H_1 by an element of $\mathfrak{D}(L_{21}^0)$, say w^1. Thus $\dot{w}^i = \theta$ and hence on setting $y^1 = v^1 + w^1$, we see that $y^i = -G_1\dot{y}^1 \in \mathfrak{D}(C_0)$ and that y^1 approximates g^1. It follows that $\mathfrak{y}^1 = [y^1, \dot{y}^1] \in \mathfrak{D}(\mathfrak{F}_{20}\mathfrak{L}_0\mathfrak{F}_{01})$ and approximates $\mathfrak{g}^1$.

We now define $\mathfrak{N}_1$ to be the $\mathfrak{Q}$-orthogonal complement of $\mathfrak{N}_0$. As before we see by Lemma 1.4.3 that $\mathfrak{N}_1$ is a graph, say of the transformation $\mathfrak{L}_{21}^1$. Obviously $\mathfrak{L}_{21}^1 \supset \mathfrak{L}_{21}^0$ and somewhat less obviously we have

LEMMA 2.2.4. *The transformation $\mathfrak{L}_{21}^1$ can be represented as*

$$(2.2.27) \qquad \mathfrak{L}_{21}^1\mathfrak{y}^1 = [L_{21}^1y^1, \dot{B}_{21}\dot{y}^1 + C_1y^i]$$

where $y^i = \beta[y^1, L_{21}^1y^1]$ and

$$(2.2.28) \qquad \mathfrak{D}(L_{21}^1) = [[y^1, \dot{y}^1]; y^1 \in \mathfrak{D}(L_{21}^1), y^i \in \mathfrak{D}(C_1), \dot{y}^1 \in \dot{H}_1].$$

Proof. Suppose $y \in \mathfrak{N}_1$. Then for each $z \in \mathfrak{N}_0$

$$0 = \mathfrak{Q}(\mathfrak{y}, \mathfrak{z}) = Q(y, z) + Q(\dot{y}, \dot{z}).$$

It is clear that $N_0 \times \theta \subset \mathfrak{N}_0$ and hence that $Q(y, z) = 0$ for each $z \in N_0$. Consequently $y \in N_1 = \mathfrak{G}(L_{21}^1)$, y^i exists, and $Q(y, z) = \mathcal{Q}(y^i, z^i) = -\mathcal{Q}(y^i, G_1\dot{z}^1)$. On the other hand

$$Q(\dot{y}, \dot{z}) = \langle \dot{y}^2, \dot{z}^1 \rangle + \langle \dot{y}^1, B_{21}\dot{z}^1 - C_1G_1\dot{z}^1 \rangle - \langle (\dot{B}_{21} + \dot{B}_{21}^* - [CG]_{12})\dot{y}^1, \dot{z}^1 \rangle$$

and since

$$\langle \dot{y}^1, C_1G_1\dot{z}^1 \rangle = \langle \dot{y}^1, [CG]_{12}\dot{z}^1 \rangle = \langle [CG]_{12}\dot{y}^1, \dot{z}^1 \rangle,$$

we see that

$$0 = -\langle W^{-1}y^i, G_1\dot{z}^1 \rangle + \langle \dot{y}^2 - \dot{B}_{21}\dot{y}^1, \dot{z}^1 \rangle$$

for all $\dot{z}^1 \in \mathfrak{D}(G_1)$. It follows that $W^{-1}y^i \in \mathfrak{D}(G_1^*)$, that is $y^i \in \mathfrak{D}(C_1)$ and $\langle W^{-1}y, ^i G_1\dot{z}^1 \rangle = \langle C_1y^i, \dot{z}^1 \rangle$. Making use of the fact that $\mathfrak{D}(G_1)$ is dense in $\dot{H}_1$, we conclude that $\dot{y}^2 = \dot{B}_{21}\dot{y}^1 + C_1y^i$. Thus each $\mathfrak{y} \in \mathfrak{N}_1$ lies in the graph of $\mathfrak{L}_{21}^1$ as given by (2.2.27) and (2.2.28); the converse assertion is readily verified by retracing the above steps.

As before we let $\mathfrak{L}_1$ denote the operator engendered by $\mathfrak{L}_{21}^1$. It is readily seen that

$$(2.2.29) \qquad \mathfrak{L}_1\mathfrak{y}^0 = [L_1y^0, B\dot{y}^0 + C_0y^i]$$

where

$$(2.2.30) \qquad \mathfrak{D}(\mathfrak{L}_1) = [[y^0, \dot{y}^0]; y^0 \in \mathfrak{D}(L_1), y^i \in \mathfrak{D}(C_0), \dot{y}^0 \in \dot{I}_{01}[\dot{H}_1]].$$

Having determined $\mathfrak{N}_0$ and $\mathfrak{N}_1$ we next study the boundary space $\mathfrak{H} = \mathfrak{N}_1/\mathfrak{N}_0$ defined as in §1.5 with generic element $\mathfrak{y}$. We denote the natural

mapping $\mathfrak{y} \in \mathfrak{N}_1 \to \mathfrak{y} \in \mathfrak{H}$ by β and again we see that the quadratic form $\mathfrak{Q}$ induces a quadratic form on $\mathfrak{H}$ with the property $\mathfrak{Q}(\mathfrak{y}, \mathfrak{z}) = \mathfrak{Q}(\beta\mathfrak{y}, \beta\mathfrak{z})$.

The boundary spaces $\mathfrak{H}$ and H are not unrelated and under certain conditions they are equivalent. In order to study this relationship we first set

$$(2.2.31) \quad \mathfrak{N}_0 = [\mathfrak{y}; \mathfrak{y} \in \mathfrak{N}_1, y^i \in \mathfrak{D}(C_1), \dot{y}^1 \in \mathfrak{D}(G_1)] \text{ and } \mathfrak{H}_o = \beta[\mathfrak{N}_o].$$

Since $\mathfrak{N}_o$ consists only of cosets of $\mathfrak{N}_o$ we see that $\mathfrak{N}_o = \beta^{-1}[\mathfrak{H}_o]$. Further it is easy to show that $\mathfrak{N}_o$ is dense in $\mathfrak{N}_1$ and it follows from this that $\mathfrak{H}_o$ is dense in $\mathfrak{H}$.

LEMMA 2.2.5. *The mapping $\beta\mathfrak{y} \to y^e \equiv y^i + G_1\dot{y}^1$ is a one-to-one linear map of $\mathfrak{H}_o$ onto $\mathfrak{D}(C_1) \subset H$, under which*

$$(2.2.32) \qquad\qquad \|\beta\mathfrak{y}\|^2 \leqq \|y^e\|^2 + \|C_1 y^e\|^2,$$

and

$$(2.2.33) \qquad\qquad \mathfrak{Q}(\beta\mathfrak{y}, \beta\mathfrak{z}) = Q(y^e, y^e).$$

Proof. The first assertion follows directly from the fact that $\mathfrak{y} \to y^e$ is a linear map on $\mathfrak{N}_o$ with kernel $\mathfrak{N}_o$ and, by (2.2.18), with image set $\mathfrak{D}(C_1)$. For a particular $y^e \in \mathfrak{D}(C_1)$ choose $\mathfrak{y} \in \mathfrak{N}_o$ such that $y^i = y^e$ and $\dot{y}^1 = \theta$. Then

$$\|\beta\mathfrak{y}\|^2 \leqq \|\mathfrak{y}\|^2 = \|y\|^2 + \|C_1 y^i\|^2.$$

Minimizing this expression over all $y \in N_1$ with $y^i = y^e$ we obtain the inequality (2.2.32). Finally for $\mathfrak{y}, \mathfrak{z} \in \mathfrak{N}_o$ we have

$$\mathfrak{Q}(\mathfrak{y}, \mathfrak{z}) = Q(y^i, z^i) + \langle C_1 y^i, \dot{z}^1 \rangle + \langle \dot{y}^1, C_1 z^i \rangle + \langle [CG]_{12}\dot{y}^1, \dot{z}^1 \rangle$$

and in view of the fact that

$$\langle C_1 y^i, \dot{z}^1 \rangle = \langle W^{-1} y^i, G_1 \dot{z}^1 \rangle = Q(y^i, G_1 \dot{z}^1),$$

$$\langle [CG]_{12}\dot{y}^1, \dot{z}^1 \rangle = \langle C_1 G_1 \dot{y}^1, \dot{z}^1 \rangle = \langle W^{-1} G_1 \dot{y}^1, G_1 \dot{z}^1 \rangle = Q(G_1 \dot{y}^1, G_1 \dot{z}^1),$$

we see that $\mathfrak{Q}(\beta\mathfrak{y}, \beta\mathfrak{z}) = \mathfrak{Q}(\mathfrak{y}, \mathfrak{z}) = Q(y^e, z^e)$.

COROLLARY. *If C is bounded, then $\mathfrak{N}_o = \mathfrak{N}_1$ and $\beta\mathfrak{y} \to y^e$ is a bicontinuous isomorphism of $\mathfrak{H}$ onto H under which (2.2.33) is valid.*

Proof. If C is bounded, then $\dot{F}$ is bounded and $\dot{H}_1$ and $\dot{H}_2$ are equivalent with $\dot{H}_0$ under the respective mappings $\dot{I}_{01}$ and $\dot{I}_{20}$. As a consequence, C_1 and G_1 are bounded with $\mathfrak{D}(C_1) = H$ and $\mathfrak{D}(G_1) = \dot{H}_1$. The above lemma therefore implies that $\beta\mathfrak{y} \to y^e$ is one-to-one and linear on all of $\mathfrak{H}$ to all of H. Further the inequality (2.2.32) shows that the inverse map is continuous and this together with the closed graph theorem yields bicontinuity. Finally the lemma asserts that (2.2.33) is valid under this correspondence.

The full statement of our result contains a description of the adjoint system and we therefore proceed to the associated adjoint transformations. According to the general theory developed in §1.5 the maximal and

minimal adjoint transformations $\mathfrak{M}_{21}^1$ and $\mathfrak{M}_{21}^0$, respectively, are defined by the relations

$$(2.2.34) \qquad \mathfrak{G}(\mathfrak{M}_{21}^1) = \mathfrak{U}[\mathfrak{G}(\mathfrak{L}_{21}^1)] \quad \text{and} \quad \mathfrak{G}(\mathfrak{M}_{21}^0) = \mathfrak{U}[\mathfrak{G}(\mathfrak{L}_{21}^0)].$$

Consequently

$$(2.2.35) \qquad \mathfrak{M}_{21\mathfrak{z}}^{1\ 1} = [M_{21}^1 z^1, \dot{B}_{21}^{*} \dot{z}^1 - [CG]_{21} \dot{z}^i - C_1 z^i]$$

where $\mathfrak{D}(\mathfrak{M}_{21}^1) = \mathfrak{D}(\mathfrak{L}_{21}^1)$ and $z^i = \beta[z^1, L_{21}^1 z^1]$. Furthermore $\mathfrak{M}_{21}^0$ is the restriction of $\mathfrak{M}_{21}^1$ with domain $\mathfrak{D}(\mathfrak{M}_{21}^0) = \mathfrak{D}(\mathfrak{L}_{21}^0)$ and hence can be represented simply as

$$(2.2.36) \qquad \mathfrak{M}_{21\mathfrak{z}}^{0\ 1} = [M_{21}^1 z^1, \dot{B}_{21}^{*} \dot{z}^1].$$

It is easy to see that the operator $\mathfrak{M}_0$ engendered by $\mathfrak{M}_{21}^0$ is

$$(2.2.37) \qquad \mathfrak{M}_{0\mathfrak{z}}^0 = [M_1 z^0, \dot{B}^{*} \dot{z}^0]$$

where

$$(2.2.38) \quad \mathfrak{D}(\mathfrak{M}_0) = [[z^0, \dot{z}^0]; z^0 \in \mathfrak{D}(M_1), \dot{z}^0 \in \mathfrak{D}(G), z^i = -G\dot{z}^0];$$

here again $z^i = \beta[I_{01}^{-1} z^0, L_{21}^1 I_{01}^{-1} z^0]$. The operator $\mathfrak{M}_1$ engendered by $\mathfrak{M}_{21}^1$ requires a somewhat more complicated description. We have

$$(2.2.39) \qquad \mathfrak{M}_{1\mathfrak{z}}^0 = [M_1 z^0, \dot{B}^{*} \dot{z}^0 - \dot{I}_{20}^{-1}\{[CG]_{21} \dot{I}_{01}^{-1} \dot{z}^0 + C_1 z^i\}]$$

where

$$(2.2.40) \qquad \begin{aligned} \mathfrak{D}(\mathfrak{M}_1) &= [[z^0, \dot{z}^0]; z^0 \in \mathfrak{D}(M_1), z^i \in \mathfrak{D}(C_1), \dot{z}^0 \in \dot{I}_{01}[\dot{H}_1], \\ & \{[CG]_{21} \dot{I}_{01}^{-1} \dot{z}^0 + C_1 z^i\} \in I_{20}[\dot{H}_0]]. \end{aligned}$$

By now we have not only verified the hypothesis of Theorem 1.5.1 but characterized the pertinent maximal and minimal operators as well. The conclusions of Theorems 1.5.1 and 1.6.1 are therefore valid and represent the principal result of this section. It will be noticed that the coupling transformation in the adjoint operator $\mathfrak{M}_1$ acts on the boundary space of L_{21}^1. By employing the transformation theory developed in §1.6, this defect is easily rectified. In fact setting

$$(2.2.41) \qquad z^{i\prime} = U z^i, \qquad C_1' = C_1 U', \quad \text{and} \quad G' = UG,$$

it is clear that $\mathfrak{M}_0$ and $\mathfrak{M}_1$ can be represented by the equivalent operators

$$(2.2.42) \qquad \begin{aligned} \mathfrak{M}_{0\ \mathfrak{z}}'^0 &= [M_1 z^0, \dot{B}^{*} \dot{z}^0], \\ \mathfrak{D}(\mathfrak{M}_0') &= [[z^0, \dot{z}^0]; z^0 \in \mathfrak{D}(M_1), \dot{z}^0 \in \mathfrak{D}(G'), z^{i\prime} = -G'\dot{z}^0]; \end{aligned}$$

and

$$\mathfrak{M}_1' \mathfrak{z}^0 = [M_1 z^0, \dot{B} * \dot{z}^0 - \dot{I}_{20}^{-1}\{[CG]_{21} \dot{I}_{01}^{-1} \dot{z}^0 + C_1' z^{i'}\}],$$

$$(2.2.43) \quad \mathfrak{D}(\mathfrak{M}_1') = [[z^0, \dot{z}^0]; z^0 \in \mathfrak{D}(M_1), z^{i'} \in \mathfrak{D}(C_1'), \dot{z}^0 \in I_{01}[H_1],$$

$$\{[CG]_{21} \dot{I}_{01}^{-1} \dot{z}^0 + C_1' z^{i'}\} \in \dot{I}_{20}[\dot{H}_0]].$$

It was shown in the proof of Theorem 1.6.1 for $z \in \mathfrak{G}(M_{21}^1)$ that $\beta U z = \beta U_0' \beta' z$ or, equivalently, $\beta' z = \beta' U_0 \beta U z$. Consequently $z^{i'} = U z^i = \beta' U_0 [\beta U z] = \beta' z$, that is

$$(2.2.44) \qquad\qquad z^{i'} = \beta'[\dot{I}_{01}^{-1} \dot{z}^0, I_{20} M_1 z^0].$$

When C is bounded, the Corollary to Lemma 2.2.5 permits us to state the boundary conditions on $\mathfrak{L}$ and $\mathfrak{M}$ in terms of the L_1 and M_1 boundary spaces, namely H and H' respectively. Thus in this case Theorem 1.5.1 takes the form: There is a one-to-one correspondence between the maximal negative subspaces $[N]$ of H, taken with respect to Q, and the closed maximal dissipative operators $\mathfrak{L}$ on H such that $\mathfrak{L}_0 \subset \mathfrak{L} \subset \mathfrak{L}_1$, this correspondence being defined by

$$(2.2.45) \qquad\qquad \mathfrak{D}(\mathfrak{L}) = [\mathfrak{y}^0; \mathfrak{y}^0 \in \mathfrak{D}(\mathfrak{L}_1), y^e \in N],$$

where $y^e = y^i + G \dot{y}^0$. Moreover the adjoint operator $\mathfrak{M} = \mathfrak{L}^*$ is again a maximal dissipative operator with dense domain, $\mathfrak{M}_0 \subset \mathfrak{M} \subset \mathfrak{M}_1$, and

$$(2.2.46) \qquad\qquad \mathfrak{D}(\mathfrak{M}) = [\mathfrak{z}^0; \mathfrak{z}^0 \in \mathfrak{D}(\mathfrak{M}_1), z^e \in P],$$

where $z^e = z^i + G \dot{z}^0$ and P is the Q-orthogonal complement of N. As in the proof of Theorem 1.6.1, $z^e \in P$ if and only if

$$(2.2.47) \qquad\qquad z^{e'} = U z^e = z^{i'} + G' \dot{z}^0$$

belongs to $U[P]$. Hence (2.2.46) is equivalent with

$$(2.2.48) \qquad\qquad \mathfrak{D}(\mathfrak{M}) = [\mathfrak{z}^0; \mathfrak{z}^0 \in \mathfrak{D}(\mathfrak{M}_1), z^{e'} \in U[P]].$$

We note that $U[P]$ is a maximal negative subspace of H'.

By way of illustrating the above development, we now sketch a treatment for the vibrating membrane on a unit disk. The equation of motion being

$$(2.2.49) \qquad u_{tt} = u_{x_1 x_1} + u_{x_2 x_2}, \qquad x_1^2 + x_2^2 < 1, \qquad\qquad t > 0,$$

we set $u_{x_1} = \eta^1$, $u_{x_2} = \eta^2$, and $u_t = \eta^3$, and obtain the system (1.2) where

$$A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad E = I, \qquad B = \Theta.$$

It is clear that $D = \Theta$ so that $H_1 = H_0 = H_2 = L_2(\Delta; I)$, $L_0 = -M_0$, and $L_1 = -M_1$. According to (1.5.8) each element of the boundary space H is

represented by a solution of

$$(2.2.50) \qquad \overset{2}{L_1} \overset{0}{y} = \overset{0}{y},$$

or, equivalently, by a weak solution of the system

$$\overset{1}{\eta}_{x_1 x_1} + \overset{2}{\eta}_{x_1 x_2} = \overset{1}{\eta},$$

$$\overset{1}{\eta}_{x_1 x_2} + \overset{2}{\eta}_{x_2 x_2} = \overset{2}{\eta},$$

$$\overset{3}{\eta}_{x_1 x_1} + \overset{3}{\eta}_{x_2 x_2} = \overset{3}{\eta}.$$

It can be shown that weak solutions correspond to regular solutions for such a system and further that the first two equations are equivalent to

$$(2.2.51) \qquad \eta^1 = \phi_{x_1}, \qquad \eta^2 = \phi_{x_2}, \qquad \phi_{x_1 x_1} + \phi_{x_2 x_2} = \phi.$$

Representing the boundary values of η^3 and $\partial\phi/\partial r$ (here $r^2 = x_1^2 + x_2^2$) by Fourier series

$$(2.2.52) \quad \eta^3 \sim (2\pi)^{-1/2} \sum_{k=-\infty}^{\infty} a_k e^{ik\sigma}, \qquad \partial\phi/\partial r \sim (2\pi)^{-1/2} \sum_{k=-\infty}^{\infty} b_k e^{ik\sigma},$$

the Bessel function expansions for η^3 and ϕ are simply

$$\eta^3(r, \sigma) = (2\pi)^{-1/2} \sum_{k=-\infty}^{\infty} a_k \{I_k(1)\}^{-1} I_k(r) e^{ik\sigma},$$

$$(2.2.53)$$

$$\phi(r, \sigma) = (2\pi)^{-1/2} \sum_{k=-\infty}^{\infty} b_k \{I_k'(1)\}^{-1} I_k(r) e^{ik\sigma};$$

and the norm of $y = [\eta^3, \partial\phi/\partial r]$ is given by

$$\|y\|^2 = \langle y^0, y^0 \rangle + \langle L_1 y^0, L_1 y^0 \rangle = \int_\Gamma \eta^3 (\partial\eta^3/\partial r)^- + \phi(\partial\phi/\partial r)^- d\sigma$$

$$(2.2.54)$$

$$= \sum_{k=-\infty}^{\infty} (\rho_k |a_k|^2 + \rho_k^{-1} |b_k|^2),$$

where $\rho_k = I_k'(1)/I_k(1)$. It follows that the boundary space H is the direct product of hilbert spaces h_1 and h_2 with elements $n^1 = \eta^3$ and $n^2 = \partial\phi/\partial r$, and norms

$$(2.2.55) \qquad \|\overset{1}{n}\|_1^2 = \sum \rho_k |a_k|^2 \quad \text{and} \quad \|\overset{2}{n}\|_2^2 = \sum \rho_k^{-1} |b_k|^2,$$

respectively. These are dual spaces and the transformation

$$(2.2.56) \qquad f_{12}\{b_k\} = \{\rho_k^{-1} b_k\}$$

on h_2 to h_1 is unitary with inverse $f_{21}\{a_k\} = \{\rho_k a_k\}$.

In the present example the third relation in (1.3.4) becomes $W[y^1, y^2]$ $= [y^2, y^1]$. We also recall that $y = [y^1, L_1 y^1] \in N_1 \cap N_0^\perp$ if and only if $L_1^2 y^1 = y^1$. On the other hand for such a y, $Wy = [L_1 y^1, y^1] = [L_1 y^1, L_1[L_1 y^1]]$ and $(L_1)^2 L_1 y^1 = L_1[L_1^2 y^1] = L_1 y^1$ so that W maps $N_1 \cap N_0^\perp$ on itself. Moreover in going from y^1 to $L_1 y^1$ we see that $\eta^1 \to \eta_{\chi_1}^3$, $\eta^2 \to \eta_{\chi_2}^3$, and $\eta^3 \to \eta_{\chi_1}^1 + \eta_{\chi_2}^2 = \Delta\phi = \phi$. Hence $W[\eta^3, \partial\phi/\partial r] = [\phi, \partial\eta^3/\partial r]$. In other words

$$(2.2.57) \qquad\qquad W[\mathbf{n}^1, \mathbf{n}^2] = [f_{12}\mathbf{n}^2, f_{21}\mathbf{n}^1].$$

Finally the relevant quadratic form on H is

$$(2.2.58)
\begin{aligned}
Q(y, y) &= \int_\Gamma \{\eta^3 (\partial\phi/\partial r)^- + [\eta^3]^- \partial\phi/\partial r\} d\sigma = \langle \mathbf{n}^1, \mathbf{n}^2 \rangle + \langle \mathbf{n}^2, \mathbf{n}^1 \rangle \\
&= \frac{1}{2} \{\| \overset{1}{\mathbf{n}} + f_{12}\overset{2}{\mathbf{n}} \|_1^2 - \| \overset{1}{\mathbf{n}} - f_{12}\overset{2}{\mathbf{n}} \|_1^2 \},
\end{aligned}$$

and the general boundary condition given by a maximal negative subspace N of H is characterized by an arbitrary contraction operator J on h_1 to itself as

$$(2.2.59) \qquad\qquad \mathbf{n}^1 + f_{12}\mathbf{n}^2 + J[\mathbf{n}^1 - f_{12}\mathbf{n}^2].$$

We next impose an elastic type boundary condition on the membrane by adjoining to the membrane system the equation

$$(2.2.60) \qquad\qquad \dot{\eta}_t = \kappa \mathbf{n}^1$$

which governs the boundary system; here κ is a multiplicative operator on h_1 (considered as a space of functions on the circle Γ) to $\dot{H}_0 = L_2(\Gamma)$. Assuming the multiplier $\kappa(\cdot)$ to be measurable and bounded, it is easy to see that κ is a bounded linear transformation. Further

$$(2.2.61) \qquad\qquad \dot{B} = \Theta \quad \text{and} \quad C = (\kappa \quad \Theta).$$

Now $\langle \kappa \mathbf{n}^1, \dot{\eta} \rangle = \langle \mathbf{n}^1, f_{12}\kappa_1 \dot{\eta} \rangle_1$, where κ_1 is the bounded multiplicative transformation on $\dot{H}_0$ to h_2 defined by the function $[\kappa(\cdot)]^-$. It follows that

$$(2.2.62) \qquad C^* = \begin{pmatrix} f_{12}\kappa_1 \\ \Theta \end{pmatrix} \quad \text{and} \quad G = WC^* = \begin{pmatrix} \Theta \\ \kappa_1 \end{pmatrix}.$$

The transformations C and G being bounded, we conclude that the mapping $\mathfrak{y} \to y^e = y^i + G\dot{y}^0$ is an isomorphism of $\mathfrak{H}$ onto H under which $Q(y^e, y^e) = \mathfrak{Q}(\mathfrak{y}, \mathfrak{y})$. In this case

$$(2.2.63) \qquad\qquad y^e = [\mathbf{n}^1, \mathbf{n}^2 + \kappa_1 \dot{\eta}].$$

The maximal negative subspaces of $\mathfrak{H}$ are defined by contraction operators J as in (2.2.59) and in particular $J = I$ gives $(y^e)^2 = \mathbf{n}^2 + \kappa_1 \dot{\eta} = \theta$ as a boundary condition. On the other hand, $\mathbf{n}^2 = \partial u/\partial r|_{r=1}$ and by (2.2.60) $\dot{\eta}_t = \kappa \mathbf{n}^1 = \kappa u_t|_{r=1}$

so that we have in essence

$$(2.2.64) \qquad \partial u/\partial r\big|_{r=1} + \kappa_1 \kappa u\big|_{r=1} = 0,$$

the familiar boundary condition associated with an elastic constraint on the boundary.

A somewhat more interesting boundary problem is obtained when we couple a spring with mass attached to the membrane in which case we require an unbounded coupling operator. The boundary space is now $\dot{H}_0 = L_2(\Gamma) \times L_2(\Gamma)$ and the boundary system is governed by the equations

$$(2.2.65) \qquad \begin{aligned} \overset{1}{\dot{\eta}}_t &= \mu \overset{2}{\dot{\eta}}, \\ \overset{2}{\dot{\eta}}_t &= -\mu \overset{1}{\dot{\eta}} - \nu^{-1}\overset{2}{\mathbf{n}}, \end{aligned}$$

where μ and ν are multiplicative operators corresponding to real measurable bounded functions $\mu(\cdot)$ and $\nu(\cdot)$, in fact we even assume that $\nu(\cdot)$ is bounded away from zero. As such μ is clearly a bounded operator on $L_2(\Gamma)$ to itself. On the other hand ν is bounded on $L_2(\Gamma)$ to h_2 and hence ν^{-1} is closed on h_2 to $L_2(\Gamma)$ with a dense domain consisting of those functions whose Fourier coefficients are square summable (without the ρ_k^{-1} factor). We now have

$$(2.2.66) \qquad B = \begin{pmatrix} \Theta & \mu \\ -\mu & \Theta \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \Theta & \Theta \\ \Theta & -\nu^{-1} \end{pmatrix}.$$

Again $\langle \nu\dot{\eta}^2, \mathbf{n}^2\rangle_2 = \langle \dot{\eta}^2, \nu_1 f_{12}\mathbf{n}^2\rangle$ where ν_1 is the bounded linear multiplicative operator on h_1 to $L_2(\Gamma)$ defined by $\nu(\cdot)$. Thus $(\nu^{-1})^* = (\nu^*)^{-1} = f_{21}\nu_1^{-1}$ and

$$(2.2.67) \qquad C^* = \begin{pmatrix} \Theta & \Theta \\ \Theta & -f_{21}\nu_1^{-1} \end{pmatrix} \quad \text{and} \quad G = WC^* = \begin{pmatrix} \Theta & -\nu_1^{-1} \\ \Theta & \Theta \end{pmatrix}.$$

Finally we note that $CWC^* = \Theta$, with domain $\mathfrak{D}(C^*)$, and this together with the fact that $F = I$ shows that Condition (i) is satisfied. Likewise (ii) is satisfied since $\dot{B} + \dot{B}^* - CWC^* = \Theta$ and we see, incidentally, that $\dot{H}_1 = \dot{H}_0 = \dot{H}_2$.

The boundary space elements are represented in the present instance by solutions of $\mathfrak{M}_1 \mathfrak{L}_1 \mathfrak{y}^1 = -\mathfrak{y}^1$ or, equivalently, by solutions of (2.2.50) plus

$$(2.2.68) \qquad B^*[B\dot{y}^1 + Cy^i] - Cz^i = -\dot{y}^1,$$

where $y^i = \beta[y^1, L_1 y^1] = [\eta^3, \partial\phi/\partial r]$ and $z^i = \beta[L_1 y^1, y^1] = [\phi, \partial\eta^3/\partial r]$. The Equations (2.2.50) are solved as before and, in terms of the functions η^3 and ϕ, we obtain as the solution of (2.2.68)

$$(2.2.69) \qquad \begin{aligned} \dot{\eta}^1 &= -\mu(1+\mu^2)^{-1}\nu^{-1}\partial\phi/\partial r, \\ \dot{\eta}^2 &= -(1+\mu^2)^{-1}\nu^{-1}\partial\eta^3/\partial r. \end{aligned}$$

From this one readily computes for $\mathfrak{y} = [\mathfrak{y}^1, \mathfrak{L}_1\mathfrak{y}^1]$

$$(2.2.70) \quad \|\mathfrak{y}\|^2 = \int_\Gamma \{ \eta^3 (\partial \eta^3 / \partial r)^- + \phi (\partial \phi / \partial r)^- $$

$$+ (1 + \mu^2)^{-1} (| \nu^{-1} \partial \eta^3 / \partial r |^2 + | \nu^{-1} \partial \phi / \partial r |^2) \} d\sigma,$$

and

$$(2.2.71) \quad \mathfrak{Q}(\mathfrak{y}, \mathfrak{y}) = 2 \operatorname{Re} \left[\int_\Gamma \{ \nu_1 \eta^3 + (1 + \mu^2)^{-1} \nu^{-1} \partial \eta^3 / \partial r \} \{ \nu^{-1} \partial \phi / \partial r \}^- d\sigma \right].$$

Finally since η^3 and ϕ are mutually independent and since the range of ν^{-1} is $L_2(\Gamma)$, we see that a maximal negative subspace is determined by the relation $\nu_1 \eta^3 + (1 + \mu^2)^{-1} \nu^{-1} \partial \eta^3 / \partial r = 0$ on Γ, in other words $\dot{\eta}^2 = \nu_1 \eta^3$ on Γ. Now by virtue of (2.2.65) we have $\mu \nu_1 u_t |_{r=1} = \mu \nu_1 \eta^3 = \dot{\eta}_t^1$. In essence then $\dot{\eta}^1 = \mu \nu_1 u |_{r=1}$ and hence by (2.2.65)

$$(2.2.72) \quad \nu_1 u_{tt} |_{r=1} + \mu^2 \nu_1 u |_{r=1} = - \nu^{-1} \partial u / \partial r |_{r=1},$$

which is the boundary condition associated with a spring plus mass constraint at the boundary.

Appendix

2.3. **General maximal dissipative extensions of L_0 and restrictions of L_1.** The body of this chapter has dealt with maximal and properly dissipative extensions of L_0, that is, maximal dissipative extensions of L_0 which are at the same time restrictions of L_1. If $F = I$, then L_0 is conservative, $L_1 = -L_0^*$, and we see from Lemma 1.1.5 that the properly dissipative operators are the only maximal dissipative extensions of L_0. However in the general case, $F \neq I$, other maximal dissipative extensions of L_0 exist and, dually, there are maximal dissipative restrictions of L_1 whose domains do not contain $\mathfrak{D}(L_0)$[7]. Such extensions of L_0 need not be differential operators; the physical models associated with operators of this kind allow for part of the energy which leaves the system through the boundary to be redistributed in the interior[8]. The adjoint M of such an operator is a restriction of M_1 with $\mathfrak{D}(M) \supsetneq \mathfrak{D}(M_0)$ and, in the associated model, energy may enter through the boundary, but only to an extent which can be compensated for by internal energy losses. In this section we shall make an exploratory study of these nonproperly-dissipative operators; however, the problem of determining the most general maximal dissipative extension of L_0 or restriction of L_1 is left open.

  We shall assume that L is a maximal dissipative extension of L_0 with domain contained in $\mathfrak{D}(L_1)$. In this case $L - L_1$ annihilates $\mathfrak{D}(L_0)$ and hence $(L - L_1) y^0$ depends only on the coset of $\mathfrak{D}(L)/\mathfrak{D}(L_0)$ to which y^0 belongs. Now

$$\mathfrak{D}(L)/\mathfrak{D}(L_0)$$

is isomorphic to a subspace N of H under the mapping

(2.3.1)
$$\gamma : y^0 \to y = \beta[I_{01}^{-1}y^0, \ L_{21}^{1}I_{01}^{-1}y^0].$$

Thus $L - L_1$ defines a linear transformation T on N to H_0 such that

(2.3.2)
$$Ly^0 = L_1y^0 + T\gamma y^0, \qquad\qquad y^0 \in \mathfrak{D}(L).$$

We shall soon restrict T further; however first it will be instructive to deduce certain properties of T and N implied by the above basic assumption.

LEMMA 2.3.1. *N is a negative subspace of H.*

Proof. Suppose there were a $y^0 \in \mathfrak{D}(L)$ such that $Q(\gamma y^0, \gamma y^0) > 0$. Then since $\mathfrak{D}(I_{20}L_0I_{01})$ is dense in H_1, we can choose a sequence $\{u_n^0\} \subset \mathfrak{D}(L_0)$ with the property $I_{01}^{-1}u_n^0 \to I_{01}^{-1}y^0$ in H_1. Setting $w_n^0 = y^0 - u_n^0$, we see that $\{w_n^0\} \subset \mathfrak{D}(L)$, $\gamma w_n^0 = \gamma y^0$, $I_{01}^{-1}w_n^0 \to \theta$ in H_1, and $w_n^0 \to \theta$ in H_0. Consequently[8]

$$\langle Lw_n^0, w_n^0\rangle + \langle w_n^0, Lw_n^0\rangle = Q(\gamma y^0, \gamma y^0) + \langle Dw_n^0, w_n^0\rangle$$

(2.3.3)
$$+ \langle T\gamma y^0, w_n^0\rangle + \langle w_n^0, T\gamma y^0\rangle$$

and this converges to $Q(\gamma y^0, \gamma y^0) > 0$ as $n \to \infty$. However this is contrary to L being dissipative.

Since $Q(\gamma y^0, \gamma y^0)$ is a measure of the rate at which energy enters through the boundary, the above lemma shows for the model associated with L that energy does not enter through the boundary. On the other hand, T has the effect of channeling energy directly into the interior at a rate depending on the boundary data.

LEMMA 2.3.2. *If $\Re(T) \subset I_{01}[H_1]$, then $Ty = \theta$ whenever $y \in N$ and $Q(y,y) = 0$.*

Proof. If the assertion were false, there would exist a $y^0 \in \mathfrak{D}(L)$ such that $Q(\gamma y^0, \gamma y^0) = 0$ and $T\gamma y^0 \neq \theta$. For a given $\epsilon > 0$, choose a sequence $\{u_n^0\} \subset \mathfrak{D}(L_0)$ with the property $I_{01}^{-1}u_n^0 \to I_{01}^{-1}(\epsilon T\gamma y^0 - y^0)$. Setting $w_n^0 = y^0 + u_n^0$ we see that $\{w_n^0\} \subset \mathfrak{D}(L)$, $\gamma w_n^0 = \gamma y^0$, $I_{01}^{-1}w_n^0 \to I_{01}^{-1}(\epsilon T\gamma y^0)$ in H_1, and $w_n^0 \to \epsilon T\gamma y^0$ in H_0. Thus (2.3.3) holds and

$$\langle Lw_n^0, w_n^0\rangle + \langle w_n^0, Lw_n^0\rangle \to \epsilon^2 \langle DT\gamma y^0, T\gamma y^0\rangle + 2\epsilon\|T\gamma y^0\|^2,$$

which is positive for a sufficiently small choice of ϵ. Again this is contrary to L being dissipative.

The physical interpretation of Lemma 2.3.2 is that T transports energy into the interior only when energy leaves through the boundary. This is reasonable enough and suggests that it would be better to define T as a transformation on the quotient space

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[8] Here we use the notation $\langle Dw^0, z^0\rangle \equiv \int_\Delta (EDw^0, z^0)d\chi = \langle D_{21}I_{01}^{-1}w^0, I_{01}^{-1}z^0\rangle$, valid for all $w^0, z^0 \in I_{01}[H_1]$.

(2.3.4) $$\widehat{N_o} \equiv N/N_o$$

where

(2.3.5) $$N_0 = [y; Q(y, y) = 0, y \in N].$$

According to Lemma 2.3.1, $-Q$ is a positive hermitian bilinear form on N. It follows that $-Q$ defines a positive definite bilinear form on $\widehat{N_o}$, namely $-\widehat{Q}$, where we set $\widehat{Q}(\widehat{y}, \widehat{z}) = Q(y, z)$ for any $y \in \widehat{y}$ and $z \in \widehat{z}$. Letting

(2.3.6) $$\langle \widehat{y}, \widehat{z} \rangle = - \widehat{Q}(\widehat{y}, \widehat{z}),$$

$\widehat{N_o}$ becomes a pre-hilbert space whose completion we denote by $\widehat{N}$. We shall denote the mapping which sends $y^0 \in \mathfrak{D}(L)$ into the coset $\widehat{y}$ containing γy^0 by the symbol $\widehat{\gamma}$.

With this preliminary discussion as motivation, we now suppose that L is of the form

(2.3.7) $$Ly^0 = L_1y^0 + \widehat{T}\widehat{\gamma}y^0, \qquad\qquad y^0 \in \mathfrak{D}(L),$$

where $\widehat{T}$ *is a linear bounded transformation on* $\widehat{N}$ *to* H_0. This amounts to a rather strong assumption on T and implies, for instance, that the adjoint transformation $\widehat{T}^*$ on H_0 to $\widehat{N}$ exists. Thus for $y^0 \in \mathfrak{D}(L)$ and $z^0 \in H_0$ we have

(2.3.8) $$\langle \widehat{T}\widehat{\gamma}y^0, z^0 \rangle = \langle \widehat{\gamma}y^0, \widehat{T}^*z^0 \rangle = - \widehat{Q}(\widehat{\gamma}y^0, \widehat{T}^*z^0).$$

Consequently for $y^0, z^0 \in \mathfrak{D}(L)$ we can write

(2.3.9)
$$\begin{aligned}
\langle Ly^0, z^0 \rangle &+ \langle y^0, Lz^0 \rangle \\
&= \widehat{Q}(\widehat{\gamma}y^0, \widehat{\gamma}z^0) + \langle Dy^0, z^0 \rangle - \widehat{Q}(\widehat{\gamma}y^0, \widehat{T}^*z^0) - \widehat{Q}(\widehat{T}^*y^0, \widehat{\gamma}z^0) \\
&= \widehat{Q}(\widehat{\gamma}y^0 - \widehat{T}^*y^0, \widehat{\gamma}z^0 - \widehat{T}^*z^0) + \langle (D + \widehat{T}\widehat{T}^*)y^0, z^0 \rangle;
\end{aligned}$$

here we have used the identity $\widehat{Q}(\widehat{T}^*y^0, \widehat{T}^*z^0) = -\langle \widehat{T}\widehat{T}^*y^0, z^0 \rangle$.

LEMMA 2.3.3. *Considered as a quadratic form on* $I_{01}[H_1]$,

(2.3.10) $$D + \widehat{T}\widehat{T} \leqq \Theta.$$

Proof. Suppose the contrary were the case. Then since $I_{01}^{-1}[\mathfrak{D}(L_0)]$ is dense in H_1, there is a $y^0 \in \mathfrak{D}(L_0)$ such that

$$\langle (D + \widehat{T}\widehat{T}^*)y^0, y^0 \rangle > 0.$$

Further, since $\widehat{T}^*y^0 \in \widehat{N}$ and $\widehat{\gamma}[\mathfrak{D}(L)]$ is dense in $\widehat{N}$ (by construction), we can find a sequence $\{u_n^0\} \subset \mathfrak{D}(L)$ such that $\widehat{\gamma}u_n^0 \to \widehat{T}^*y^0$ in $\widehat{N}$ and again making use of the fact that $I_{01}^{-1}[\mathfrak{D}(L_0)]$ is dense in H_1, we may further suppose that $I_{01}^{-1}u_n^0 \to \theta$ in H_1. Then

$$w_n^0 = y^0 + u_n^0 \in \mathfrak{D}(L), \qquad \widehat{\gamma}w_n^0 = \widehat{\gamma}u_n^0 \to \widehat{T}^*y^0, \qquad I_{01}^{-1}w_n^0 \to I_{01}^{-1}y^0 \text{ in } H_1,$$

and

$$\overset{0}{w_n} \to \overset{0}{y} \text{ in } H_0.$$

It now follows from (2.3.9) that

$$\langle L\overset{0}{w_n}, \overset{0}{w_n} \rangle + \langle \overset{0}{w_n}, L\overset{0}{w_n} \rangle \to \langle (D + \hat{T}\hat{T}^*)\overset{0}{y}, \overset{0}{y} \rangle > 0,$$

which is impossible for L dissipative.

LEMMA 2.3.4. *Let N^- denote the closure of $N = \gamma[\mathfrak{D}(L)]$. Then N^- is a maximal negative subspace of H and*

$$(2.3.11) \qquad \mathfrak{D}(L) = [y^0; y^0 \in \mathfrak{D}(L_1) \text{ and } \gamma y^0 \in N^-].$$

Proof. By assumption $\mathfrak{D}(L_0) \subset \mathfrak{D}(L) \subset \mathfrak{D}(L_1)$, and it follows from this fact alone that $\mathfrak{D}(L)$ is of the form (2.3.11) with N^- replaced by $N = \gamma[\mathfrak{D}(L)]$. According to Lemma 2.3.1, N is a negative subspace of H. Hence if N is not maximal negative, it is certainly contained in some maximal negative subspace, say N^-. We set

$$\mathfrak{D}^- = [y^0; y^0 \in \mathfrak{D}(L_1) \text{ and } \gamma y^0 \in N^-]$$

and show that $\mathfrak{D}(L) = \mathfrak{D}^-$. Obviously $\mathfrak{D}(L) \subset \mathfrak{D}^-$. Again defining

$$N_0^- = [y; Q(y, y) = 0, y \in N^-],$$

we proceed as before and construct the pre-hilbert space

$$\widehat{N_o^-} = N^-/N_0^-$$

of residue classes $\widehat{y^-}, \widehat{z^-}$ having as inner produce $\langle \widehat{y^-}, \widehat{z^-} \rangle = -\widehat{Q}^-(\widehat{y^-}, \widehat{z^-})$, where $\widehat{Q}^-(\widehat{y^-}, \widehat{z^-}) = Q(y, z)$ for all $y \in \widehat{y^-}$ and $z \in \widehat{z^-}$. Likewise we denote the mapping which takes $y^0 \in \mathfrak{D}^-$ into the coset $\widehat{y^-}$ containing γy^0 by the symbol $\widehat{\gamma}^-$. It is clear that $N_0 = N_0^- \cap N$ and it follows from this that $\widehat{N_0}$ can be embedded in $\widehat{N_o^-}$ under the mapping which takes $\widehat{y}$ into $\widehat{y^-}$ if the residue class $\widehat{y}$ is contained in the residue class $\widehat{y^-}$; in particular, the inner product is preserved under this mapping. Moreover, $\widehat{\gamma}^-$ is in this sense an extension of $\widehat{\gamma}$. Finally, if $\widehat{N}^-$ denotes the completion of $\widehat{N_o^-}$, then it is clear that $\widehat{N}$ can be thought of as a closed subspace of $\widehat{N}^-$, and that $\hat{T}$ can be thought of as a transformation of this subspace to H_0. We note that each $\widehat{y^-} \in \widehat{N}^-$ has a unique representation $\widehat{y^-} = \widehat{y_1^-} + \widehat{y_2^-}$, where $\widehat{y_1^-} \in \widehat{N}$ and $\widehat{y_2^-} \in \widehat{N}^- \cap N^\perp$. We now extend $\hat{T}$ onto $\widehat{N}^-$ by setting

$$\hat{T}^- \widehat{y^-} = \hat{T}\widehat{y_1^-}.$$

Then $\hat{T}^-$ is a bounded linear transformation on $\widehat{N}^-$ to H_0. Next we define what is clearly an extension of L, namely,

$$L^- y^0 = L_1 y^0 + \hat{T}^- \widehat{\gamma}^- y^0, \qquad\qquad y^0 \in \mathfrak{D}^-,$$

and we show that L^- is dissipative. In fact, suppose $y^0 \in \mathfrak{D}^-$ and let $\widehat{\gamma}^- y^0 = \widehat{y^-} = \widehat{y_1^-} + \widehat{y_2^-}$. Then

$$\widehat{\boldsymbol{Q}}^-(\widehat{\boldsymbol{y}}^-, \widehat{\boldsymbol{y}}^-) = \widehat{\boldsymbol{Q}}(\widehat{\boldsymbol{y_1}}, \widehat{\boldsymbol{y_1}}) + \widehat{\boldsymbol{Q}}^-(\widehat{\boldsymbol{y_2}}, \widehat{\boldsymbol{y_2}})$$

and

$$\langle \hat{T}^- \widehat{\boldsymbol{y}}^-, y^0 \rangle = \langle \hat{T}\widehat{\boldsymbol{y_1}}, y^0 \rangle = \langle \widehat{\boldsymbol{y_1}}, \hat{T}^* y^0 \rangle = -\widehat{\boldsymbol{Q}}(\widehat{\boldsymbol{y_1}}, \hat{T}^* y^0).$$

Consequently, we obtain in analogy to (2.3.9)

$$\langle L^- y^0, y^0 \rangle + \langle y^0, L^- y^0 \rangle$$
$$= \widehat{\boldsymbol{Q}}^-(\widehat{\boldsymbol{y_2}}, \widehat{\boldsymbol{y_2}}) + \widehat{\boldsymbol{Q}}(\widehat{\boldsymbol{y_1}} - \hat{T}^* y^0, \widehat{\boldsymbol{y_1}} - \hat{T}^* y^0) + \langle (D + \hat{T}\hat{T}^*) y^0, y^0 \rangle$$

which is $\leqq 0$ since each term in the right member is nonpositive, the last being so by Lemma 2.3.3. Thus L^- is dissipative and it follows from the fact that L was assumed to be maximal dissipative that $L = L^-$ and hence that $\mathfrak{D}(L) = \mathfrak{D}^-$. Thus $\mathfrak{D}(L)$ is of the type considered in Theorem 1.5.1 and hence there is a maximal dissipative operator L' such that $\mathfrak{D}(L) = \mathfrak{D}(L')$ and $L_0 \subset L' \subset L_1$. However, L' is engendered from a transformation L'_{21} for which $\beta[\mathfrak{G}(L'_{21})] = N^-$ and, according to Theorem 1.3.1, $\mathfrak{G}(I_{20} L' I_{01})$ is dense in $\mathfrak{G}(L'_{21})$. It follows that $\gamma[D(L)] = \beta[\mathfrak{G}(I_{20} L' I_{01})]$ is dense in N^-. Incidentally, this shows that $\widehat{N}_0$ is dense in $\widehat{N_0^-}$ and hence that $\widehat{N} = \widehat{N}^-$.

We summarize these results in

THEOREM 2.3.1. *Let L be a maximal dissipative extension of L_0 with $\mathfrak{D}(L)$ $\subset \mathfrak{D}(L_1)$. Then $N = \gamma[\mathfrak{D}(L)]$ is a negative subspace of H and we can construct the hilbert space $\widehat{N}$ from the pre-hilbert space defined by (2.3.4) and (2.3.6). Suppose further that L is of the form (2.3.7), where $\hat{T}$ is a bounded linear transformation on $\widehat{N}$ to H_0. Then $\hat{T}$ necessarily satisfies the Condition (2.3.10), the closure of N, namely N^-, is a maximal negative subspace of H defining the same hilbert space $\widehat{N}$ as N, and $\mathfrak{D}(L)$ is given by (2.3.11).*

In the converse direction we have

THEOREM 2.3.2. *Suppose N is a maximal negative subspace of H and construct the hilbert space $\widehat{N}$ from the pre-hilbert space defined by (2.3.4) and (2.3.6). Let $\hat{T}$ be a bounded linear transformation on $\widehat{N}$ to H_0 satisfying the Condition (2.3.10). Then*

$$(2.3.12) \qquad \begin{aligned} Ly^0 &= L_1 y^0 + \hat{T}\hat{\gamma} y^0, \\ \mathfrak{D}(L) &= [y^0; y^0 \in \mathfrak{D}(L_1) \text{ and } \gamma y^0 \in N] \end{aligned}$$

defines a maximal dissipative extension of L_0.

Proof. Since $\hat{\gamma}[\mathfrak{D}(L_0)] = \hat{\boldsymbol{\theta}}$, it is clear that $L \supset L_0$ and it follows from (2.3.9) that L is dissipative. It remains, therefore, to show that L is maximal dissipative and for this it suffices to prove that $\Re(\lambda I - L) = H_0$ for some $\lambda > 0$, according to the corollary to Theorem 1.1.1. We shall establish this fact with the help of the maximal dissipative operator L' defined in Theorem 1.5.1 with $\mathfrak{D}(L') = \mathfrak{D}(L)$ and $L_0 \subset L' \subset L_1$. We show, first of all, that $R(\lambda; L')$

can be thought of as defining a bounded linear transformation on H_0 to $\widehat{N}$ which we shall denote by S_λ; more precisely

$$(2.3.13) \qquad S_\lambda f^0 = \hat{\gamma}[R(\lambda; L')f^0].$$

Making use of the relation $\widehat{Q}(S_\lambda f^0, S_\lambda f^0) = Q(\gamma R(\lambda; L')f^0, \gamma R(\lambda; L')f^0)$ and the identity $(\lambda I - L')R(\lambda; L') = I$, we have

$$\begin{aligned}
\widehat{Q}(S_\lambda f^0, S_\lambda f^0) &= \langle R(\lambda; L')f^0, L'R(\lambda; L')f^0 \rangle \\
&\quad + \langle L'R(\lambda; L')f^0, R(\lambda; L')f^0 \rangle - \langle DR(\lambda; L')f^0, R(\lambda; L')f^0 \rangle \\
&= 2\lambda \langle R(\lambda; L')f^0, R(\lambda; L')f^0 \rangle - \langle R(\lambda; L')f^0, f^0 \rangle \\
&\quad - \langle f^0, R(\lambda; L')f^0 \rangle - \langle DR(\lambda; L')f^0, R(\lambda; L')f^0 \rangle.
\end{aligned}$$

If L'_{21} denotes the transformation with maximal negative graph which engenders L' and if $y^1 = I_{01}^{-1}R(\lambda; L')f^0$, then $\lambda I_{21}y^1 - L'_{21}y^1 = I_{20}f^0$ and Lemma 1.3.2 implies that

$$(2.3.14) \qquad \begin{aligned}
c(\lambda)^2 \big| \langle DR(\lambda; L')f^0, R(\lambda; L')f^0 \rangle \big| &\leq c(\lambda)^2 \big\| y^1 \big\|_1^2 \\
&\leq \big\| I_{20}f^0 \big\|_2^2 \leq \big\| f^0 \big\|^2.
\end{aligned}$$

It readily follows that

$$\big\| S_\lambda f^0 \big\|^2 = -\widehat{Q}(S_\lambda f^0, S_\lambda f^0) \leq k(\lambda) \big\| f^0 \big\|^2.$$

We now show that $k(\lambda) \leq (2\lambda)^{-1/2}$ by estimating the bound of the adjoint transformation S_λ^* on $\widehat{N}$ to H_0. To this end, let $f^0 \in H_0$ and $\widehat{y} \in \widehat{N}_0$ so that $\widehat{y} = \hat{\gamma}y^0$ for some $y^0 \in \mathfrak{D}(L')$. Then $\widehat{Q}(S_\lambda f^0, \widehat{y}) = Q(\gamma R(\lambda, L')f^0, \gamma y^0)$ and hence

$$\widehat{Q}(S_\lambda f^0, \widehat{y}) = \langle R(\lambda; L')f^0, L'y^0 \rangle + \langle L'R(\lambda; L')f^0, y^0 \rangle - \langle DR(\lambda; L')f^0, y^0 \rangle.$$

As the relation (2.3.14) shows $R_{10}(\lambda; L') \equiv I_{01}^{-1}R(\lambda; L')$ is a bounded linear transformation on H_0 to H_1. Hence we can rewrite the above equation as

$$(2.3.15) \qquad \widehat{Q}(S_\lambda f^0, \widehat{y}) = \langle R_{10}(\lambda; L')f^0, (\lambda I_{21} - M_{21}^1)I_{01}^{-1}y^0 \rangle - \langle f^0, y^0 \rangle;$$

here we have again made use of the identity $(\lambda I - L')R(\lambda; L') = I$. Next set $M' = (L')^*$. Then $R^*(\lambda; L') = R(\lambda; M')$ so that if $z^0 \in \mathfrak{D}(M')$ then

$$\langle f^0, z^0 \rangle = \langle R(\lambda; L')f^0, (\lambda I - M')z^0 \rangle = \langle R_{10}(\lambda; L')f^0, (\lambda I_{21} - M_{21}^1)I_{01}^{-1}z^0 \rangle.$$

It is seen from the proof of Theorem 1.5.1 that M' is engendered from a transformation M'_{21} whose graph is a maximal negative subspace of H_{12}; in fact L' is similarly engendered from L'_{21} and $U[\mathfrak{G}(M'_{21})]$ is the Q-orthogonal complement to $\mathfrak{G}(L'_{21})$ and hence maximal positive. Moreover $\mathfrak{G}(I_{20}M'I_{01})$ is dense in $\mathfrak{G}(M'_{21})$ and it follows from this and the above displayed relation that

$$\langle f^0, I_{01}z^1 \rangle = \langle R_{10}(\lambda; L')f^0, (\lambda I_{21} - M_{21}^1)z^1 \rangle$$

for all $z^1 \in \mathfrak{D}(M'_{21})$. On the other hand, the $\Re(\lambda I_{21} - M'_{21}) = H_2$ by Lemma 1.3.4. Hence there is a $z^1 \in \mathfrak{D}(M'_{21})$ such that

$$(\lambda I_{21} - M^1_{21}) I^{-1}_{01} \overset{0}{y} = (\lambda I_{21} - M^1_{21}) \overset{1}{z}.$$

Combining this with (2.3.15) we obtain

$$(2.3.16) \qquad \langle S_\lambda f^0, \widehat{y} \rangle = - \widehat{Q}(S_\lambda f^0, \widehat{y}) = - \langle f^0, u^0 \rangle,$$

where $u^0 = I_{01} z^1 - y^0$. Setting $u^1 = I^{-1}_{01} u^0$ we see that $M^1_{21} u^1 = \lambda I_{21} u^1 \in I_{20}[H_0]$, from which it follows that $u^0 \in \mathfrak{D}(M_1)$ and hence that $M_1 u^0 = \lambda u^0$. Finally (2.3.16) shows that $S^*_\lambda \widehat{y} = - u^0$.

Next we set $z^0 = I_{01} z^1$, $y = \gamma y^0$, $z = \gamma z^0$, and $u = \gamma u^0$. Then $y \in \beta[\mathfrak{G}(L'_{21})]$ and $z \in \beta[U[\mathfrak{G}(M'_{21})]]$. Consequently y and z are Q-orthogonal and

$$(2.3.17) \qquad \begin{aligned} Q(u, u) &= Q(y, y) + Q(z, z), \\ Q(y, y) &\leqq 0 \leqq Q(z, z). \end{aligned}$$

Further

$$\begin{aligned}
(2.3.18) \qquad -Q(u, u) &= \langle M^1_{21} \overset{1}{u}, \overset{1}{u} \rangle + \langle \overset{1}{u}, M^1_{21} \overset{1}{u} \rangle - \langle D_{21} \overset{1}{u}, \overset{1}{u} \rangle \\
&= 2\lambda \langle I_{21} \overset{1}{u}, \overset{1}{u} \rangle - \langle D_{21} \overset{1}{u}, \overset{1}{u} \rangle \geqq 2\lambda \langle \overset{0}{u}, \overset{0}{u} \rangle \geqq 0.
\end{aligned}$$

Combining (2.3.17) and (2.3.18) we obtain

$$2\lambda \langle u^0, u^0 \rangle \leqq - Q(u, u) \leqq - Q(y, y) = - \widehat{Q}(\widehat{y}, \widehat{y}) = \langle \widehat{y}, \widehat{y} \rangle$$

which shows that $2\lambda \| S^*_\lambda \widehat{y} \|^2 \leqq \| \widehat{y} \|^2$ for all $\widehat{y} \in \widehat{N}_o$. Since $\widehat{N}_o$ is dense in $\widehat{N}$, this gives the required estimate on $\| S^*_\lambda \|$ and it follows that the same estimate holds for $\| S_\lambda \| = \| S^*_\lambda \|$. As a consequence $\| \widehat{T} S_\lambda \| < 1$ for $\lambda > \| \widehat{T} \|^2/2$ and it is easy to prove from this that $R(\lambda; L)$ exists for such values of λ (see, for instance, E. Hille and R. S. Phillips [7, Theorem 5.10.4]). In fact,

$$(2.3.19) \qquad R(\lambda; L) = R(\lambda; L') \sum_{n=0}^{\infty} [\widehat{T} S_\lambda]^n, \qquad \lambda > \| \widehat{T} \|^2/2,$$

and for such λ, $\Re(\lambda I - L) = \mathfrak{D}[R(\lambda; L)] = H_0$. This concludes the proof of Theorem 2.3.2.

COROLLARY. $\Re[S^*_\lambda] \subset I_{01}[\mathfrak{D}(M^1_{21})]$.

Proof. In the course of the proof of the above theorem, it was shown for each $\widehat{y} \in \widehat{N}_o$ that $u^0 = S^*_\lambda \widehat{y} \in \mathfrak{D}(M_1)$ and that $M_1 u^0 = \lambda u^0$. Combining (2.3.17) and (2.3.18), we see that

$$0 \leqq - \langle D u^0, u^0 \rangle \leqq - Q(u, u) \leqq - Q(y, y) = \langle \widehat{y}, \widehat{y} \rangle.$$

Thus if $\{\widehat{y}_n\} \subset \widehat{N}_o$ converges to $\widehat{y} \in \widehat{N}$, then $u_n^0 \equiv S_\lambda^* \widehat{y}_n \to u^0 \equiv S_\lambda^* \widehat{y}$ in H_0 and, by the above inequality,

$$\langle D(u_n^0 - u_m^0), (u_n^0 - u_m^0) \rangle \to 0$$

as n, $m \to \infty$. Setting $u_n^1 = I_{01}^{-1} u_n^0$, it follows that $\{u_n^1\}$ converges in H_1, say to u^1, and that

$$[u_n^1, M_{21}^1 u_n^1] = [u_n^1, \lambda I_{21} u_n^1] \to [u^1, \lambda I_{21} u^1] \text{ in } H_{12}.$$

It is clear that $u^0 = I_{01} u^1$ and, since M_{21}^1 is closed, we see that $u^1 \in \mathfrak{D}(M_{21}^1)$.

We next characterize the adjoint M of the maximal dissipative operator L defined as in Theorem 2.3.2. According to Theorem 1.1.2, M is again a maximal dissipative operator with dense domain. Further, $R(\lambda; M) = R^*(\lambda; L)$ and this together with (2.3.19) implies

$$(2.3.20) \quad R(\lambda; M) = R(\lambda; M') + S_\lambda^* \widehat{T}^* \left[\sum_{n=0}^{\infty} [S_\lambda^* \widehat{T}^*]^n \right] R(\lambda; M'), \quad \lambda > \|\widehat{T}\|^2/2,$$

where M' is the adjoint of L', defined as in the proof of Theorem 2.3.2. Since $M_0 \subset M' \subset M_1$, it follows from the corollary to Theorem 2.3.2 that

$$(2.3.21) \qquad \mathfrak{D}(M) = \mathfrak{R}[R(\lambda; M)] \subset I_{01}[\mathfrak{D}(M_{21}^1)].$$

On the other hand, $L_0 \subset L$ so that for $y^0 \in \mathfrak{D}(L_0)$, $z^0 \in \mathfrak{D}(M)$, $y^1 = I_{01}^{-1} y^0$, and $z^1 = I_{01}^{-1} z^0$ (well defined by (2.3.21)), we have

$$\langle I_{20} M z^0, y^1 \rangle = \langle M z^0, y^0 \rangle = \langle z^0, L_0 y^0 \rangle = \langle z^1, L_{21}^0 y^1 \rangle.$$

Since $\mathfrak{G}(I_{20} L_0 I_{01})$ is dense in $\mathfrak{G}(L_{21}^0)$, we conclude that $M_{21}^1 z^1 = I_{20} M z^0$ and hence that $M \subset M_1$.

Comparing the relations (2.3.2) and (2.3.7) we see that

$$(2.3.22) \qquad\qquad T = \widehat{T} \widehat{\gamma} \gamma^{-1}.$$

Now for $y \in N$ we have

$$\langle \widehat{\gamma} \gamma^{-1} y, \widehat{\gamma} \gamma^{-1} y \rangle = - \widehat{Q}(\widehat{\gamma} \gamma^{-1} y, \widehat{\gamma} \gamma^{-1} y) = - Q(y, y) \leqq 2\langle y, y \rangle.$$

It follows that $\widehat{\gamma} \gamma^{-1}$ is a bounded linear transformation on N to $\widehat{N}$ and hence that T is bounded on N to H_0. Since $N \subset H$, the adjoint transformation T^* can be thought of as a bounded transformation on H_0 to H, and for $y^0 \in \mathfrak{D}(L)$ and $z^0 \in H_0$

$$\langle T \gamma y^0, z^0 \rangle = \langle \gamma y^0, T^* z^0 \rangle = Q(\gamma y^0, W T^* z^0)$$

by (1.6.4). Finally for $y^0 \in \mathfrak{D}(L)$ and $z^0 \in \mathfrak{D}(M_1)$

$$\langle Ly^0, z^0 \rangle - \langle y^0, M_1 z^0 \rangle = Q(\gamma y^0, \gamma z^0) + \langle T\gamma y^0, z^0 \rangle$$
$$= Q(\gamma y^0, \gamma z^0 + WT^* z^0),$$

and this vanishes for all $y^0 \in \mathfrak{D}(L)$ if and only if $\gamma z^0 + WT^* z^0$ belongs to the Q-orthogonal complement of N. We have therefore proved

THEOREM 2.3.3. *Let L be defined as in Theorem 2.3.2 and set $M = L^*$. Then M is a maximal dissipative restriction of M_1 with dense domain and*

$$(2.3.23) \qquad \mathfrak{D}(M) = [z^0; z^0 \in \mathfrak{D}(M_1) \text{ and } \gamma z^0 + WT^* z^0 \in P],$$

where P is the Q-orthogonal complement of N.

The relation (2.3.23) shows that the presence of T permits the rate at which energy enters the system through the boundary, namely $Q(U\gamma z^0, U\gamma z^0) = -Q(\gamma z^0, \gamma z^0)$, to be positive. However the inequality (2.3.10) together with the condition $\gamma z^0 + WT^* z^0 \in P$ require that the rate at which energy is dissipated in the interior of the system be at least as large as the flow in through the boundary.

2.4. **An example.** It will be instructive to carry through the previous development for the case of the telegraphist's equation

$$(2.4.1) \qquad\qquad u_{tt} + u_t - u_{xx} = 0, \qquad\qquad a < x < b, t > 0.$$

This can be brought into the form of a system by setting

$$(2.4.2) \qquad\qquad \begin{aligned} \eta^1 &= u_x + u_t, \\ \eta^2 &= u_x - u_t, \end{aligned}$$

in which case

$$(2.4.3) \qquad\qquad \begin{aligned} \overset{1}{\eta_t} &= \overset{1}{\eta_x} - 2^{-1}(\overset{1}{\eta} - \overset{2}{\eta}), \\ \overset{2}{\eta_t} &= -\overset{2}{\eta_x} + 2^{-1}(\overset{1}{\eta} - \overset{2}{\eta}). \end{aligned}$$

In our usual notation with $y^0 = [\eta^1, \eta^2]$,

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

this system is of the form (1.2). Since $I < F = I - D < 3I$, we see that the spaces H_0, H_1, and H_2 are essentially the same and in the following we shall not distinguish between them. The boundary integral takes the form

$$(2.4.4) \qquad\qquad (Ay^0, y^0)\Big|_a^b = \{|\eta^1(x)|^2 - |\eta^2(x)|^2\}\Big|_a^b.$$

It can be shown (see [12]) that

$$\mathfrak{D}(L_1) = [\overset{0}{y}; \overset{0}{y}{}'(\chi) \text{ absolutely continuous, } \overset{0}{y} \text{ and } \overset{0}{y}_x \in L_2[(a, b); I]],$$

(2.4.5)

$$\mathfrak{D}(L_0) = [\overset{0}{y}; \overset{0}{y}{}' \in \mathfrak{D}(L_1), \overset{0}{y}(a) = \theta = \overset{0}{y}(b)].$$

Moreover if $y^0 \in \mathfrak{D}(L_1)$ and $a = -\infty$ $[b = \infty]$, then of necessity

$$y^0(a) = \theta[y^0(b) = \theta].$$

Thus for $(a, b) = (-\infty, \infty)$, $\mathfrak{D}(L_0) = \mathfrak{D}(L_1)$, the boundary space H consists of only the zero element, and $L = L_0 = L_1$ is the only maximal dissipative extension of L_0 as well as the only maximal dissipative restriction of L_1.

When $(a, b) = (0, \infty)$ the problem becomes more interesting. The boundary space H is now two dimensional and $y = \beta[y^0, L_1 y^0]$ can be represented componentwise by $[\eta^1(0), \eta^2(0)]$, in which case

(2.4.6) $Q(y, z) = \eta^2(0)[\zeta^2(0)]^- - \eta^1(0)[\zeta^1(0)]^-.$

It is readily seen that each maximum negative subspace N of H is determined by an α of absolute value ≤ 1 and is of the form

(2.4.7) $N = [[\eta^1(0), \eta^2(0)]; \eta^2(0) = \alpha \eta^1(0)].$

The hilbert space $\widehat{N}$ is one dimensional unless $|\alpha| = 1$, in which case it consists of only the zero element. Representing $\hat{\gamma} y^0$ by $\eta^1(0)$ we see that

(2.4.8) $\widehat{Q}(\hat{\gamma} y^0, \hat{\gamma} z^0) = (|\alpha|^2 - 1)\eta^1(0)[\zeta^1(0)]^-.$

It is clear that the most general form of $\hat{T}$ is given by

$$\hat{T}\hat{\gamma} y^0 = c\eta^1(0) \quad \text{where} \quad c = \begin{pmatrix} c_1(\chi) \\ c_2(\chi) \end{pmatrix}$$

and $c_i \in L_2(0, \infty)$. Condition (2.3.10) becomes

(2.4.9) $(1 - |\alpha|^2)^{-1}|\langle c, y^0 \rangle|^2 \leq -\langle Dy^0, y^0 \rangle,$ $y^0 \in H_0.$

In order to see what restriction this imposes on c we diagonalize D, say by the rotation U, and obtain

$$U^* DU = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{where} \quad U = (u_{jk}) = 2^{-1/2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Setting $w^0 = U^* y^0$, the relation (2.4.9) takes the form

(2.4.10) $|\langle d_1, \omega^1 \rangle + \langle d_2, \omega^2 \rangle|^2 \leq 2(1 - |\alpha|^2)\langle \omega^2, \omega^2 \rangle$

where $d_k = \sum_j \bar{u}_{jk} c_j$, $k = 1, 2$. It follows that (2.3.10) is equivalent with $d_1 = 0$ and $\langle d_2, d_2 \rangle \leq 2(1 - |\alpha|^2)$ and hence with

(2.4.11) $c_1 = -c_2$ and $\langle c_1, c_1 \rangle \leq 1 - |\alpha|^2.$

According to Theorem 2.3.2, the operator

$$(2.4.12) \qquad \begin{aligned} Ly^0 &= L_1 y^0 + c\eta^1(0), \\ \mathfrak{D}(L) &= [y^0;\, y^0 \in D(L_1),\, \eta^2(0) = \alpha\eta^1(0)] \end{aligned}$$

where $c = [c_1,\, -c_1]$ and $\langle c_1, c_1\rangle \leq 1 - |\alpha|^2 \geq 0$, defines a maximal dissipative extension of L_0. The adjoint operator $M = L^*$ is described in Theorem 2.3.3. Thus M is a restriction of

$$M_1 z^0 = -(Az^0)_x + B^* z^0,\ \mathfrak{D}(M_1) = \mathfrak{D}(L_1),$$

in fact the domain of M is readily found from the relation

$$\begin{aligned} \langle Ly^0, z^0\rangle - \langle y^0, M_1 z^0\rangle &= \eta^2(0)[\zeta^2(0)]^- - \eta^1(0)[\zeta^1(0)]^- + \langle c, z^0\rangle\eta^1(0) \\ &= \eta^1(0)[\alpha[\zeta^2(0)]^- - [\zeta^1(0)]^- + \langle c, z^0\rangle]; \end{aligned}$$

and this vanishes for all $y^0 \in \mathfrak{D}(L)$ (and hence for all $\eta^1(0)$) if and only if

$$(2.4.13) \qquad \langle z^0, c\rangle \equiv \langle(\zeta^1 - \zeta^2), c_1\rangle = \zeta^1(0) - \bar\alpha\zeta^2(0).$$

The global lateral Condition (2.4.13) serves to define $\mathfrak{D}(M) \subset \mathfrak{D}(M_1)$ and assuming $\langle c_1, c_1\rangle \leq 1 - |\alpha|^2 \geq 0$, M is a maximal dissipative restriction of M_1 by Theorem 2.3.3.

The case $-\infty < a < b < \infty$ is slightly more complex. Here the boundary space $\boldsymbol{H}$ is four dimensional and $\boldsymbol{y} = \beta[y^0, L_1 y^0]$ can be represented componentwise by $[\eta^1(b), \eta^2(b), \eta^1(a), \eta^2(a)]$, in which case

$$(2.4.14)\ \ Q(y,z) = \eta^1(b)[\zeta^1(b)]^- - \eta^2(b)[\zeta^2(b)]^- - \eta^1(a)[\zeta^1(a)]^- + \eta^2(a)[\zeta^2(a)]^-.$$

Each maximal negative subspace $\boldsymbol{N}$ of $\boldsymbol{H}$ is now two dimensional and can be characterized by four constants $\delta, \epsilon, \mu, \nu$ as

$$(2.4.15) \qquad \begin{aligned} \eta^1(b) &= \delta\eta^2(b) + \epsilon\eta^1(a), \\ \eta^2(a) &= \mu\eta^2(b) + \nu\eta^1(a), \end{aligned}$$

provided

$$(2.4.16)\ \ |\delta\eta^2(b) + \epsilon\eta^1(a)|^2 + |\mu\eta^2(b) + \nu\eta^1(a)|^2 - |\eta^2(b)|^2 - |\eta^1(a)|^2$$

is a negative quadratic form in $\eta^2(b)$, $\eta^1(a)$. Necessary and sufficient conditions for this are

$$(2.4.17) \qquad \begin{aligned} |\delta|^2 + |\mu|^2 &\leq 1, \\ |\delta|^2 + |\epsilon|^2 + |\mu|^2 + |\nu|^2 &\leq 1 + |\delta\nu - \epsilon\mu|^2. \end{aligned}$$

Replacing the $\leq$ signs by $<$ signs, the Conditions (2.4.17) become necessary and sufficient for the form to be negative definite. It is convenient at this point to diagonalize the form (2.4.16) by a rotation of coordinates: $[\eta_i^1, \eta_i^2] = V[\eta^2(b), \eta^1(a)]$, in terms of which we have for $\gamma y^0, \gamma z^0 \in \boldsymbol{N}$

$$(2.4.18) \qquad Q(\gamma y^0, \gamma z^0) = -[q_1 \eta_i^1[\zeta_i^1]^- + q_2 \eta_i^2[\zeta_i^2]^-].$$

If $\boldsymbol{N}$ is (maximal) negative, then $q_j \geq 0$, $j = 1, 2$. The hilbert space $\widehat{\boldsymbol{N}}$ will now

be two, one, or zero dimensional according as the q_j's are both positive, only one is positive, or both are zero. If both q_j's are positive we may write $\hat{\gamma}y^0 = \gamma y^0$ and define

$$(2.4.19) \qquad \hat{T}\hat{\gamma}y^0 = C\begin{pmatrix} \eta_i^1 \\ \eta_i^2 \end{pmatrix}, \quad \text{where} \quad C = (c_{kj}(x)).$$

Actually this formalism creates no difficulty even if say $q_j=0$ provided we then set $c_{kj}=0$ for $k=1, 2$. Condition (2.3.10) now becomes

$$(2.4.20) \qquad \sum_{j=1}^2 q_j^{-1}\left|\sum_{k=1}^2 \langle \eta^k, c_{kj}\rangle\right|^2 \leqq -\langle Dy^0, y^0\rangle.$$

Again, diagonalizing D with $w^0 = U^*y^0$, the Condition (2.4.20) becomes

$$(2.4.21) \qquad \sum_{j=1}^2 q_j^{-1}\left|\sum_{k,l=1}^2 \langle u_{kl}\omega^l, c_{kj}\rangle\right|^2 \leqq 2\langle\omega^2, \omega^2\rangle.$$

For $\omega^2=0$ we see that this implies $\sum_k u_{k1}c_{kj} \equiv 2^{-1/2}(c_{1j}+c_{2j}) = 0$, providing that $q_j\neq 0$. Hence setting $c_1 = -c_{11}$, $c_2 = c_{22}$, (2.4.21) becomes

$$q_1^{-1}|\langle\omega^2, c_1\rangle|^2 + q_2^{-1}|\langle\omega^2, c_2\rangle|^2 \leqq \langle\omega^2, \omega^2\rangle.$$

If, say, $q_1=0\neq q_2$, then as above $c_1=0$ and the Condition (2.3.10) becomes

$$(2.4.22) \qquad \langle c_2, c_2\rangle \leqq q_2.$$

On the other hand if both q_i's are positive, then Condition (2.3.10) is equivalent with

$$(2.4.23) \qquad d_{11} + d_{22} + [(d_{11} + d_{22})^2 - 4d]^{1/2} \leqq 2,$$

where $d_{ij} = (q_iq_j)^{-1/2}\langle c_i, c_j\rangle$ and $d = d_{11}d_{22} - |d_{12}|^2$. Assuming c_1 and c_2 to be so restricted, assuming N to be a maximal negative subspace defined as above, and assuming $\hat{T}$ to be given by (2.4.19), then the operator L defined as in Theorem 2.3.2 is a maximal dissipative extension of L_0.

The adjoint operator $M = L^*$ is again maximal dissipative and a restriction of M_1. Moreover $\mathfrak{D}(M)$ is delimited by global lateral conditions and again these conditions are readily obtained from the relation

$$\begin{aligned}\langle Ly^0, z^0\rangle - \langle y^0, M_1z^0\rangle = {}&(\delta\eta^2(b) + \epsilon\eta^1(a))[\zeta^1(b)]^- - \eta^2(b)[\zeta^2(b)]^- \\ &- \eta^1(a)[\zeta^1(a)]^- + (\mu\eta^2(b) + \nu\eta^1(a))[\zeta^2(a)]^- \\ &+ \eta^2(b)[\langle c_{11}', \zeta^1\rangle + \langle c_{21}', \zeta^2\rangle] \\ &+ \eta^1(a)[\langle c_{12}', \zeta^1\rangle + \langle c_{22}', \zeta^2\rangle],\end{aligned}$$

where $(c_{ij}') = CV$. The left member vanishes for all $y^0\in\mathfrak{D}(L)$ (and hence for all $[\eta^2(b), \eta^1(a)]$) if and only if

$$(2.4.24) \quad \begin{aligned} \langle \zeta^1, c_{11}' \rangle + \langle \zeta^2, c_{21}' \rangle &= \zeta^2(b) - \bar{\delta}\zeta^1(b) - \bar{\mu}\zeta^2(a), \\ \langle \zeta^1, c_{12}' \rangle + \langle \zeta^2, c_{22}' \rangle &= \zeta^1(a) - \bar{\epsilon}\zeta^1(b) - \bar{\nu}\zeta^2(a). \end{aligned}$$

These are the required lateral conditions determining $\mathfrak{D}(M) \subset \mathfrak{D}(M_1)$. We recall that $c_j = 0$ if $q_j = 0$ and hence when both q_j's vanish, then $C = \Theta$ and the left members of (2.4.24) are each zero so that the lateral conditions become boundary conditions; in this case both L and M are properly dissipative operators.

<h2 style="text-align:center">REFERENCES</h2>

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