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On *Gibbs*'s Phenomenon.

By *Maxime Bôcher* in Cambridge (Mass.).

In a letter to the English journal *Nature* in 1899 (vol. 59, p. 606) *J. Willard Gibbs* made a remarkable observation concerning the way in which the *Fourier*'s series*)

(1.) $$\Psi(x) = \sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \cdots$$

behaves near the points of discontinuity of $\Psi(x)$. His statement, while remarkably lucid and precise, so far as it goes, is extreemly brief, touches on only the central aspects of the question, and contains nothing in the way of proof. So far as I know, this observation of *Gibbs* remained absolutely unnoticed except by some of his immediate colleagues at New Haven, one of whom, Professor *E. B. Wilson*, called it to my attention a few years later. Its high degree of interest was at once apparent to me, and I saw, what had doubtless been obvious to others, that it applied to the *Fourier*'s development of a large class of other discontinuous functions. Consequently, in a monograph on *Fourier*'s series which I wrote soon after**) I devoted a section to a detailed discussion of the peculiar conditions in question, to which I gave the name of *Gibbs*'s phenomenon. This exposition and development brought the matter to the attention of other mathematicians, and is, I believe, the direct or indirect source from which all further work on the subject flowed***).

In accordance with the didactic purpose of the article, I did not

*) To be exact, it is the function $\Psi(\pi - x)$ which *Gibbs* considers.

**) Annals of Mathematics, 2nd series, vol. 7 (1906) p. 81—152; also printed separately by Harvard University.

***) Reference should also be made to *Runge*'s *Theorie und Praxis der Reihen*, pp. 77—80 where a certain other special *Fourier*'s series is discussed in a similar way, but where there is no indication that the phenomenon is one of frequent occurrence and of importance, — the whole matter appears in the light of an illustrative example. This book was published five years after *Gibbs*'s remark and almost two years before my article, but it did not become known to me until my article was in type.

think it desirable to formulate any very broad conditions for the class of functions whose *Fourier*'s developments exhibit this phenomenon*); and it seemed to me the less necessary to do so because any mature mathematician on reading my treatment could not fail to observe its degree of generality. The fact is simply that the results established for the series (1.) may be extended at once to the *Fourier*'s development of the function $\frac{D}{\pi}\Psi(x-\alpha)$ which has at the point $x=\alpha$ a jump of magnitude D. If, by subtracting this function from a function $f(x)$ which also has a jump of magnitude D at α, we get a function whose *Fourier*'s development converges uniformly throughout the neighborhood of α, it is clear that all the facts apply, without essential change, to the *Fourier*'s development of $f(x)$. This is, in substance, the method I used, and from it we may infer that *Gibbs's phenomenon is valid for every real function, $f(x)$, with a discontinuity at α, provided it is possible by adding to $f(x)$ on one side or the other of α a suitable constant to form another function whose Fourier's development converges uniformly throughout the neighborhood of α.*

Corresponding to every class of functions for which we can assert the uniform convergence of the *Fourier*'s series in the neighborhood of α, we thus have a class of discontinuities for which we can assert the validity of *Gibbs*'s phenomenon. The case I mentioned explicitly is one such case; a case recently mentioned by *Gronwall***), namely that in which

*) I did not, however, assume that the derivative of the function satisfies *Dirichlet*'s conditions as is erroneously stated by *Gronwall* in an article presently to be cited. Indeed the class of functions with which I was explicitly concerned do not even themselves necessarily satisfy *Dirichlet*'s conditions.

**) Über die *Gibbs*'sche Erscheinung, Math. Ann. vol. 72, (1912) p. 228. *Gronwall*'s reference to my article (foot-note p. 234) is such as to make it appear that his method of treatment is different from and more general than mine. This is not the case; the two methods have the same degree of generality and are not essentially different. The main difference is that in treating series (1.) I considered the points

$$\frac{\pi}{n+\frac{1}{2}},\ \frac{2\pi}{n+\frac{1}{2}},\ \frac{3\pi}{n+\frac{1}{2}},\cdots$$

where the remainder of the series is a maximum or a minimum, while *Gronwall* considers the points

$$\frac{\pi}{n+1},\ \frac{2\pi}{n},\ \frac{3\pi}{n+1},\ \frac{4\pi}{n},\cdots$$

where the sum of the first n terms is a maximum or a minimum. It is at once

$f(x)$ satisfies *Dirichlet*'s conditions, is another. Neither of these cases includes the other, nor can either be regarded as more than a very special illustrative example.

If the conditions just stated are satisfied, $f(x)$ approaches definite limits, $f(\alpha+0)$ and $f(\alpha-0)$, as x approaches α from above or below. The difference

$$D = f(\alpha+0) - f(\alpha-0)$$

we call the magnitude of the jump at α. I now reproduce verbatim the statement of *Gibbs*'s phenomenon from my article already cited (p. **131**) in which the peculiarities are brought out not only with much greater generality, but also with much greater detail than had been done by *Gibbs*.

If $S_n(x)$ denotes the sum of the first $n+1$ terms of the Fourier's expansion of $f(x)$, the curve $y = S_n(x)$ will, for large values of n, pass in almost a vertical direction through a point whose abscissa is α and whose ordinate is almost equal to $\frac{1}{2}[f(\alpha+0)+f(\alpha-0)]$. The curve then rises and falls abruptly on the two sides of this point to the neighborhood of the curve $y = f(x)$, and oscillates about this curve lying alternately above and below it. The highest (or lowest) point of the k^{th} waves to the right and left of α will, for large values of n, lie approximately at the points

$$\alpha \pm \frac{2k\pi}{2n+1}$$

and the height of these waves will be approximately $\frac{DP_k}{\pi}$.

Here (cf. formula (54.))

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clear, when we consider the shapes of the arches of the approximation curves, that for large values of n the difference between the ordinates of the curve at these points is extremely small, and that this difference is quite immaterial to the work in hand. There can at most be a question of a minute simplification of the formal work. In making this comparison it should be further noticed that *Gronwall* establishes only a part of *Gibbs*'s phenomenon.

Gronwall does, it is true, give various results for the series (1.) which I had not given, but some of which had heen published several months earlier by *Dunham Jackson*, Rendiconti di Palermo, vol. 32, (1911) p. 257. These results are of such a character as to be essentially restricted to the special series (1.) and to be without importance for *Gibbs*'s phenomenon in general.

$$P_k = \frac{\pi}{2} - \int_0^{k\pi} \frac{\sin x}{x} dx.$$

Ample determination and discussion of the numerical values of these constants is also given; for instance the fact that the height of the first wave is approximately 9% of the magnitude of the jump.

Within the last year articles have appeared which contain, along with some other material, methods and results concerning *Gibbs*'s phenomenon which differ so slightly from those of my paper, or are such obvious corollaries of it, that I cannot let them pass without mention. For instance, a part of the theorem last stated, when written as a formula, would read:

$$\lim_{n=\infty} S_n\left(\alpha \pm \frac{2k\pi}{2n+1}\right) = f(\alpha \pm 0) \mp \frac{DP_k}{\pi}.$$

By subtracting these two formulas from one another, we find

$$(2.)\quad D = \frac{\pi}{\pi - 2P_k} \lim_{n=\infty} \left[S_n\left(\alpha + \frac{2k\pi}{2n+1}\right) - S_n\left(\alpha - \frac{2k\pi}{2n+1}\right)\right],$$

k being any fixed positive integer. In a recent number of this journal*) *Fejér* enunciates and solves by various methods the problem: To express the magnitude of the jump D in terms of the *Fourier*'s series. The formula just given solves this problem by such an immediate and obvious step that this answer to *Fejér*'s question may be regarded as merely a corollary of my statement of *Gibbs*'s phenomenon.

The formula used by *Fejér* for answering his question is a little different from that just given but it also flows with the greatest ease from a formula established in my paper. This I will now show, deducing at once a more general formula which includes *Fejér*'s result, and formula (2.) above, as special cases.

The remainder of series (1.) after the n^{th} term is written in my paper (formula (51.) page 125) in the form

$$(3.)\qquad R_n(x) = \frac{\pi}{2} - \int_0^{(n+\frac{1}{2})x} \frac{\sin x}{x} dx + I_n(x),$$

and it is proved (page 126) that $I_n(x)$ approaches zero uniformly when $0 < x < b$, $b < 2\pi$ as n becomes infinite. Now let us write

*) Bd. 142 (1913), S. 165.

$$F(x) = \int_x^\infty \frac{\sin x}{x}\, dx,$$

and select a set of positive constants $q_1, q_2, \ldots$ such that $\lim_{n=\infty} q_n = \infty$, and

$$(4.) \qquad \lim_{n=\infty} \left(\frac{n}{q_n}\right) = c,$$

where c is some positive constant or $+\infty$. We then have

$$\lim_{n=\infty} F\left(\frac{n+\frac{1}{2}}{q_n}\right) = F(c).$$

Consequently, since (3.) may be written

$$R_n(x) = F((n+\tfrac{1}{2})\,x) + I_n(x),$$

we find

$$\lim_{n=\infty} R_n\left(\frac{1}{q_n}\right) = F(c).$$

If now, we use the method of my paper as indicated above for passing from the special series (1.) to the *Fourier*'s development of a function $f(x)$ with a jump of magnitude D at α and satisfying the condition above stated for *Gibbs*'s phenomenon, we infer at once, $S_n(x)$ denoting the sum of the first $n+1$ terms of this development,

$$\lim_{n=\infty} S_n\left(\alpha \pm \frac{1}{q_n}\right) = f(\alpha \pm 0) \mp \frac{D}{\pi} F(c).$$

Subtracting these equations from one another, we find

$$(5.) \qquad D = \frac{\pi}{\pi - 2F(c)} \lim_{n=\infty} \left[S_n\left(\alpha + \frac{1}{q_n}\right) - S_n\left(\alpha - \frac{1}{q_n}\right)\right].$$

This is the general formula referred to above. I mention the following special cases:

(a.) $q_n = \dfrac{2n+1}{2k\pi}$, k a positive integer. Here $c = k\pi$ and (5.) reduces to (2.).

(b.) $q_n = \dfrac{n}{k\pi}$, k a positive integer. Here again $c = k\pi$ and (5.) reduces to a form very similar to (2.) but corresponding more nearly to *Gronwall*'s manner of approaching *Gibbs*'s phenomenon. Cf. the foot-note on page 42.

(c.) $q_n = n^p$, $0 < p < 1$. Here $c = +\infty$, $F(c) = 0$, and (5.) takes the very simple form

$$D = \lim_{n=\infty} [S_n(\alpha + n^{-p}) - S_n(\alpha - n^{-p})].$$

(d.) $q_n = \frac{n}{g}$, g any root of $F(x) = 0$. Here $c = g$, and (5.) reduces to *Fejér*'s formula (21.).

If we write out the sums $S_n\left(\alpha \pm \frac{1}{q_n}\right)$ and apply the addition theorems for the sine and cosine, as was done by *Fejér* in the special case last mentioned, we find at once

$$(6.) \qquad D = \frac{2\pi}{\pi - 2F(c)} \lim_{n=\infty} \sum_{\nu=1}^{n} (b_\nu \cos \nu\alpha - a_\nu \sin \nu\alpha) \sin \frac{\nu}{q_n},$$

where we suppose the *Fourier*'s development of $f(x)$ to have been

$$f(x) = \sum_{\nu=0}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

Corresponding to the four cases (a.)—(d.) just mentioned, we have four special cases of (6.), the last of which is the formula given by *Fejér* at the bottom of page 183.

In the last section of his paper, *Fejér* obtains an additional piece of information, not contained in my statement of *Gibbs*'s phenomenon, as to the nature of the convergence of a *Fourier*'s series near a finite jump*). It is true that, for this purpose, the function must be subjected to additional restrictions. The following method of looking at this question is perhaps not without interest:

In the case of the series (1.), the derivative of the sum of the first n terms, when $x = 0$, is obviously equal to n. Consequently, the derivative of the sum of the first n terms of the *Fourier*'s development of $\frac{D}{\pi} \Psi(x - \alpha)$ has, when $x = \alpha$, the value $\frac{nD}{\pi}$. If the function $f(x)$ has a finite jump of magnitude D at α, we consider the *Fourier*'s development of

$$(7.) \qquad f(x) - \frac{D}{\pi} \Psi(x - \alpha).$$

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*) *Fejér* regards this result as giving another expression for the magnitude of the jump. To me it seems of greater interest to regard it as giving an asymptotic expression for the steepness of the approximation curves at the point where the jump occurs.

I pass over without discussion other parts of *Fejér*'s paper which have no close relation to my work on *Gibbs*'s phenomenon. To avoid misunderstanding, I add explicitly that I have not, in what precedes, referred in any way to the *methods* used by *Fejér*, but merely to his *results*.

If the derivative of the sum of its first $n+1$ terms, at the point $x=\alpha$, remains finite as n becomes infinite, or at any rate always remains small enough so that when divided by n it approaches zero as its limit, it is clear that

$$\lim_{n=\infty}\left(\frac{S_n'(\alpha)}{n}\right)=\frac{D}{\pi} \qquad (8.)$$

where S_n' denotes the derivative of the sum of the first $n+1$ terms of the *Fourier*'s development of $f(x)$. It is this relation (8.) which constitutes *Fejér*'s result. The condition just enunciated for its validity is obviously both a necessary and a sufficient one. Another sufficient condition, more serviceable for some purposes, may be obtained by means of the following observation:

If S_n denote the sum of the first $n+1$ terms of any series, and σ_n the arithmetic mean of $S_0, S_1, \dots S_n$, then

$$\frac{S_n}{n}=\frac{n+1}{n}\,\sigma_n-\sigma_{n-1}.$$

Consequently, if σ_n approaches a finite limit as n becomes infinite, $\lim\limits_{n=\infty}\frac{S_n}{n}=0$. Hence

*If the series obtained by differentiating term by term the Fourier's development of the function**) (7.) *is summable, when* $x=\alpha$, *by the method of the first arithmetic mean, then Fejér's result,* (8.), *is valid.*

Various conditions on the function $f(x)$ may readily be specified which are sufficient to secure this summability.

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*) (7.) may obviously be replaced here by any function obtained by subtracting from $f(x)$ a function with period 2π which, in any finite interval, consists of a finite number of a analytic pieces and which has a finite jump of magnitude D at α. The analytic pieces just mentioned are supposed to be analytic at the ends of their intervals of definition as well as within these intervals.