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Spectral Properties of the Laplace Operator with respect to Electric and Magnetic Boundary Conditions

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1. INTRODUCTION

Let Ω be the exterior of a finite collection of disjoint bodies with smooth boundaries $S_1, ..., S_n$ and set $\partial\Omega = S_1 + \cdots + S_n$. In two preceding papers ([15, 16]) we have considered selfadjoint extensions A and A' of the vector Laplacian $-\Delta$ in Ω with respect to electric boundary conditions

$$n \times E = 0, \qquad \nabla \cdot E = 0 \qquad \text{on} \quad \partial\Omega \tag{1.1}$$

and magnetic boundary conditions

$$n \times (\nabla \times H) = 0, \qquad n \cdot H = 0 \qquad \text{on} \quad \partial\Omega, \tag{1.2}$$

respectively, where n denotes the exterior normal unit vector on $\partial\Omega$. The definition of the operators A and A' is given in [15, Sect. 3] and briefly recalled in [16, Sect. 1]. We have shown in [15] that A and A' are positive and selfadjoint. By using the functional calculus for unbounded, selfadjoint operators, we obtained in [15, Sect. 7] weak solutions E and H of the initial and boundary value problems for the vector wave equation in Ω with respect to the boundary conditions (1.1) and (1.2). In [16] we studied regularity properties of A and A'. In particular, we have shown that E and H are classical solutions if the initial data and the boundary $\partial\Omega$ are sufficiently smooth. As we have pointed out in [15, Sect. 2], our analysis includes the initial and boundary value problem of perfect reflection for electromagnetic wave fields in the case that Ω is filled by an isotropic, homogeneous medium.

The main object of this paper is the investigation of the spectra of the operators A and A'. We start with the discussion of the null spaces $N(A)$ and $N(A')$ of A and A' and show in Sections 2 and 3 that the dimensions of $N(A)$ and $N(A')$ are n and $p = p_1 + \cdots + p_n$, respectively, where n is the number of reflecting bodies and p_j denotes the topological genus of the boundary S_j of the jth reflector. It turns out that the eigenelements belonging to the eigen-

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value $\lambda=0$ are harmonic vector fields satisfying the boundary condition $n \times E=0$ in the case of the operator A and $n \cdot H=0$ in the case of A'.

In Sections 4 and 5 we discuss the spectral families $\{P_\lambda\}$ and $\{P'_\lambda\}$ of A and A', by using the well-known relationship between the spectral family and the resolvent R_z of a selfadjoint operator. If F is a sufficiently smooth vector field with bounded support, then $R_z F=(A-zI)^{-1} F$ can be identified for $z \notin [0, \infty)$ with the $\mathbf{L}_2$-solution $E=E_\kappa[F]$ of the boundary value problem

$$\begin{gathered} \Delta E+\kappa^2 E=-F \quad \text{in} \quad \Omega, \\ n \times E=0, \qquad \nabla \cdot E=0 \quad \text{on} \quad \partial\Omega, \end{gathered} \tag{1.3}$$

where $\kappa^2=z$ and $\operatorname{Im} \kappa>0$. This boundary value problem has been studied in [11], by employing integral equation methods. In particular, we have shown in [11] that the principle of limiting absorption holds in the following form: There exists an open subset B_0 of the complex κ-plane, containing the set $\{\kappa: \operatorname{Im} \kappa \geqslant 0,\ \kappa \neq 0\}$, such that $E_\kappa[F](x)$, as a function of κ, can be analytically extended onto B_0 for every $x \in \Omega$. Furthermore, the extended function $E=E_\kappa[F]$ satisfies, for real $\kappa \neq 0$, Eq. (1.3) and the radiation condition

$$E=O(1/r), \qquad \left(\frac{\partial}{\partial r}-i\kappa\right) E=o(1/r) \qquad \text{as} \quad r=|x| \to \infty. \tag{1.4}$$

In agreement with the physical interpretation of the radiation condition, a solution E of (1.3) with real $\kappa \neq 0$ is called outgoing (incoming) if (1.4) holds with $\kappa>0$ ($\kappa<0$). By using the principle of limiting absorption, we derive in Section 4 the formula

$$(P_\lambda F)(x)=(P_{+0} F)(x)+\frac{1}{2\pi i} \int_0^\lambda \left(E_{\sqrt{\sigma}}[F](x)-E_{-\sqrt{\sigma}}[F](x)\right) d\sigma \tag{1.5}$$

for $\lambda>0$ and $x \in \bar{\Omega}$ under suitable smoothness assumptions on F and $\partial\Omega$ (compare Theorem 4.1). Note that P_{+0} is the projection of $\mathbf{L}_2(\Omega)$ onto $N(A)$. The integrand in (1.5) may be discontinuous at $\sigma=0$, but we shall show that the (improper) integral in (1.5) converges uniformly in every bounded subset of $\bar{\Omega}$. In particular, we obtain $P_\lambda F \in C(\bar{\Omega})$. The formula (1.5) relates the spectral family $\{P_\lambda\}$ of A to the outgoing and incoming solutions of (1.3) with real κ. Furthermore, our analysis yields: $\lambda=0$ is the only eigenvalue of A, and the continuous spectrum of A consists of the half axis $[0, \infty)$.

Similar results can be derived for the operator A'. As we shall show in Section 5, the methods developed in [11] can also be applied to the magnetic boundary value problem

$$\begin{gathered} \Delta H+\kappa^2 H=-F \quad \text{in} \quad \Omega, \\ n \times (\nabla \times H)=0, \qquad n \cdot H=0 \quad \text{on} \quad \partial\Omega, \end{gathered} \tag{1.6}$$

so that we obtain an analogous formula relating the spectral family $\{P'_\lambda\}$ of A' to the outgoing and incoming solutions of (1.6) for real κ.
In Sections 6–8 we use the results of Sections 4 and 5 to derive generalized eigenfunction expansions with respect to the boundary value problems (1.3) and (1.6). The kernels of the generalized Fourier integrals are solutions of (1.3) and (1.6), respectively, with $F=0$ and real κ, which behave at infinity like plane waves $ae^{ip\cdot x}$ with $a, p \in R^3$ and $|p|^2 = \kappa^2$ (*distorted plane waves*). Results of this type have been obtained for the Schrödinger equation in R^3 by Ikebe [2] and for the scalar wave equation in exterior domains with smooth boundaries by Shenk [6] and, under more general assumptions on the boundary, by Wilcox [18]. First results on generalized eigenfunction expansions in the vector case are due to Grieb [1]. In addition to [1], the unitary character of the expansions will be established in Section 8, by employing a method developed by Wilcox [18] in the scalar case. The generalized Fourier transforms, studied in Sections 6–8, can be used to derive orthogonal decompositions of the Hilbert space $\mathbf{L}_2(\Omega)$ into closed subspaces, consisting of irrotational or solenoidal vector fields, respectively, as we shall show in Section 9. Section 10 contains a proof of a regularity statement which is used several times in the preceding sections.

The results of this paper can be applied to the initial and boundary value problems for the vector wave equation studied in [15, 16] and allow a discussion of the behavior of the solutions as $t \to \infty$. In particular, it is possible to derive necessary and sufficient conditions for the validity of the principle of limiting amplitude. We shall discuss this problem and related applications to the time-dependent theory in a subsequent paper.

2. The Null Space of A

In the following we assume that $\partial\Omega \in C^6$. It follows from the classical theory of the exterior Dirichlet problem for the scalar Laplace equation that there exist uniquely determined functions $\varphi_1, \ldots, \varphi_n \in C^1(\bar{\Omega}) \cap C^\infty(\Omega)$ such that $\Delta\varphi_i = 0$ in Ω, $\varphi_i = \delta_{ij}$ on S_j ($\delta_{ij} :=$ Kronecker's symbol), and $D^p\varphi_i = O(r^{-|p|-1})$ as $r = |x| \to \infty$ for every differential operator $D^p = \partial_1^{p_1}\partial_2^{p_2}\partial_3^{p_3}$ of order $|p| = p_1 + p_2 + p_3$ (compare, for example, [9, Satz 4] and [9, Lemma 15]). The vector field

$$E_i = \nabla\varphi_i \qquad (i = 1, \ldots, n) \tag{2.1}$$

satisfies the equations $\nabla \cdot E_i = 0$, $\nabla \times E_i = 0$ and hence

$$\Delta E_i = \nabla(\nabla \cdot E_i) - \nabla \times (\nabla \times E_i) = 0$$

in Ω. Furthermore, E_i satisfies the boundary conditions (1.1) on $\partial\Omega$ and the asymptotic relation $D^p E_i = O(r^{-|p|-2})$ as $r = |x| \to \infty$. Lemma 2.1 shows that E belongs to $\mathbf{C}^2(\bar{\Omega})$.

LEMMA 2.1. *Assume that* $k \geqslant 2$, $\partial\Omega \in C^{k+4}$, $E \in \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^k(\Omega)$, $\nabla \times E \in \mathbf{C}(\bar{\Omega})$, $\nabla \cdot E \in C(\bar{\Omega})$, $n \times E = 0$ *and* $\nabla \cdot E = 0$ *on* $\partial\Omega$, *and* $F := -\Delta E - \lambda E \in \mathbf{H}_k(\Omega)$ *for a suitable complex number* λ. *Then we have* $E \in \mathbf{C}^k(\bar{\Omega})$.

A proof will be given in Section 10.

Recall the definition of the linear space $\mathbf{S}$ introduced in [15], Section 3:

$$\mathbf{S} := \{E \in \mathbf{C}^2(\bar{\Omega}) \colon n \times E = 0 \text{ and } \nabla \cdot E = 0 \text{ on } \partial\Omega;$$
$$E_i, \partial_i E, \partial_i \partial_k E = O(r^{-2}) \text{ for } i, k = 1, 2, 3 \text{ and } r = |x| \to \infty\}.$$

The Properties of E_i collected above imply that $E_i \in \mathbf{S} \subset D(A)$ and $AE_i = 0$ since $A = -\Delta$ on $D(A)$. Hence $E_1, ..., E_n$ belong to the null space $N(A)$ of A. The fields $E_1, ..., E_n$ are linearly independent. In fact, assume that $c_1 E_1 + \cdots + c_n E_n = 0$ and set $\varphi := c_1 \varphi_1 + \cdots + c_n \varphi_n$. Formula (2.1) implies $\Delta\varphi = 0$ and hence $\varphi = 0$, since $\varphi(x) \to 0$ as $|x| \to \infty$. Since $\varphi = c_i$ on S_i, we obtain $c_1 = \cdots = c_n = 0$. In order to verify that the fields $E_1, ..., E_n$ form a basis of $N(A)$, we consider an arbitrary element $E \in N(A)$. Since $E \in D(A)$, $\Delta E = 0$, and $\partial\Omega \in C^6$, we have $E \in \mathbf{C}^2(\bar{\Omega})$ by [14], Theorem 6.3 (with $F = 0$, $\lambda = 0$). Furthermore, E satisfies the boundary conditions (1.1) by [14, Theorem 7.1]. Now we use the following elementary fact on L_2-functionals:

LEMMA 2.2. *Assume that* $u \in C(\bar{\Omega}) \cap L_2(\Omega)$, *where* Ω *is the exterior of a finite collection of disjoint bodies with smooth boundaries. Let* F_u *denote the linear functional on* $C_0^\infty(\Omega)$, *generated by* u,

$$F_u \varphi := \int_\Omega u\varphi \, dx \qquad \text{for} \quad \varphi \in C_0^\infty(\Omega). \tag{2.2}$$

Then the improper integral $\int_\Omega |u|^2 \, dx$ *exists, and we have*

$$\|F_u\|^2 = \int_\Omega |u|^2 \, dx. \tag{2.3}$$

$\| \cdots \|$ *denoting the norm in* $L_2(\Omega)$.

Remark, As in [12], the elements of $L_2(\Omega)$ are interpreted as distributions, and every function $u \in C(\bar{\Omega})$ is identified with the functional F_u defined by (2.2). In particular, $u \in L_2(\Omega)$ means that F_u is bounded with respect to the norm $\|\varphi\| = [\int |\varphi|^2 \, dx]^{1/2}$ in $C_0^\infty(\Omega)$, and the norm of F_u is defined by

$$\|F_u\| := \sup\{|F_u \varphi| : \varphi \in C_0^\infty(\Omega), \|\varphi\| = 1\} \tag{2.4}$$

(compare [12, Sect. 2], in particular, Definition 2.1).

Proof of Lemma 2.2. Let F_R be the restriction of the functional F_u to $C_0^\infty(\Omega_R)$ with $\Omega_R := \{x \in \Omega : |x| < R\}$. By [12, Lemma 2.5] we have

$$\int_{\Omega_R} |u|^2 \, dx = \|F_R\|^2_{L_2(\Omega_R)} \leqslant \|F_u\|^2$$

for every $R > R_0 := \max\{|x| : x \in \partial\Omega\}$. By applying the monotone convergence theorem of elementary calculus, it follows that the improper integral $\int_\Omega |u|^2 \, dx$ exists and that

$$\int_\Omega |u|^2 \, dx \leqslant \|F_u\|^2.$$

The opposite inequality follows from (2.4), (2.2), and Schwarz's inequality.

Since $E \in D(A) \subset \mathbf{H}_1(\Omega)$ and $E \in \mathbf{C}^2(\bar{\Omega})$, Lemma 2.2 implies that

$$\int_\Omega |E|^2 \, dx < \infty \qquad \text{and} \qquad \int_\Omega |\partial_i E|^2 \, dx < \infty \qquad (i = 1, 2, 3), \tag{2.5}$$

and hence, by Schwarz's inequality,

$$\int_\Omega |E| \cdot [|\nabla \times E| + |\nabla \cdot E|] \, dx < \infty. \tag{2.6}$$

Set

$$f(r) = \int_{\Sigma_r} |E| \cdot [|\nabla \times E| + |\nabla \cdot E|] \, dS \tag{2.7}$$

for $r > R_0$ with $\Sigma_r := \{x \in R^3 : |x| = r\}$. By (2.6) we have

$$\int_R^\infty f(r) \, dr < \infty$$

for $R > R_0$. Hence, there exists a sequence $\{r_k\}$ such that $r_k \to \infty$ and $f(r_k) \to 0$. By using $\nabla \times (\nabla \times E) - \nabla(\nabla \cdot E) = -\Delta E = 0$ and the boundary conditions (1.1), we obtain for $r > R_0$

$$\int_{\Omega_r} [|\nabla \times E|^2 + |\nabla \cdot E|^2]\, dx$$

$$= \int_{\Omega_r} \nabla \cdot [-\bar{E} \times (\nabla \times E) + \bar{E} \nabla \cdot E]\, dx$$

$$= \int_{\Sigma_r} [-(n \times \bar{E}) \cdot (\nabla \times E) + (n \cdot \bar{E})\, \nabla \cdot E]\, dS \leqslant f(r).$$

Since $r_k \to \infty$ and $f(r_k) \to 0$, it follows that

$$\int_{\Omega} [|\nabla \times E|^2 + |\nabla \cdot E|^2]\, dx = 0$$

and hence

$$\nabla \times E = 0, \qquad \nabla \cdot E = 0 \qquad \text{in} \quad \Omega. \tag{2.8}$$

Now we want to show that

$$E = O(r^{-2}) \qquad \text{as} \quad r = |x| \to \infty. \tag{2.9}$$

The asymptotic relation (2.9) follows from (2.8) and the following representation theorem for harmonic vector fields in exterior domains:

LEMMA 2.3. *Let Ω be the exterior of a finite collection of disjoint bodies with smooth boundaries and assume that $E \in \mathrm{C}^2(\bar{\Omega}) \cap \mathrm{L}_2(\Omega)$ and that E satisfies the equations $\nabla \times E = 0$ and $\nabla \cdot E = 0$. Then we have for every $x \in \Omega$*

$$E(x) = -\frac{1}{4\pi} \int_{\partial\Omega} \left[n(y) \cdot E(y)\, \nabla_x \frac{1}{|x-y|} \right.$$
$$\left. + (n(y) \times E(y)) \times \nabla_x \frac{1}{|x-y|} \right] dS_y.$$

Proof. Consider positive numbers r, ρ such that the sphere $\sigma_\rho(x) := \{|y - x| = \rho\}$ is contained in $\Omega_r = \{y \in \Omega : |y| < r\}$ and denote the region between $\sigma_\rho(x)$ and $\partial\Omega_r$ by Ω_0. By applying the integral theorem of Gauss to fields $E_1, E_2 \in \mathrm{C}^2(\bar{\Omega}_0)$ with $\nabla(\nabla \cdot E_j) - \nabla \times (\nabla \times E_j) = \Delta E_j = 0$, we obtain

$$\int_{\partial\Omega_0} n \cdot [E_1 \nabla \cdot E_2 - E_2 \nabla \cdot E_1 + E_1 \times (\nabla \times E_2) - E_2 \times (\nabla \times E_1)]\, dS$$

$$= \int_{\Omega_0} \nabla \cdot [E_1 \nabla \cdot E_2 - E_2 \nabla \cdot E_1 + E_1 \times (\nabla \times E_2) - E_2 \times (\nabla \times E_1)]\, dx = 0.$$

Now set $E_1 := E$ and $E_2(y) := e_i/|x-y|$, where e_i denotes the ith unit vector. Since $\nabla \times E = 0$ and $\nabla \cdot E = 0$, we obtain as $\rho \to 0$

$$\int_{\Sigma_r - \partial\Omega} \left[(n(y) \cdot E(y)) \frac{\partial}{\partial y_i} \frac{1}{|x-y|} + (n(y) \times E(y)) \left(\nabla_y \frac{1}{|x-y|} \times e_i \right) \right] dS_y$$

$$= E(x) \cdot \lim_{\rho \to 0} \int_{\sigma_\rho(x)} \left[-n(y) \left(e_i \cdot \frac{y-x}{|y-x|^3} \right) + n(y) \times \left(\frac{y-x}{|y-x|^3} \times e_i \right) \right] dS_y$$

$$= E(x) \cdot \int_{|z|=1} [-z(e_i \cdot z) + z \times (z \times e_i)] \, dS = -4\pi E(x) \cdot e_i$$

and hence

$$E(x) = -\frac{1}{4\pi} \sum_{i=1}^{3} e_i \int_{\Sigma_r - \partial\Omega} \left((n(y) \cdot E(y)) \frac{\partial}{\partial y_i} \frac{1}{|x-y|} \right.$$

$$\left. + (n(y) \times E(y)) \cdot \left(\nabla_y \frac{1}{|x-y|} \times e_i \right) \right] dS_y$$

or, equivalently,

$$E(x) = \frac{1}{4\pi} \int_{\Sigma_r - \partial\Omega} \left[(n(y) \cdot E(y)) \nabla_x \frac{1}{|x-y|} \right.$$

$$\left. + (n(y) \times E(y)) \times \nabla_x \frac{1}{|x-y|} \right] dS_y. \tag{2.10}$$

Now we discuss the part of the right-hand side in (2.10) which is integrated over Σ_r. Schwarz's inequality implies that

$$\left| \int_{\Sigma_r} [\cdots] \, dS_y \right| \leqslant 2 \left[\int_{\Sigma_r} |E|^2 \, dy \right]^{1/2} \left[\int_{\Sigma_r} \frac{dS_y}{|x-y|^4} \right]^{1/2}. \tag{2.11}$$

Now assume that $r > 2|x|$. Since $|y| = r$, we have $|y-x| > r/2$ so that the last integral in (2.11) can be estimated by $4\pi r^2 (2/r)^4 = 64\pi r^{-2}$. This yields

$$\left| \int_{\Sigma_r} [\cdots] \, dS_y \right| \leqslant \frac{16\sqrt{\pi}}{r} \left[\int_{\Sigma_r} |E|^2 \, dy \right]^{1/2} \quad \text{for} \quad r > 2|x|. \tag{2.12}$$

Set

$$g(r) := \int_{|x|=r} |E|^2 \, dy.$$

Since

$$\int_R^\infty g(r) \, dr = \int_{|x|>R} |E|^2 \, dx < \infty$$

for $R > R_0$ by Lemma 2.2, there exists a sequence $\{r_k\}$ with $r_k \to \infty$ and $g(r_k) \to 0$ and hence, by (2.12),

$$\int_{\Sigma_{r_k}} [\cdots]\, dS_y \to 0 \qquad \text{as} \quad k \to \infty.$$

By inserting this estimate into (2.10), we obtain Lemma 2.3. After these preparations, the basis property of the fields $E_1, \ldots, E_n$ follows from

LEMMA 2.4. *Assume that* $E \in \mathbf{C}^1(\bar{\Omega})$, $\nabla \times E = 0$ *and* $\nabla \cdot E = 0$ *in* Ω, $n \times E = 0$ *on* $\partial\Omega$, *and* $E = O(r^{-2})$ *as* $r = |x| \to \infty$. *Then* E *is a linear combination of the fields* $E_1, \ldots, E_n$ *introduced in the beginning of this section.*

Proof. The proof of [11, Lemma 14] shows that the $n \times n$ matrix

$$(\eta_{ik}) := \left(\int_{S_i} \frac{\partial \varphi_k}{\partial n}\, dS\right) = \left(\int_{S_i} n \cdot E_k\, dS\right)$$

has rank n. Hence we can find real numbers $c_1, \ldots, c_n$ such that

$$\sum_{k=1}^{n} c_k \int_{S_i} n \cdot E_k\, dS = \int_{S_i} n \cdot E\, dS \qquad (i = 1, \ldots, n).$$

Set

$$E_0 := E - \sum_{k=1}^{n} c_k E_k. \tag{2.13}$$

We have $E_0 \in \mathbf{C}^2(\bar{\Omega})$, $\nabla \times E_0 = 0$ and $\nabla \cdot E_0 = 0$ in Ω, $n \times E_0 = 0$ on $\partial\Omega$, $E_0 = O(R^{-2})$ as $r = |x| \to \infty$, and

$$\int_{S_i} n \cdot E_0\, dS = 0 \qquad (i = 1, \ldots, n). \tag{2.14}$$

In order to show that E_0 vanishes in $\bar{\Omega}$, we consider a fixed point $x_0 \in \Omega$ and form the potential

$$\psi(x) := \int_{x_0}^{x} E_0 \cdot t\, ds + \alpha_0, \tag{2.15}$$

where x is an arbitrary point in $\bar{\Omega}$, C is a smooth curve connecting x_0 and x within Ω, and α_0 is a suitably chosen number. Since $\nabla \times E_0 = 0$ in Ω and $n \times E_0 = 0$ on $\partial\Omega$, it follows from the integral theorem of Stokes that the integral in (2.15) does not depend on the choice of C. Since $E_0 = O(r^{-2})$ as $r \to \infty$, α_0 can be chosen in such a way that $\psi = O(r^{-1})$ as $r \to \infty$. Note that $\nabla \psi = E_0$. Hence ψ satisfies the assumptions of [11, Lemma 14] with

$\alpha_1 = \cdots = \alpha_n = 0$ (because of (2.14)). Therefore the uniqueness part of [11, Lemma 14] implies $\psi = 0$ and hence $E_0 = \nabla\psi = 0$. Thus, by (2.13), E is a linear combination of $E_1, ..., E_n$.

We collect our results in

THEOREM 2.1. *Assume that Ω is the exterior of n disjoint bodies with boundaries $S_1, ..., S_n \in C^6$. Then the null space $N(A)$ of the operator A introduced in* [13, *Sect.* 3] *has the dimension n. E belongs to $N(A)$ if and only if $E \in \mathbf{C}^2(\bar{\Omega})$, $\nabla \times E = 0$ and $\nabla \cdot E = 0$ in Ω, $n \times E = 0$ on $\partial\Omega$, and $E = O(r^{-2})$ as $r = |x| \to \infty$. A basis of $N(A)$ is given by the fields $E_1, ..., E_n$ introduced in the beginning of this section.*

3. THE NULL SPACE OF A'

Recall that Ω is the exterior of a finite collection of disjoint bodies $B_1, ..., B_n$ with smooth boundaries. More precisely, we assume that $B_1, ..., B_n$ are bounded, open, connected subsets of R^3 with $\bar{B}_i \cap \bar{B}_k = \varnothing$ for $i \neq \kappa$ and that Ω is the complement of $\bar{B}_1 \cup \cdots \cup \bar{B}_n$ in R^3. Let p_i be the topological genus of $S_i := \partial B_i$ and set $p := p_1 + \cdots + p_n$. In particular, B_i has p_i handles. Each of the p handles $H_1, ..., H_p$ of the bodies $B_1, ..., B_n$ may possess nontrivial knots, and different handles may be intertwisted like olympic rings.

Consider p smooth closed curves $C_1, .., C_p$ in $\bar{\Omega}$ with the property that C_i runs around the ith handle H_i exactly once, without circulating around any other handle H_j. It follows from the integral theorem of Stokes that the curves $C_1, ..., C_p$ form a homology basis for $\bar{\Omega}$ in the following sense: For every closed curve C in $\bar{\Omega}$ there exist integers $a_1, ..., a_p$ such that

$$\int_C H \cdot t\, ds = \sum_{j=1}^{p} a_j \int_{C_j} H \cdot t\, ds \tag{3.1}$$

for every field $H \in \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^1(\Omega)$ with $\nabla \times H = 0$. Furthermore, consider p smooth closed curves $C_1^*, ..., C_p^*$ in $R^3 - \bar{\Omega} = B_1 \cup \cdots \cup B_n$ such that C_i^* runs along the ith handle H_i exactly once, without circulating along any other handle H_j. Choose the orientations of C_i and C_i^* such that C_i encircles C_i^* in a positive sense. According to the law of Biot–Savart, we consider the fields

$$H_i'(x) := \frac{1}{4\pi} \nabla \times \int_{C_i^*} t(y) \frac{1}{|x-y|}\, ds_y. \tag{3.2}$$

It is well known that

$$\nabla \times H_i' = 0 \quad \text{in } \bar{\Omega} \qquad \text{and} \qquad \int_{C_i} H_j' \cdot t\, ds = \delta_{ij} \qquad (i, j = 1, ..., p). \tag{3.3}$$

A proof of (3.3) is contained in [4, Sect. 1.2]. A different proof will be sketched at the end of this section.

It follows from the classical theory of the exterior Neumann problem for the scalar Laplace equation that there exist uniquely determined functions $\psi_1,\dots,\psi_p \in C^1(\bar{\Omega}) \cap C^\infty(\Omega)$ such that $\Delta\psi_i = 0$ in Ω, $(\partial/\partial n)\,\psi_i = -n \cdot H_i'$ on $\partial\Omega$, and $D^p\psi_i = O(r^{-|p|-1})$ as $r = |x| \to \infty$ for every differential operator D^p of order $|p|$ (compare, for example, [9], in particular the Corollary after Satz 3 and the remark [9, p. 50]). Set

$$H_i := H_i' + \nabla\psi_i \qquad (i = 1,\dots, p). \tag{3.4}$$

H_i satisfies the equations $\nabla \cdot H_i = 0$, $\nabla \times H_i = 0$ and hence $\Delta H_i = 0$ in Ω. Furthermore, H_i satisfies the boundary conditions (1.2) on $\partial\Omega$ and the asymptotic relation $D^p H_i = O(r^{-|p|-2})$ as $r = |x| \to \infty$ for every differential operator of order $|p|$. As in Section 2, we assume that $\partial\Omega \in C^6$. The following variant of Lemma 2.1 shows that H belongs to $\mathbf{C}^2(\bar{\Omega})$:

LEMMA 3.1. *Assume that* $k \geqslant 2$, $\partial\Omega \in C^{k+4}$, $H \in \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^k(\Omega)$, $\nabla \times H \in \mathbf{C}(\bar{\Omega})$, $\nabla \cdot H \in C(\bar{\Omega})$, $n \times (\nabla \times H) = 0$ *and* $n \cdot H = 0$ *on* $\partial\Omega$, *and* $F := -\Delta H - \lambda H \in \mathbf{H}_k(\Omega)$ *for a suitable complex number* λ. *Then we have* $H \in \mathbf{C}^k(\bar{\Omega})$.

The properties of H_i collected above imply that $H_i \in \mathbf{S}' \subset D(A')$ and $AH_i = -\Delta H_i = 0$. Hence $H_1,\dots, H_p$ belong to the null space $N(A')$ of A'. Note that (3.3) and (3.4) imply that

$$\int_{C_i} H_j \cdot t\, ds = \delta_{ij} \qquad (i, j = 1,\dots, p). \tag{3.5}$$

The fields $H_1,\dots, H_p$ are linearly independent, since it follows from $c_1 H_1 + \cdots + c_p H_p = 0$ and (3.5) by integration over C_i that $c_i = 0$ for $i = 1,\dots, p$.

In order to show that the fields $H_1,\dots, H_p$ form a basis of $N(A')$, we consider an arbitrary element $H \in N(A')$. Since $H \in D(A')$, $\Delta H = 0$, and $\partial\Omega \in C^6$, we have $H \in \mathbf{C}^2(\bar{\Omega})$ by [14, Theorem 6.3], and H satisfies boundary conditions (1.2) by [14, Theorem 7.1]. The same argument as in Section 2 implies that

$$\nabla \times H = 0, \qquad \nabla \cdot H = 0 \qquad \text{in} \quad \Omega \tag{3.6}$$

and

$$H = O(r^{-2}) \qquad \text{as} \quad r = |x| \to \infty. \tag{3.7}$$

Note that the argument leading to (2.8) remains valid if boundary conditions (1.1) are replaced by (1.2) and that Lemma 2.3 does not depend on the assigned boundary data. Set

$$H_0 := H - \sum_{j=1}^{p} a_j H_j \tag{3.8}$$

with

$$a_j := \int_{C_j} H \cdot t\, ds. \tag{3.9}$$

It follows from (3.5) that

$$\int_{C_i} H_0 \cdot t\, ds = 0 \qquad \text{for} \quad i = 1,..., p$$

and hence, by (3.1),

$$\int_{C} H_0 \cdot t\, ds = 0 \tag{3.10}$$

for every closed curve C in $\bar{\Omega}$. Thus the potential

$$\psi(x) := C \int_{x_0}^{x} H_0 \cdot t\, ds + \alpha_0 \qquad (x_0, x \in \bar{\Omega}) \tag{3.11}$$

has the same value for every smooth curve connecting x_0 and x within Ω. Since $H_0 = O(r^{-2})$ as $r \to \infty$ by (3.2) and (3.7), the number α_0 can be chosen such that $\psi = O(r^{-1})$ as $r \to \infty$. Note that $\psi \in C^1(\bar{\Omega})$, $\Delta\psi = 0$ in Ω, and $(\partial/\partial n)\,\psi = 0$ on $\partial\Omega$. Hence the uniqueness theorem for the exterior Neumann problem yields $H_0 = \nabla\psi = 0$ so that H is a linear combination of the fields $H_1,..., H_p$. Thus we obtain:

THEOREM 3.1. *Assume that Ω is the exterior of n disjoint bodies with boundaries $S_1,..., S_n \in C^6$. Then the null space $N(A')$ of the operator A' introduced in* [13, *Sect.* 3] *has the dimension $p = p_1 + \cdots + p_n$, where p_i denotes the topological genus of S_i. H belongs to $N(A')$ if and only if $H \in \mathbf{C}^2(\bar{\Omega})$, $\nabla \times H = 0$ and $\nabla \cdot H = 0$ in Ω, $n \cdot H = 0$ on $\partial\Omega$, and $H = O(r^{-2})$ as $r = |x| \to \infty$. A basis of $N(A')$ is given by the fields $H_1,..., H_p$ introduced in the beginning of this section.*

We conclude this section with some remarks on formula (3.3). Suppose first that C_i^* is the boundary of a piecewise smooth surface S_i^*. S_i^* can be chosen such that C_i intersects S_i^* in exactly one point x_i and $C_j \cap S_i^* = \varnothing$ for $j \neq i$. The integral theorem of Stokes implies for smooth functions f

$$\int_{C_i^*} ft \, ds = \int_{S_i^*} n \times \nabla f \, dS, \tag{3.12}$$

with suitably chosen orientation of S_i^*, since

$$\int_{C_i^*} ft \cdot e_j \, ds = \int_{S_i^*} n \cdot [\nabla \times (fe_j)] \, dS = \int_{S_i^*} (n \times \nabla f) \cdot e_j \, dS$$

for every unit vector e_j ($j = 1, 2, 3$). Hence (3.2) yields for $x \notin S_i^*$

$$\begin{aligned} H_i'(x) &= \frac{1}{4\pi} \nabla \times \int_{S_i^*} n(y) \times \nabla_y \frac{1}{|x-y|} \, dS_y \\ &= \frac{1}{4\pi} \nabla \times \left[\nabla \times \int_{S_i^*} n(y) \frac{1}{|x-y|} \, dS_y \right] \\ &= \frac{1}{4\pi} \nabla \left[\nabla \cdot \int_{S_i^*} n(y) \frac{1}{|x-y|} \, dS_y \right] = \nabla \varphi_i(x) \end{aligned}$$

with

$$\varphi_i(x) := -\frac{1}{4\pi} \int_{S_i^*} \frac{\partial}{\partial n_y} \frac{1}{|x-y|} \, dS_y. \tag{3.13}$$

Note that $\nabla \times H_i' = 0$ in $\bar{\Omega}$ and

$$\int_{C_j} H_i' \cdot t \, ds = 0 \qquad \text{for} \quad j \neq i$$

since $H_i' \in \mathbf{C}^1(\bar{\Omega})$, $H_i' = \nabla \varphi_i$ in $R^3 - S_i^*$, and $C_j \cap S_i^* = \varnothing$. The jump relation for the double potential (3.13) yields

$$\int_{C_i} H_i' \cdot t \, ds = \int_{C_i} \nabla \varphi_i' \cdot t \, ds = \varphi_i^-(x_i) - \varphi_i^+(x_i) = 1.$$

This concludes the proof of (3.3) in the special situation considered above. In order to extend the argument to curves C_i^* with nontrivial knot structure, we choose a finite number of closed curves $C_i^1, \ldots, C_i^m$ without knots such that $C_i^* = C_i^1 + \cdots + C_i^m$, by subdividing C_i^* and inserting auxiliary curves which are passed in both directions, and apply the above argument to $C_i^1, \ldots, C_i^m$. In particular, we choose piecewise smooth surfaces S_i^k with $\partial S_i^k = C_i^k$. The surfaces S_i^k can be choosen such that C_i^* intersects S_i^1 in exactly one point x_i, $C_i^* \cap S_i^k = \varnothing$ for $k > 1$, and $C_j^* \cap S_i^k = \varnothing$ for $j \neq i$. In the case of a

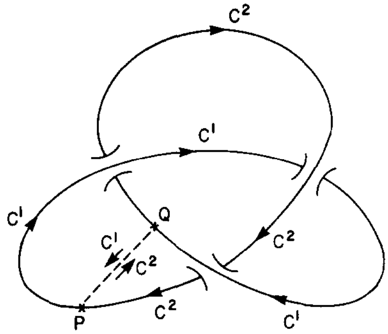

FIGURE 1

trefoil $C = C_1^*$, closed curves C^1 and C^2 with the required properties can be chosen as in Fig. 1, by using two subdivision points P and Q and one auxiliary curve which connects P and Q and is passed in both directions.

4. THE SPECTRAL FAMILY OF A

In this section we investigate the spectral family $\{P_\lambda\}$, $\lambda \geqslant 0$ of the selfadjoint, positive operator A. The basic properties of the projection operators P_λ have been collected in [15, p. 152–153]. Note that formula [15, Eq. (7.11)] holds also for piecewise continuous functions f (compare [8, Section 9]). Furthermore, we shall use the following elementary fact: For every $\lambda \geqslant 0$ there exists a projection operator $P_{\lambda+0}$ such that $P_\mu F \to P_{\lambda+0} F$ as $\mu \downarrow \lambda$ for every $F \in \mathbf{L}_2(\Omega)$, and $P_{\lambda+0} - P_\lambda$ is the orthogonal projection of $\mathbf{L}_2(\Omega)$ onto the null space of $A - \lambda I$. At first we show:

LEMMA 4.1. *Assume that* $\partial\Omega \in C^{2j+4}$. *Then we have* $P_\mu F \in \mathbf{C}^{2j}(\bar{\Omega})$ *for every* $F \in \mathbf{L}_2(\Omega)$ *and every* $\mu \geqslant 0$. *Furthermore,* $P_\mu F$ *satisfies boundary conditions* (1.1) *if* $j \geqslant 1$.

Proof. Note that

$$P_\mu F = \int_0^\mu d(P_\lambda F). \tag{4.1}$$

By using [15, (7.11)] k-times, we obtain

$$P_\mu F \in D(A^k), \qquad A^k P_\mu F = \int_0^\mu \lambda^k \, d(P_\lambda F) \tag{4.2}$$

for every positive integer k. In particular, we have $A^k(P_\mu F) \in D(A)$ for $k = 0, 1, \ldots, j+1$. By applying [16, Theorem 6.1] (with $G = A^j(P_\mu F)$, $k = 0$, $\lambda = 0$), it follows that

$$A^j(P_\mu F) \in \mathbf{H}_2(\Omega') \qquad \text{for every} \quad \Omega' \sqsubset \Omega \tag{4.3}$$

(compare the notation introduced in the beginning of [16, Sect. 6]). By a second application of [16, Theorem 6.1] (with $G = A^{j-1}(P_\mu F)$, $k = 2$, $\lambda = 0$), (4.3) implies that

$$A^{j-1}(P_\mu F) \in \mathbf{H}_4(\Omega') \qquad \text{for every} \quad \Omega' \sqsubset \Omega. \tag{4.4}$$

By repeating this argument $(j-2)$ times, we obtain $A(P_\mu F) \in \mathbf{H}_{2j}(\Omega')$ for every $\Omega' \sqsubset \Omega$ and hence, by [16, Theorem 6.3], $P_\mu F \in \mathbf{C}^{2j}(\bar{\Omega})$. The boundary conditions (1.1) follow from [16, Lemma 7.1].

In the following we assume as in the preceding sections that $\partial\Omega \in C^6$. The resolvent $R_z = (A - zI)^{-1}$ and the spectral family $\{P_\lambda\}$ of A are related by

$$R_z F = \int_0^\infty (\lambda - z)^{-1}\, d(P_\lambda F) \qquad \text{for} \quad F \in \mathbf{L}_2(\Omega) \text{ and } z \notin [0, \infty) \tag{4.5}$$

(compare [15, Eq. (7.11)]). It follows from (4.5), by using Plemelj's inversion formula for Cauchy–Stieltjes integrals ([7, Sect. 29]), that

$$\begin{aligned} &((P_\beta + P_{\beta+0}) F, G) - ((P_\alpha + P_{\alpha+0}) F, G) \\ &\quad = \lim_{\sigma \downarrow 0} \frac{1}{\pi i} \int_\alpha^\beta (R_{\lambda+i\sigma} F - R_{\lambda-i\sigma} F, G)\, d\lambda \qquad \text{for} \quad F, G \in \mathbf{L}_2(\Omega). \end{aligned} \tag{4.6}$$

This well-known formula is the basis of our further discussion of the spectral family $\{P_\lambda\}$.

In order to study the behavior of the resolvent R_z as $\operatorname{Im} z \to 0$, we consider the following boundary value problem for sufficiently smooth vector fields F with bounded support and for $\operatorname{Im} \kappa \geqslant 0$, $\kappa \neq 0$:

(A) *Find a vector field* $E \in \mathbf{C}^2(\bar{\Omega})$ *such that*

(i) $\Delta E + \kappa^2 E = -F$ *in* Ω,

(ii) $n \times E = 0$, $\nabla \cdot E = 0$ *on* $\partial\Omega$,

(iii) $E = O(r^{-1})$, $(\partial/\partial r - i\kappa) E = o(r^{-1})$ *as* $r = |x| \to \infty$.

We show:

LEMMA 4.2. *Assume that* F *has bounded support and belongs to* $\mathbf{C}^2(\bar{\Omega})$. *Then problem* (A) *has a uniquely determined solution for every* κ *with* $\operatorname{Im} \kappa \geqslant 0$ *and* $\kappa \neq 0$.

Proof. The uniqueness argument is contained in [11] (compare, in particular, the beginning of Section II and the first part of the proof of [11, Lemma 6]). Now set

$$T(x) := \frac{1}{4\pi} \int_{\Omega} F(y) \frac{e^{i\kappa|x-y|}}{|x-y|} \, dy \tag{4.7}$$

and

$$c := -n \times T, \qquad \gamma := -\nabla \cdot T \qquad \text{on} \quad \partial\Omega. \tag{4.8}$$

By the classical theory of volume potentials, we have $T \in \mathbf{C}^1(\bar{\Omega}) \cap \mathbf{C}^2(\Omega)$, $\Delta T + \kappa^2 T = -F$ in Ω, and ∇T satisfies a Hölder condition uniformly on $\partial\Omega$. In particular, c and γ satisfy the assumptions made in [11] (compare the formulation of problem (B) on [11, p. 355]). Hence [11, Theorem I] implies that there exists a field E' such that $E' \in \mathbf{C}^\infty(\Omega) \cap \mathbf{C}(\bar{\Omega})$, $\nabla \times E' \in \mathbf{C}(\bar{\Omega})$, $\nabla \cdot E' \in C(\bar{\Omega})$, $\Delta E' + \kappa^2 E' = 0$ in Ω, $n \times E' = c$ and $\nabla \cdot E' = \gamma$ on $\partial\Omega$, and $E' = O(r^{-1})$ and $(\partial/\partial r - i\kappa) E = o(r^{-1})$ as $r = |x| \to \infty$. The field

$$E := T + E' \tag{4.9}$$

satisfies the properties (i)–(iii) stated above. Furthermore, we have $E \in \mathbf{C}^2(\Omega) \cap \mathbf{C}(\bar{\Omega})$, $\nabla \times E \in \mathbf{C}(\bar{\Omega})$, $\nabla \cdot E \in C(\bar{\Omega})$ and $F \in \mathbf{C}^2(\bar{\Omega}) \subset \mathbf{H}_2(\Omega)$. Hence Lemma 2.1 yields $E \in \mathbf{C}^2(\bar{\Omega})$. This concludes the proof of Lemma 4.2.

It follows from (4.7) and the analysis in [11] (compare, in particular, the representation [11, Eq. (2.3)] for E') that E, $\partial_i E$ and $\partial_i \partial_\kappa E$ decay exponentially if $\operatorname{Im} \kappa > 0$ and $\operatorname{supp} F$ is bounded. This implies that $E \in \mathbf{S} \subset D(A)$ and $(A - \kappa^2) E = -(\Delta + \kappa^2) E = F$ in this case. Thus we obtain:

LEMMA 4.3. *Assume that F has bounded support and belongs to $C^2(\bar{\Omega})$. Let $E = E_\kappa[F]$ be the solution of problem* (A). *Then the resolvent R_z of A satisfies*

$$R_z F = E_\kappa[F] \qquad \textit{with} \quad \kappa^2 = z \quad \textit{and} \quad \operatorname{Im} \kappa > 0 \tag{4.10}$$

for every complex number $z \notin [0, \infty)$.

Note that T, c, and γ depend analytically on κ in the whole κ-plane and that the corresponding power series expansions converge uniformly on $\partial\Omega$. Hence the argument in [11, Sect. IV] shows that $E_\kappa[F]$ depends analytically on κ for $\operatorname{Im} \kappa \geqslant 0$ and $\kappa \neq 0$. More precisely, we obtain:

LEMMA 4.4. *Under the assumptions of Lemma* 4.3, *$E_\kappa[F](x)$ depends analytically on κ in $B_0 := \{\kappa \in C\colon \operatorname{Im} \kappa \geqslant 0, \kappa \neq 0\}$ for every $x \in \Omega$.*

Furthermore, for every $\kappa_0 \in B_0$ there exists a $\rho(\kappa_0) > 0$ such that the power series expansion of $E_\kappa[F](x)$ at κ_0 converges uniformly in $\{\kappa: |\kappa - \kappa_0| \leqslant \rho(\kappa_0)\} \times M$ for every compact subset M of Ω.

Remark. It is possible to prove the analyticity of $E_\kappa[F](x)$ also for $x \in \partial\Omega$ and to extend the second part of Lemma 4.4 to bounded subsets M of $\bar{\Omega}$, by using the methods developed in [10]. This situation requires additional considerations since the first term in [11, Formula (2.3)],

$$E_1(x) = \nabla \times \int_{\partial\Omega} a(y) \frac{e^{i\kappa|x-y|}}{|x-y|} dS_y \qquad (x \in \Omega), \tag{4.11}$$

cannot be continuously extended onto $\bar{\Omega}$ for arbitrary continuous tangential fields a. The argument in [10] suggests to replace the Banach space B_1 of continuous tangential fields in [11, Sect. III] by the Banach space B_1^α of Hölder-continuous tangential fields a with Hölder exponent α, $0 < \alpha < 1$, where the norm is defined by

$$\|a\|_\alpha := \operatorname{supr}_{x \in \partial\Omega} |a(x)| + \operatorname{supr}_{x, y \in \partial\Omega} \frac{|a(x) - a(y)|}{|x-y|^\alpha}. \tag{4.12}$$

It follows as in the proof of [10, Lemma 13] that the operator T introduced in [11, Sect. III] is completely continuous as operator from the Banach space $B_\alpha := B_1^\alpha \times B_2 \times B_3$ into itself. It can be shown that T, and hence $(I+T)^{-1}$, depends analytically on κ with respect to the operator norm for bounded operators acting from B_α into itself. Since also c depends analytically on κ with respect to the norm (4.12), it follows that $(a, b, \lambda) = (I+T)^{-1}(c, 0, \gamma)$ depends analytically on κ in B_α. In particular, a depends analytically on κ with respect to the norm (4.12). Now consider the region Ω_δ between $\partial\Omega$ and a sufficiently close (exterior) parallel surface $\partial\Omega_\delta = \{x = z + \delta n(z): z \in \partial\Omega\}$. Note that

$$\begin{aligned} E_1(x) = &\int_{\partial\Omega} [a(y) - a(z)] \times \nabla_y \frac{e^{i\kappa|x-y|}}{|x-y|} dS_y \\ &+ a(z) \times \int_{\partial\Omega} \nabla_y \frac{e^{i\kappa|x-y|}}{|x-y|} dS_y \qquad \text{for} \quad x = z + tn(z) \in \Omega_\delta. \end{aligned} \tag{4.13}$$

The last integral can be continuously extended onto $\bar{\Omega}$ (see [5, Lemma 70]). Since a depends analytically on κ with respect to the norm (4.12), the representation (4.13) shows that $E_1(x)$ depends analytically on κ in $\bar{\Omega}_\delta$, and hence in $\bar{\Omega}$. We omit a detailed presentation of the proofs and add the remark that the following weaker statement can be obtained by the same argument as the corollary to [10, Sect. 5, Satz 1] and by observing [10, Lemma 3]:

LEMMA 4.5. *Under the assumptions of Lemma* 4.3, $E_\kappa[F](x)$ *depends continuously on* (x, κ) *in* $\bar{\Omega} \times B_0$, *where* $B_0 = \{\kappa \in C: \operatorname{Im} \kappa \geqslant 0, \kappa \neq 0\}$.

After these preparations we return to the discussion of the spectral family $\{P_\lambda\}$ of A. We assume as above that $F \in \mathbf{C}^2(\bar{\Omega})$ and that the support of F is bounded. Suppose that $0 < \alpha < \beta < \infty$ and $G \in \mathbf{C}_0^\infty(\Omega)$. By (4.6) we have

$$((P_\beta + P_{\beta+0}) F, G) - ((P_\alpha + P_{\alpha+0}) F, G) = \lim_{\sigma \downarrow 0} \frac{1}{\pi i} \int_\alpha^\beta \left[\int_\Omega (R_{\lambda+i\sigma} F - R_{\lambda-i\sigma} F) \cdot \bar{G} \, dx \right] d\lambda. \tag{4.14}$$

Since the integrand depends continuously on (λ, σ, x) in the compact subset $[\alpha, \beta] \times [0, 1] \times \operatorname{supp} G$ of R^5 by Lemmas 4.3 and 4.4, the order of the integrations and the limit $\sigma \downarrow 0$ can be interchanged. Note that

$$R_{\lambda \pm i\sigma} F \to E_{\pm\sqrt{\lambda}}[F] \quad \text{as} \quad \sigma \downarrow 0 \tag{4.15}$$

by Lemmas 4.3 and 4.4. Thus we obtain for every $G \in \mathbf{C}_0^\infty(\Omega)$ and $0 < \alpha < \beta$

$$((P_\beta + P_{\beta+0}) F, G) - ((P_\alpha + P_{\alpha+0}) F, G) = \frac{1}{\pi i} \int_\Omega \left[\int_\alpha^\beta (E_{\sqrt{\lambda}} F - E_{-\sqrt{\lambda}}[F]) \, d\lambda \right] \cdot \bar{G} \, dx. \tag{4.16}$$

As a first consequence of (4.16), we obtain:

LEMMA 4.6. *The operator* A *has no positive eigenvalues.*

Proof. Assume that $\alpha > 0$. Formula (4.16) implies, since the integrand depends continuously on λ and x by Lemma 4.4, that

$$((P_\beta + P_{\beta+0}) F, G) \to ((P_\alpha + P_{\alpha+0}) F, G) \quad \text{as} \quad \beta \downarrow \alpha \tag{4.17}$$

for $F, G \in \mathbf{C}_0^\infty(\Omega)$. On the other hand, we have $P_\beta F \to P_{\alpha+0} F$ and hence $P_{\beta+0} F \to P_{\alpha+0} F$ as $\beta \downarrow \alpha$ so that the left-hand side of (4.17) converges to $2(P_{\alpha+0} F, G)$ as $\beta \downarrow \alpha$. Thus (4.17) implies $(P_{\alpha+0} F, G) = (P_\alpha F, G)$ for $F, G \in \mathbf{C}_0^\infty(\Omega)$. Since $\mathbf{C}_0^\infty(\Omega)$ is dense in $\mathbf{L}_2(\Omega)$, we obtain $P_{\alpha+0} F = P_\alpha F$ for every $F \in \mathbf{C}_0^\infty(\Omega)$, and hence for every $F \in \mathbf{L}_2(\Omega)$, since the projections P_α and $P_{\alpha+0}$ are bounded. Since $P_{\alpha+0} - P_\alpha$ is the orthogonal projection of $\mathbf{L}_2(\Omega)$ onto the null space $N(A - \alpha I)$ of $A - \alpha I$, we conclude that $N(A - \alpha I) = \{0\}$, so that α is not an eigenvalue of A.

Since $P_{\lambda+0} = P_\lambda$ for $\lambda > 0$, (4.16) can be simplified to

$$(P_\beta F - P_\alpha F, G) = \frac{1}{2\pi i} \int_\Omega \left[\int_\alpha^\beta (E_{\sqrt{\lambda}}[F] - E_{-\sqrt{\lambda}}[F]) \, d\lambda \right] \cdot \bar{G} \, dx \tag{4.18}$$

for every $G \in \mathbf{C}_0^\infty(\Omega)$ and $0 < \alpha < \beta$. Note that $P_\beta F - P_\alpha F$ and the expression $[\cdots]$ on the right-hand side of (4.18) are continuous in Ω by Lemmas 4.1 and 4.4, respectively. Hence (4.18) yields

$$(P_\beta F)(x) - (P_\alpha F)(x) = \frac{1}{2\pi i} \int_\alpha^\beta (E_{\sqrt{\lambda}}[F](x) - E_{-\sqrt{\lambda}}[F](x))\, d\lambda \quad (4.19)$$

for $x \in \Omega$ and $0 < \alpha < \beta$. By Lemma 4.5, (4.19) holds also for $x \in \bar{\Omega}$.

Now we assume in addition that $n \times F = 0$ and $\nabla \cdot F = 0$ on $\partial\Omega$. In particular, we have $F \in \mathbf{S} \subset D(A)$. Hence it follows immediately from the functional calculus for selfadjoint operators (compare [15, Eqs. (7.11)]) that

$$A(P_\alpha F) = P_\alpha(AF) = \int_0^\alpha \lambda \, d(P_\lambda F). \quad (4.20)$$

Recall that

$$\|P_\alpha G - P_{+0} G\| \to 0 \qquad \text{as} \quad \alpha \downarrow 0 \quad (4.21)$$

for every $G \in \mathbf{L}_2(\Omega)$. By setting $G = AF$, we conclude from (4.20) and (4.21) that

$$\|A(P_\alpha F) - P_{+0}(AF)\| \to 0 \qquad \text{as} \quad \alpha \downarrow 0. \quad (4.22)$$

Note that $AF \in N(A)^\perp$ since $(AF, G) = (F, AG) = 0$ for every $G \in N(A)$. Since P_{+0} is the projection of $\mathbf{L}_2(\Omega)$ onto $N(A)$, we obtain $P_{+0}(AF) = 0$ and hence, by (4.22),

$$\|A(P_\alpha F)\| \to 0 \qquad \text{as} \quad \alpha \downarrow 0. \quad (4.23)$$

Since $P_{+0} F \in N(A)$, (4.23) can be rewritten as

$$\|A(P_\alpha F - P_{+0} F)\| \to 0 \qquad \text{as} \quad \alpha \downarrow 0. \quad (4.24)$$

Now we apply [16, Theorem 6.2] (with $k = 0$) to the field $G := P_\alpha F - P_{+0} F \in D(A)$. It follows that, for every bounded subset M of $\bar{\Omega}$, there exists a $c > 0$ such that

$$|(P_\alpha F)(x) - (P_{+0} F)(x)| \leqslant c(\|P_\alpha F - P_{+0} F\| + \|A(P_\alpha F - P_{+0} F)\|) \quad (4.25)$$

for every $x \in M$. This inequality implies by (4.21) and (4.24) that $(P_\alpha F)(x)$ converges to $(P_{+0} F)(x)$ as $\alpha \downarrow 0$ uniformly in every bounded subset of $\bar{\Omega}$. Hence we obtain, by letting $\alpha \downarrow 0$ in (4.19),

$$(P_\beta F)(x) = (P_{+0} F)(x) + \frac{1}{2\pi i} \int_0^\beta (E_{\sqrt{\lambda}}[F](x) - E_{-\sqrt{\lambda}}[F](x))\, d\lambda \quad (4.26)$$

for every $x \in \bar{\Omega}$. Formula (4.26) relates the spectral family $\{P_\lambda\}$ of the operator A to the solutions $E_\kappa[F]$ of problem (A) for real $\kappa \neq 0$ and to the nullspace $N(A)$ of A characterized by Theorem 2.1. The integral in (4.26) may be improper since Lemmas 4.4 and 4.5 guarantee the continuity of the integrand only in the open interval $\lambda > 0$. The analysis above shows that the convergence of the improper integral in (4.26) is uniform with respect to x in every bounded subset of $\bar{\Omega}$.

In a similar way we can perform the limit $\beta \to \infty$. Note that

$$\| P_\beta G - G \| \to 0 \qquad \text{as} \quad \beta \to \infty \tag{4.27}$$

for every $G \in \mathbf{L}_2(\Omega)$ since $\{P_\lambda\}$ is the spectral family of a selfadjoint operator. By setting $G = AF$ in (4.27) and observing (4.20), we obtain also

$$\|A(P_\beta F - F)\| \to 0 \qquad \text{as} \quad \beta \to \infty \tag{4.28}$$

and hence, by using [14, Theorem 6.2],

$$P_\beta F(x) \to F(x) \qquad \text{as} \quad \beta \to \infty \tag{4.29}$$

uniformly in every bounded subset of $\bar{\Omega}$. By combining (4.26) and (4.29), we get the identity

$$F(x) = (P_{+0}F)(x) + \frac{1}{2\pi i} \int_0^\infty (E_{\sqrt{\lambda}}[F](x) - E_{-\sqrt{\lambda}}[F](x))\, d\lambda, \tag{4.30}$$

the improper integral converging uniformly in every bounded subset of $\bar{\Omega}$.

It follows from (4.26) that

$$\frac{d}{d\lambda}(P_\lambda F)(x) = \frac{1}{2\pi i}(E_{\sqrt{\lambda}}[F](x) - E_{-\sqrt{\lambda}}[F](x)) \tag{4.31}$$

for $\lambda > 0$ and $x \in \bar{\Omega}$. In particular, (4.31) and Lemma 4.4 show that $(P_\lambda F)(x)$, as a function of λ, has derivatives of arbitrary order for $\lambda > 0$ and $x \in \Omega$ under the assumptions on F made above.

The preceding results can be used to locate the spectrum of A. We show:

LEMMA 4.7. *The spectrum $\sigma(A)$ of A consists of the positive half-axis: $\sigma(A) = [0, \infty)$.*

Proof. Assume that $\mu > 0$ and $\mu \notin \sigma(A)$. We use the elementary fact that P_λ is constant in $(\lambda - \varepsilon, \mu + \varepsilon)$ for a suitable $\varepsilon > 0$ (compare, for example, [8, Bemerkung 19.13]). By (4.31) we have

$$E_{\sqrt{\lambda}}[F](x) - E_{-\sqrt{\lambda}}[F](x) = 0 \tag{4.32}$$

for $F \in \mathbf{C}_0^\infty(\Omega)$, $x \in \Omega$ and $\lambda \in (\mu - \varepsilon, \mu + \varepsilon)$. By Lemma 4.4, the left-hand side of (4.32) depends analytically on λ in $(0, \infty)$. Hence it follows by analytic continuation that (4.32) holds for every $\lambda > 0$. By inserting (4.32) into (4.30), we obtain $F = P_{+0}F \in N(A)$ for every $F \in \mathbf{C}_0^\infty(\Omega)$, in contradiction to the fact that $N(A)$ is finite-dimensional by Theorem 2.1.

The main results of this section are collected in the following theorem:

THEOREM 4.1. *Assume that Ω is the exterior of n disjoint bodies with boundaries $S_1, ..., S_n \in C^6$. Let A be the selfadjoint operator introduced in* [15, *Sect.* 3] *and denote the spectral family of A by $\{P_\lambda\}$. Then the following statements hold:*

(a) *$\lambda = 0$ is the only eigenvalue of A. The spectrum of A is given by $\sigma(A) = [0, \infty)$. Furthermore, we have $P_\lambda F \in \mathbf{C}^2(\bar{\Omega})$ for every $F \in \mathbf{L}_2(\Omega)$ and $\lambda \geqslant 0$, and $P_\lambda F$ satisfies the electric boundary conditions* (1.1) *on $\partial\Omega$.*

(b) *Assume, in addition, that $F \in \mathbf{C}^2(\bar{\Omega})$, $\operatorname{supp} F$ is bounded, and $n \times F = 0$ and $\nabla \cdot F = 0$ on $\partial\Omega$. Then $\{P_\lambda\}$ and the solutions $E_\kappa[F]$ of the exterior boundary value problem* (A) *are related by*

$$(P_\lambda F)(x) = (P_{+0}F)(x) + \frac{1}{2\pi i}\int_0^\lambda (E_{\sqrt{\sigma}}[F](x) - E_{-\sqrt{\sigma}}[F](x))\, d\sigma$$

for $\lambda > 0$ and $x \in \bar{\Omega}$, where P_{+0} denotes the projection of $\mathbf{L}_2(\Omega)$ onto the null space $N(A)$ of A characterized in Theorem 2.1. Furthermore, the identity

$$F(x) = (P_{+0}F)(x) + \frac{1}{2\pi i}\int_0^\infty (E_{\sqrt{\sigma}}[F](x) - E_{-\sqrt{\sigma}}[F](x))\, d\sigma$$

holds for $x \in \bar{\Omega}$. The improper integrals on the right-hand sides converge uniformly with respect to x in every bounded subset of $\bar{\Omega}$.

5. THE SPECTRAL FAMILY OF A'

The spectral family $\{P'_\lambda\}$ of A' can be discussed by the methods developed in Section 4. In contrast to problem (A), however, the related exterior boundary value problem (A′), which corresponds to the magnetic boundary conditions (1.2), has not yet been discussed in the literature from a similar point of view, so that some additional remarks seem to be appropriate. We begin with the formulation of problem (A′):

(A′) *Find a vector field $H \in \mathbf{C}^2(\bar{\Omega})$ such that*

(i) *$\Delta H + \kappa^2 H = -F$ in Ω,*

(ii) $n \times (\nabla \times H) = 0$, $n \cdot H = 0$ *on* $\partial\Omega$,

(iii) $H = O(r^{-1})$, $(\partial/\partial r - i\kappa) H = o(r^{-1})$ *as* $r = = |x| \to \infty$.

We show:

LEMMA 5.1. *Assume that* Ω *is the exterior of a finite collection of disjoint bodies with surfaces* $S_1, \ldots, S_n \in C^6$ *and that* F *has bounded support and belongs to* $\mathbf{C}^2(\bar{\Omega})$. *Then problem* (A′) *has a uniquely determined solution for every* κ *with* $\operatorname{Im} \kappa \geqslant 0$ *and* $\kappa \neq 0$.

Proof. The uniqueness follows as in [11]. Consider the volume potential T defined by (4.7) and set

$$c := -n \times (\nabla \times T), \qquad \gamma := -n \cdot T \qquad \text{on} \quad \partial\Omega. \tag{5.1}$$

We shall construct below a field H' with the following properties:

(a) $H' \in \mathbf{C}^\infty(\Omega) \cap \mathbf{C}(\bar{\Omega})$, $\nabla \times H' \in \mathbf{C}(\bar{\Omega})$, $\nabla \cdot H' \in C(\bar{\Omega})$;

(b) $\Delta H' + \kappa^2 H' = 0$ in Ω;

(c) $n \times (\nabla \times H') = c$, $n \cdot H' = \gamma$ on $\partial\Omega$;

(d) $H' = O(r^{-1})$, $(\partial/\partial r - i\kappa) H' = o(r^{-1})$ as $r = |x| \to \infty$.

It follows from Lemma 3.1, that $H := T + H'$ is a solution of (A′). Set $\Omega_i := R^3 - \bar{\Omega}$. We try to choose surface and volume layers a, b, λ such that the field

$$H'(x) = \int_{\partial\Omega} a(y)\, \Phi(x,y)\, dS_y - \frac{1}{2} \int_{\Omega_i} b(y)\, \Phi(x,y)\, dy$$

$$-\nabla \int_{\partial\Omega} \lambda(y)\, \Phi(x,y)\, dS_y, \qquad \Phi(x,y) := \frac{1}{2\pi} \frac{e^{i\kappa|x-y|}}{|x-y|} \tag{5.2}$$

has the required properties. The field a in (5.2) is supposed to be tangential on $\partial\Omega$. By the jump relations of potential theory (compare [11, Formulas (2.5)–(2.7)), boundary conditions (c) are equivalent to the integral equations

$$a(x) + \int_{\partial\Omega} n(x) \times [\nabla_x \Phi(x,y) \times a(y)]\, dS_y$$

$$-\tfrac{1}{2} \int_{\Omega_i} n(x) \times [\nabla_x \Phi(x,y) \times b(y)]\, dy = c(x) \qquad \text{for} \quad x \in S \tag{5.3}$$

and

$$\lambda(x) + \int_{\partial\Omega} n(x) \cdot a(y)\, \Phi(x,y)\, dS_y - \frac{1}{2} \int_{\Omega_i} n(x) \cdot b(y)\, \Phi(x,y)\, dy$$

$$-\int_{\partial\Omega} \lambda(y) \frac{\partial}{\partial n_x} \Phi(x,y)\, dS_y = \gamma(x) \qquad \text{for} \quad x \in S. \tag{5.4}$$

Now we choose a function $\varphi \in C^1(\bar{\Omega}_i)$ with $\varphi > 0$ in Ω_i and $\varphi = 0$ on $\partial\Omega$ and set $\tau = \tau(\kappa) = 1$ if $\operatorname{Re}\kappa \geqslant 0$ and $\tau(\kappa) = -1$ if $\operatorname{Re}\kappa < 0$. We try to determine the vector field b in (5.2) such that H' satisfies the equation

$$\Delta H' + (\kappa^2 + i\tau\varphi)\, H' = 0 \qquad \text{in} \quad \Omega_i. \tag{5.5}$$

By Poisson's formula for the second derivatives of volume potentials, (5.5) is equivalent to the integral equation

$$b(x) + i\tau\varphi(x) \left[\int_{\partial\Omega} a(y)\, \Phi(x, y)\, dS_y - \tfrac{1}{2} \int_{\Omega_i} b(y)\, \Phi(x, y)\, dy \right.$$
$$\left. - \int_{\partial\Omega} \lambda(y)\, \nabla_x \Phi(x, y)\, dS_y \right] = 0 \qquad \text{for} \quad x \in \Omega_i. \tag{5.6}$$

As in [11], we consider the Banach spaces B_1 of continuous tangential fields a on $\partial\Omega$, B_2 of continuous vector fields b in $\bar{\Omega}_i$ and B_3 of continuous functions λ on $\partial\Omega$, all equipped with the corresponding maximum norm, and set $B = B_1 \times B_2 \times B_3$, the norm in B being defined by

$$\|(a, b, \lambda)\|_B := \|a\|_{B_1} + \|b\|_{B_2} + \|\lambda\|_{B_3}. \tag{5.7}$$

The system of integral equations (5.3), (5.4), and (5.6) can be rewritten as operator equation

$$(a, b, \lambda) + K(a, b, \lambda) = (c, 0, \gamma), \tag{5.8}$$

where K is a 3×3 matrix of integral operators K^{ij} acting from B_j into B_i. It follows by standard arguments (compare the proofs of [11, Lemma 5; 10, Lemma 16]) that each operator K^{ij}, and hence K as an operator from B into itself, is completely continuous. In the case of the operator K^{23}, given by

$$(K^{23}\lambda)(x) := -i\tau\varphi(x) \int_{\partial\Omega} \lambda(y)\, \nabla_x \Phi(x, y)\, dS_y \qquad \text{for} \quad x \in \Omega_i, \tag{5.9}$$

the verification of the complete continuity is based on the fact that φ vanishes on $\partial\Omega$ so that an estimate of the form

$$|\varphi(x)\, \nabla_x \Phi(x, y)| \leqslant A/|x - y| \tag{5.10}$$

holds for every $x \in \bar{\Omega}_i$ and every $y \in \partial\Omega$. As the next step in the proof of Lemma 5.1, we verify:

LEMMA 5.2. *Let* $(a, b, \lambda) \in B$ *be a solution of* (5.8). *Then the field* H' *defined by* (5.2) *has the properties* (a)–(d). *Furthermore, we have* $H' \in \mathbf{C}(\bar{\Omega}_i)$, $\nabla \times H' \in \mathbf{C}(\bar{\Omega}_i)$, *and* $\nabla \cdot H' \in C(\bar{\Omega}_i)$.

Remark. H' is defined by (5.2) only in $\Omega \cup \Omega_i$, but not on $\partial\Omega$. The statement $H' \in \mathbf{C}(\bar{\Omega})$ means that H' can be continuously extended onto $\bar{\Omega}$. Note that $H' \in C(\bar{\Omega})$ and $H' \in C(\bar{\Omega}_i)$ do not imply that the continuous extensions of H' from the exterior Ω and the interior Ω_i coincide on $\partial\Omega$.

Proof of Lemma 5.2. Properties (b)–(d) are obvious. The field a satisfies a Hölder condition uniformly on $\partial\Omega$ by (5.1), (5.3), and [11, Lemma 1]. The same is true for λ by (5.1), (5.4), and [9, Lemma 5]. Hence [11, Lemma 3] implies that the first two terms in (5.2) have continuously differentiable extensions onto $\bar{\Omega}$ and $\bar{\Omega}_i$, while the third term,

$$H_3(x) = -\nabla \int_{\partial\Omega} \lambda(y)\, \Phi(x, y)\, dS_y \tag{5.11}$$

can be continuously extended onto $\bar{\Omega}$ and $\bar{\Omega}_i$. The same is true for $\nabla \times H_3$ and $\nabla \cdot H_3$ since $\nabla \times H_3 = 0$ and $\Delta\Phi(\cdot, y) = -\kappa^2\Phi(\cdot, y)$ in $\Omega \cup \Omega_i$. These remarks conclude the proof of Lemma 5.2.

In order to complete the proof of Lemma 5.1, we show

LEMMA 5.3. *The homogeneous equation* $(a, b, \lambda) + K(a, b, \lambda) = 0$ *has only the trivial solution* $(a, b, \lambda) = (0, 0, 0)$.

Proof. Consider a triple $(a, b, \lambda) \in B$ with $(I + K)(a, b, \lambda) = 0$. By Lemma 5.2, the field H' defined by (5.2) has the properties (a)–(d) with $c = 0$ and $\gamma = 0$. Hence the uniqueness part of Lemma 5.1 yields $H' = 0$ in Ω. Note that

$$\nabla \cdot H'(x) = \int_{\partial\Omega} a(y) \cdot \nabla_x \Phi(x, y)\, dS_y - \tfrac{1}{2} \int_{\Omega_i} b(y) \cdot \nabla_x \Phi(x, y)\, dy$$

$$+ \kappa^2 \int_{\partial\Omega} \lambda(y)\, \Phi(x, y)\, dS_y \qquad \text{in} \quad \Omega \cup \Omega_i. \tag{5.12}$$

The tangential field a and λ satisfy Hölder conditions uniformly on $\partial\Omega$ (compare the proof of Lemma 5.2). Hence it follows from (5.2), (5.12), and the jump relation for the gradient of a single potential (see, for example, [11, Lemma 3]) that

$$[n \times H']_i = [n \times H']_e \qquad \text{and} \qquad [\nabla \cdot H']_i = [\nabla \cdot H']_e \qquad \text{on} \quad \partial\Omega, \tag{5.13}$$

where $[\ ]_i$ and $[\ ]_e$ denote the limit values on $\partial\Omega$ taken from the interior and the exterior side, respectively. Thus we have $H' \in \mathbf{C}(\bar{\Omega}_i)$, $\nabla \times H' \in \mathbf{C}(\bar{\Omega}_i)$, $\nabla \cdot H' \in C(\bar{\Omega}_i)$, and $[n \times H']_i = 0$ and $[\nabla \cdot H']_i = 0$ on $\partial\Omega$. Furthermore, (5.6) yields $b \in C^1(\Omega_i)$, and hence $H' \in \mathbf{C}^2(\Omega_i)$ and $(\Delta + \kappa^2 + i\tau\varphi) H' = 0$ in Ω_i. Hence Green's formula implies that

$$\begin{aligned} 0 &= \int_{\partial\Omega} [(\overline{n \times H'}) \times (\nabla \times H') + (\overline{n \cdot H'})\nabla \cdot H']_i \, dS \\ &= \int_{\Omega_i} \nabla \cdot [\bar{H}' \times (\nabla \times H') + \overline{H'}(\nabla \cdot H')] \, dx \\ &= \int_{\Omega_i} (|\nabla \times H'|^2 + |\nabla \cdot H'|^2 + \overline{H'} \cdot \Delta H) \, dx \\ &= \int_{\Omega_i} [|\nabla \times H'|^2 + |\nabla \cdot H'|^2 - (\kappa^2 + i\tau\varphi) |H'|^2] \, dx. \end{aligned}$$

Here we use that the integrand of the last integral, and hence of all three volume integrals, is continuous in $\bar{\Omega}_i$. By taking the imaginary part, we obtain

$$\int_{\Omega_i} [\mathrm{Im}(\kappa^2) + \tau\varphi] \, |H'|^2 \, dx = 0. \tag{5.14}$$

Since $\mathrm{Im}(\kappa^2) + \tau\varphi > 0$ in Ω_i for $\mathrm{Im}\,\kappa \geqslant 0$ by our choice of φ and τ, (5.14) implies that H' vanishes also in Ω_i. Hence it follows from (5.2) by the jump relations and Poisson's formula for volume potentials that

$$\begin{aligned} a &= \tfrac{1}{2}([n \times (\nabla \times H')]_e - [n \times (\nabla \times H')]_i) = 0 \qquad \text{on} \quad \partial\Omega, \\ b &= (\Delta + \kappa^2) H' = 0 \qquad \text{in} \quad \Omega_i, \\ \lambda &= \tfrac{1}{2}([n \cdot H']_e - [n \cdot H']_i) = 0 \qquad \text{on} \quad \partial\Omega. \end{aligned}$$

This concludes the proof of Lemma 5.3.

Continuation of the proof of Lemma 5.1. By applying Fredholm's alternative theorem to the completely continuous operator T, we conclude from Lemma 5.3 that Eq. (5.8) has a uniquely determined solution (a, b, λ) in B. Lemma 5.2 implies that the field H' defined by (5.2) has the properties (a)–(d). Hence $H = T + H'$ is a solution of (A$'$) by the remarks at the beginning of this proof. This completes the proof of Lemma 5.1.

The preceding analysis has shown that problem (A$'$) can be reduced to a uniquely solvable system of Fredholm integral equations. Hence we can apply the methods of [11, Sect. IV] to obtain results on the dependence of the solution $H = H_\kappa[F]$ of problem (A$'$) on κ. In particular, Lemma 4.4 remains valid if $E_\kappa[F]$ is replaced by $H_\kappa[F]$. Also Lemma 4.5 can be extended to problem (A$'$). In this case B_3 has to be replaced by the Banach space B_3^α of functions λ satisfying a Hölder condition uniformly on $\partial\Omega$ with exponent α, $0 < \alpha < 1$, the norm being defined as in (4.12). It can be shown as in the proof of [10, Lemma 13] that T is a completely continuous operator from the Banach space $B'_\alpha = B_1 \times B_2 \times B_3^\alpha$ into itself. It follows as in the proof of the corollary to [10, Satz 1], that $H_\kappa[F](x)$ depends continuously on (x, κ) in $\bar{\Omega} \times B_0$, where $B_0 := \{\kappa \in C\colon \operatorname{Im} \kappa \geqslant 0, \kappa \neq 0\}$. The remaining parts of Section 4 can be immediately extended to the magnetic case so that we restrict our presentation to the formulation of the main results.

THEOREM 5.1. *Assume that Ω is the exterior of n disjoint bodies with boundaries $S_1, \ldots, S_n \in C^6$ and set $p = p_1 + \cdots + p_n$, where p_i denotes the topological genus of S_i. Let A' be the selfadjoint operator introduced in* [13, Sect. 3] *and denote the spectral family of A' by $\{P'_\lambda\}$. Then the following statements hold:*

(a) *If $p = 0$, then A' has no eigenvalues. If $p > 0$, then $\lambda = 0$ is the only eigenvalue of A'. The spectrum of A' is given by $\sigma(A') = [0, \infty)$. Furthermore, we have $P'_\lambda F \in \mathbf{C}^2(\bar{\Omega})$ for every $F \in \mathbf{L}_2(\Omega)$ and $\lambda \geqslant 0$, and $P'_\lambda F$ satisfies the magnetic boundary conditions* (1.2) *on $\partial\Omega$.*

(b) *Assume, in addition, that $F \in \mathbf{C}^2(\bar{\Omega})$, $\operatorname{supp} F$ is bounded, and $n \times (\nabla \times F) = 0$ and $n \cdot F = 0$ on $\partial\Omega$. Then $\{P'_\lambda\}$ and the solution $H_\kappa[F]$ of the exterior boundary value problem* (A′) *are related by*

$$(P'_\lambda F)(x) = (P'_{+0} F)(x) + \frac{1}{2\pi i} \int_0^\lambda (H_{\sqrt{\sigma}}[F](x) - H_{-\sqrt{\sigma}}[F](x))\, d\sigma \quad (5.15)$$

for $\lambda > 0$ and $x \in \bar{\Omega}$, where P'_{+0} denotes the projection of $\mathbf{L}_2(\Omega)$ onto the null space $N(A')$ of A' characterized in Theorem 3.1. *Furthermore, the identity*

$$F(x) = (P'_{+0} F)(x) + \frac{1}{2\pi i} \int_0^\infty (H_{\sqrt{\sigma}} F[(x)] - H_{-\sqrt{\sigma}}[F](x))\, d\sigma \quad (5.16)$$

holds for $x \in \bar{\Omega}$. The improper integrals on the right-hand sides converge uniformly with respect to x in every bounded subset of $\bar{\Omega}$.

6. GENERALIZED FOURIER TRANSFORM

Ikebe has developed a generalized Fourier transformation theory for the Schrödinger operator $-\Delta + q(x)$ in R^3 in his fundamental paper [2]. Analogous results have been obtained by Shenk [6] for the scalar Laplace operator in an exterior domain Ω. In the following we shall extend some of these results to the vector Laplace operator in Ω with respect to electric or magnetic boundary conditions. Our representation is influenced in many points by Wilcox's treatment of the scalar case (see [18, Sect. 6]).

Consider the plane wave solutions

$$w_0(x, p; a) := (2\pi)^{-3/2}\, a e^{ix \cdot p} \tag{6.1}$$

of the reduced (vector) wave equation

$$(\Delta + |p|^2)\, w = 0 \tag{6.2}$$

with $x \in R^3$ and $p, a \in R^3 - \{0\}$. Plane wave solutions of the corresponding scalar equation form the kernel of the classical Fourier transform. In analogy to [2, 6], we replace w_0 by *distorted plane waves* in Ω which arise from the reflection of w_0 at $\partial\Omega$ with regard to the boundary conditions (1.1) or (1.2). In each case we consider two types of distorted plane waves. In the electric case, the distorted plane waves $w_+(x, p; a)$ and $w_-(x, p; a)$ are defined as the solutions of the following boundary value problems:

(i) $w_\pm(\cdot, p; a) \in \mathbf{C}^2(\bar{\Omega})$,

(ii) $(\Delta_x + |p|^2)\, w_\pm(x, p; a) = 0$ for $x \in \Omega$,

(iii) $n(x) \times w_\pm(x, p; a) = 0$, $\nabla_x \cdot w_\pm(x, p; a) = 0$ for $x \in \partial\Omega$,

(iv) $w_\pm(x, p; a) - w_0(x, p; a) = O(r^{-1})$ and $(\partial/\partial r \mp i|p|)[w_\pm(x, p; a) - w_0(x, p; a)] = o(r^{-1})$ as $r = |x| \to \infty$.

For fixed $a, p \in R^3$, $w_\pm$ is given by $w_\pm = w_0 + E$, where E is the solution of problem (B) on [11, p. 355] with $\kappa = \pm|p|$, $c = -n \times w_0$, $\gamma = -\nabla \cdot w_0$ satisfying the radiation condition [11, Eq. (1.25)]. Hence the existence and dependence theory developed in [11] can be applied, and we obtain, by observing Lemma 2.1 and the remarks leading to Lemma 4.5:

LEMMA 6.1. *Assume that Ω is the exterior of a finite collection of disjoint bodies with surfaces $S_1, \ldots, S_n \in C^6$. Then problem* (i)–(iv) *has a uniquely determined solution $w_\pm(x, p; a)$ for all p, $a \in R^3 - \{0\}$, and $w_\pm(x, p; a)$ depends continuously on (x, p) in $\bar{\Omega} \times (R^3 - \{0\})$ and has derivatives of arbitrary order with respect to x and p in $\Omega \times (R^3 - \{0\})$. Furthermore, $w_\pm(x, p; a)$ depends linearly on a.*

Note that the linear dependence on a follows immediately from the uniqueness part of Lemma 6.1.

Now we define operators Φ_+ and Φ_- from $\mathbf{C}_0^\infty(\Omega)$ into $\mathbf{C}^\infty(R^3 - \{0\})$ by

$$(\Phi_\pm f)(p) := \sum_{j=1}^{3} e_j \int_\Omega f(x) \cdot \overline{w_\pm(x, p; e_j)}\, dx$$

for

$$f \in \mathbf{C}_0^\infty(\Omega) \qquad \text{and} \qquad p \in R^3 - \{0\}, \tag{6.3}$$

where e_j denotes the jth unit vector. Note that $\Phi_\pm f \in C^\infty(R^3 - \{0\})$ by Lemma 6.1. By replacing Ω by R^3 and $w_\pm$ by w_0, the right-hand side of (6.3) changes into

$$\frac{1}{(2\pi)^{3/2}} \sum_{j=1}^{3} e_j \int_{R^3} (f(x) \cdot e_j)\, e^{-ix\cdot p}\, dx$$
$$= \frac{1}{(2\pi)^{3/2}} \int_{R^3} f(x)\, e^{-ix\cdot p}\, dx =: \hat{f}(p) \tag{6.4}$$

so that Φ_+ and Φ_- can be considered as generalizations of the classical Fourier transform in R^3.

In a similar way we can define generalized Fourier transforms Φ'_+ and Φ'_- in the magnetic case, by replacing $w_\pm$ in (6.3) by the solution $w'_\pm(x, p; a)$ of the corresponding magnetic boundary value problem, satisfying (i), (ii), (iv), and

$$\text{(iii}'\text{)} \quad n(x) \times [\nabla_x \times w'_\pm(x, p; a)] = 0, \; n(x) \cdot w'_\pm(x, p; a) = 0 \text{ for } x \in \partial\Omega.$$

The argument in Section 5 shows that Lemma 6.1 is also valid in the magnetic case. In the following we restrict our considerations to the electric case since the magnetic case can be treated by the same argument. The main purpose of this section is to prove that $\Phi_\pm f$ belongs to $\mathbf{L}_2(R^3)$ for $f \in \mathbf{C}_0^\infty(\Omega)$ and that

$$\| \Phi_\pm f \|^2_{\mathbf{L}_2(R^3)} = \| f \|^2_{\mathbf{L}_2(\Omega)} - \| P_{+0} f \|^2_{\mathbf{L}_2(\Omega)} \leqslant \| f \|^2_{\mathbf{L}_2(\Omega)}, \tag{6.5}$$

where P_{+0} is the projection of $\mathbf{L}_2(\Omega)$ onto the null space of A. This observation allows us to extend the definition of Φ_+ and Φ_- onto the whole space $\mathbf{L}_2(\Omega)$ since $\mathbf{C}_0^\infty(\Omega)$ is dense in $\mathbf{L}_2(\Omega)$.

In order to verify (6.5), we choose a $r_0 > 0$ such that $R^3 - \Omega$ is contained in the sphere $\{x: |x| < r_0\}$. As in [18, Lecture 6], we choose a function $j \in C^\infty(R^3)$ with $j(x) = 0$ for $|x| < r_0$ and $j(x) = 1$ for $|x| > r_0 + 1$, and set

$$v_\pm(x, p; a) := w_\pm(x, p; a) - j(x)\, w_0(x, p; a). \tag{6.6}$$

Note that

$$(\Delta_x + |p|^2)\, v_\pm(x, p; a) = M(x, p; a) \tag{6.7}$$

with

$$M(x, p; a) := -w_0(x, p; a)\, \Delta j(x) - 2(2\pi)^{-3/2}\, a(\nabla j(x) \cdot \nabla_x e^{ix\cdot p}). \tag{6.8}$$

In particular, we have $M \in C^\infty(R^3 \times R^3 \times R^3)$ and $\operatorname{supp} M(\cdot, p; a) \subset \{x: r_0 \leqslant |x| \leqslant r_0 + 1\}$. Furthermore, $v_\pm$ satisfies the electric boundary conditions

(1.1) and the radiation condition $v_\pm = O(r^{-1})$, $(\partial/\partial r \mp i\,|p|)\, v_\pm = o(r^{-1})$ as $r = |x| \to \infty$. Hence we obtain, by using Lemmas 4.3 and 4.4,

$$v_\pm(x, p; a) = \lim_{\sigma \downarrow 0} v_{|p|^2 \pm i\sigma}(x, p; a) \tag{6.9}$$

for $x \in \Omega$ and p, $a \in R^3 - \{0\}$, where

$$v_z(\cdot, p; a) := -R_z M(\cdot, p; a). \tag{6.10}$$

By Lemma 4.4, the convergence in (6.9) is uniform with respect to x in every compact subset of Ω. According to (6.6), we set

$$w_z(x, p; a) := v_z(x, p; a) + j(x)\, w_0(x, p; a). \tag{6.11}$$

It follows from (6.6) and (6.9) that

$$w_\pm(x, p; a) = \lim_{\sigma \downarrow 0} w_{|p|^2 \pm i\sigma}(x, p; a) \tag{6.12}$$

for $x \in \Omega$ and $p, a \in R^3 - \{0\}$ uniformly in every compact subset of Ω. Furthermore, we set, according to (6.3),

$$(\Phi_z f)(p) := \sum_{j=1}^{3} e_j \int_\Omega f(x) \cdot \overline{w_z(x, p; e_j)}\, dx \tag{6.13}$$

for $f \in \mathbf{C}_0^\infty(\Omega)$, $p \in R^3 - \{0\}$, and $z \notin [0, \infty)$. In analogy to the scalar case (compare [18, Lemma 6.3]), the identity

$$(\Phi_z f)(p) = (|p|^2 - \bar{z})(JR_{\bar{z}} f)^\wedge\,(p) \tag{6.14}$$

holds for $f \in \mathbf{C}_0^\infty(\Omega)$, $z \notin [0, \infty)$ and $p \in R^3 - \{0\}$, where Ju, for $u \in C(\bar{\Omega})$, is defined by

$$\begin{aligned} (Ju)(x) &:= j(x)\, u(x), && \text{for} \quad x \in \Omega, \\ &:= 0, && \text{for} \quad x \in R^3 - \Omega, \end{aligned} \tag{6.15}$$

and $(Ju)^\wedge$ denotes the (classical) Fourier transform of Ju in R^3. The proof of (6.14) proceeds as follows: We conclude from (6.13), by using (6.11) and (6.1), that

$$\begin{aligned} (\Phi_z f)(p) &= (Jf)^\wedge\,(p) + \sum_{j=1}^{3} e_j \int_\Omega f(x) \cdot \overline{v_z(x, p; e_j)}\, dx \\ &= (Jf)^\wedge\,(p) - \sum_{j=1}^{3} e_j (f, R_z M(\cdot, p; e_j)) \qquad \text{(by (6.10))} \end{aligned}$$

$$= (Jf)^\wedge (p) - \sum_{j=1}^{3} e_j(R_{\bar z}f, M(\cdot, p; e_j))$$

$$= (Jf)^\wedge (p) + \sum_{j=1}^{3} e_j \int_\Omega (R_{\bar z}f)(x) \cdot (\Delta + |p|^2)[j(x)\overline{w_0(x, p; e_j)}]\, dx$$

(by (6.8)).

Since the integrand has compact support in Ω by (6.8), Green's formula yields

$$(\Phi_z f)(p) = (Jf)^\wedge (p) + \sum_{j=1}^{3} e_j \int_\Omega j(x)\overline{w_0(x, p; e_j)} \cdot (\Delta + |p|^2)(R_{\bar z} f(x))\, dx.$$

Now (6.14) follows from (6.1) and (6.15), since

$$(\Delta + |p|^2) R_{\bar z} f = [(\Delta + \bar z) + |p|^2 - \bar z] R_{\bar z} f = -f + (|p|^2 - \bar z) R_{\bar z} f.$$

The further analysis is based upon the formula

$$((P_\beta - P_\alpha)f, g) = \frac{1}{2\pi i} \lim_{\sigma \downarrow 0} \int_\alpha^\beta (R_{\lambda + i\sigma} f - R_{\lambda - i\sigma} f, g)\, d\sigma \tag{6.16}$$

for $0 < \alpha < \beta < \infty$ and $f,\ g \in \mathbf{C}_0^\infty(\Omega)$ which follows from (4.6) and the remark that $P_{\lambda+0} = P_\lambda$ for $\lambda > 0$ by Lemma 4.6. By the elementary identity $R_z - R_{z'} = (z - z') R_z R_{z'}$, we have

$$(R_{\lambda + i\sigma} f - R_{\lambda - i\sigma} f, g) = 2i\sigma(R_{\lambda \pm i\sigma} R_{\lambda \mp i\sigma} f, g) = 2i\sigma(R_{\lambda \mp i\sigma} f, R_{\lambda \mp i\sigma}\, g)$$

so that

$$((P_\beta - P_\alpha)f, g) = \lim_{\sigma \downarrow 0} \frac{\sigma}{\pi} \int_\alpha^\beta (R_{\lambda \mp i\sigma} f, R_{\lambda \mp i\sigma}\, g)\, d\lambda$$

$$= \lim_{\sigma \downarrow 0} \frac{\sigma}{\pi} \int_\alpha^\beta (JR_{\lambda \mp i\sigma} f, JR_{\lambda \mp i\sigma}\, g)\, d\lambda.$$

The last equation holds by (6.15), since $1 - j(x)^2$ vanishes for $|x| > r_0 + 1$ and

$$\frac{\sigma}{\pi} [1 - j(x)^2](R_{\lambda \mp i\sigma} f)(x)\overline{(R_{\lambda \mp i\sigma}\, g)(x)}$$

converges uniformly to 0 as $\sigma \downarrow 0$ in $\bar\Omega \times [\alpha, \beta]$ by Lemma 4.3 and Lemma 4.5. Note that $JR_{\bar z} f$, for fixed $z \notin [0, \infty)$, is continuous in R^3 and exponentially decreasing at infinity by (6.15) and Lemma 4.3 so that $(JR_{\bar z} f)^\wedge \in$

$C(R^3) \cap L_2(R^3)$. By Lemma 2.2 we have $\int |(JR_{\bar{z}} f)^{\wedge}|^2 \, dp < \infty$. Hence, Parseval's identity and [12, Lemma 2.5] imply

$$(JR_{\bar{z}} f, JR_{\bar{z}} g) = \int_{R^3} (JR_{\bar{z}} f)^{\wedge}\ (p) \cdot \overline{(JR_{\bar{z}} g)^{\wedge}\ (p)} \, dp.$$

Thus we obtain, by using (6.14),

$$((P_\beta - P_\alpha) f, g) = \lim_{\sigma \downarrow 0} \frac{\sigma}{\pi} \int_\alpha^\beta \left[\int_{R^3} \frac{(\Phi_{\lambda \pm i\sigma} f)(p)}{|p|^2 - \lambda \pm i\sigma} \cdot \frac{\overline{(\Phi_{\lambda \pm i\sigma} g)(p)}}{|p|^2 - \lambda \mp i\sigma} \, dp \right] d\lambda. \quad (6.17)$$

In order to investigate the limit in (6.17), we show:

LEMMA 6.2. *Assume that* $f \in \mathbf{C}_0^\infty(\Omega)$ *and let* $\Phi_{\lambda \pm i\sigma} f$ *be defined by* (6.13) *for* $\sigma > 0$. *Then* $(\Phi_{\lambda \pm i\sigma} f)(p)$ *can be extended to a function depending continuously on* (λ, σ, p) *in* $(0, \infty) \times [0, \infty) \times R^3$. *Furthermore, for every* $f \in \mathbf{C}_0^\infty(\Omega)$ *and every pair* (α, β) *with* $0 < \alpha < \beta < \infty$, *there exists a* $c > 0$ *such that*

$$|\Phi_{\lambda \pm i\sigma} f(p)| \leqslant c \quad (6.18)$$

for every $(\lambda, \sigma, p) \in [\alpha, \beta] \times [0, 1] \times R^3$.

Proof. The calculation after (6.15), in connection with (6.1), implies that

$$(\Phi_{\lambda \pm i\sigma} f)(p) = (Jf)^{\wedge}\ (p) + \frac{1}{(2\pi)^{3/2}} \times \int_{r_0 < |x| < r_0 + 1} e^{-ix \cdot p} [\Delta j(x) - 2ip \cdot \nabla j(x)] \, (R_{\lambda \mp i\sigma} f)(x) \, dx. \quad (6.19)$$

This representation, together with Lemmas 4.3 and 4.4, shows that $\Phi_{\lambda \pm i\sigma} f$ can be continuously extended onto $\sigma \geqslant 0$ and that the first part of Lemma 6.2 holds. By (6.15) and the definition of the classical Fourier transform, the first term in (6.19) is bounded. Also the integral

$$\int_{r_0 < |x| < r_0 + 1} e^{-ix \cdot p} (\Delta j(x)) (R_{\lambda \mp i\sigma} f)(x) \, dx$$

is bounded in every region $[\alpha, \beta] \times [0, 1] \times R^3$ with $0 < \alpha < \beta < \infty$ since the integrand is continuous by Lemmas 4.3 and 4.4. In order to discuss the remaining term in (6.19), we apply the integral theorem of Gauss. Since $\nabla j = 0$ for $|x| = r_0$ and $|x| = r_0 + 1$, we have for $k = 1, 2, 3$

$$-\int_{r_0<|x|<r_0+1} e^{-ix\cdot p} ip_k(\partial_k j(x))(R_{\lambda\mp i\sigma}f)(x)\,dx$$
$$= \int_{r_0<|x|<r_0+1} \left(\frac{\partial}{\partial x_k} e^{-ix\cdot p}\right)(\partial_k j(x))(R_{\lambda\mp i\sigma}f)(x)\,dx$$
$$= -\int_{r_0<|x|<r_0+1} e^{-ix\cdot p}\,\partial_k[(\partial_k j(x))(R_{\lambda\mp i\sigma}f)(x)]\,dx.$$

Note that also $\partial_k R_{\lambda\mp i\sigma}f$ depends continuously on (λ, σ, x) in $[\alpha, \beta] \times [0, 1] \times \{x: r_0 \leqslant |x| \leqslant r_0 + 1\}$ by the existence and dependence theory developed in Section 3 and in [11]. In fact, the representation [11, Eq. (2.3)] shows that not only the solution E of problem (B) on [11, p. 355], but also arbitrary derivatives of E depend continuously on x and κ in every compact subset of Ω. This remark implies that also the remaining term in (6.19) has the required boundedness property.

By Lemma 6.2, the inner integral in (6.17) converges, for fixed $\sigma > 0$, uniformly with respect to λ in $[\alpha, \beta]$. Hence the order of the integrations can be interchanged, and we obtain

$$((P_\beta - P_\alpha)f, g) = \lim_{\sigma\downarrow 0} \frac{\sigma}{\pi} \int_{R^3} \left[\int_\alpha^\beta \frac{(\Phi_{\lambda\pm i\sigma}f)(p)\cdot\overline{(\Phi_{\lambda\pm i\sigma}g)(p)}}{(\lambda - |p|^2)^2 + \sigma^2}\,d\lambda\right] dp \tag{6.20}$$

for $f, g \in C_0^\infty(\Omega)$ and $0 < \alpha < \beta < \infty$. Now choose, for fixed α and β, a $m > 0$ such that $|\lambda - |p|^2| \geqslant |p|^2/2$ for $|p| \geqslant m$ and $\lambda \in [\alpha, \beta]$. If $|p| > m$, then the integrand in (6.20) can be estimated by $c_1 |p|^{-4}$ with a suitable $c_1 > 0$, by using Lemma 6.2. Hence we obtain

$$\left|\frac{\sigma}{\pi}\int_{|p|>m}\left[\int_\alpha^\beta \cdots\, d\lambda\right] dp\right| \leqslant \frac{\sigma}{\pi}(\beta - \alpha)\, c_1 \int_{|p|>m} \frac{dp}{|p|^4} \to 0 \qquad \text{as} \quad \sigma \downarrow 0,$$

so that (6.20) is reduced to

$$((P_\beta - P_\alpha)f, g) = \lim_{\sigma\downarrow 0} \frac{\sigma}{\pi} \int_{|p|<m} \left[\int_\alpha^\beta \frac{(\Phi_{\lambda\pm i\sigma}f)(p)\cdot\overline{(\Phi_{\lambda\pm i\sigma}g)(p)}}{(\lambda - |p|^2)^2 + \sigma^2}\,d\lambda\right] dp. \tag{6.21}$$

Now we use the following elementary fact:

LEMMA 6.3. *Assume that φ is continuous in $[\alpha, \beta] \times [0, 1]$. Set*

$$h(\gamma, \sigma) := \frac{\sigma}{\pi}\int_\alpha^\beta \frac{\varphi(\lambda, \sigma)}{(\lambda - \gamma)^2 + \sigma^2}\,d\lambda. \tag{6.22}$$

Then the following statements hold:

(a) *$h(\gamma,\sigma)$ is bounded in $R \times [0,1]$;*

(b) *$h(\gamma,\sigma) \to 0$ as $\sigma \downarrow 0$ uniformly in every compact subset of $R-[\alpha,\beta]$;*

(c) *$h(\gamma,\sigma) \to \varphi(\gamma,0)$ as $\sigma \downarrow 0$ uniformly in every compact subset of (α,β).*

Proof. Statements (a) and (b) are obvious. In order to verify (c), we choose, for a given $\varepsilon > 0$ and a given compact subset K of (α,β), $\delta \in (0,1)$ such that $[\gamma-\delta,\gamma+\delta] \subset (\alpha,\beta)$ for every $\gamma \in K$ and $|\varphi(\lambda,\sigma)-\varphi(\gamma,0)| < \varepsilon/2$ for $\gamma \in K$ and $(\lambda,\sigma) \in [\gamma-\delta,\gamma+\delta] \times [0,\delta]$. Let M be the maximum of $|\varphi|$ in $[\alpha,\beta] \times [0,1]$. We have for $\gamma \in K$ and $0 \leqslant \sigma \leqslant \delta$

$$\left| \frac{\sigma}{\pi} \left[\int_{\alpha}^{\gamma-\delta} \frac{\varphi(\lambda,\sigma)}{(\lambda-\gamma)^2+\sigma^2}\, d\lambda + \int_{\gamma+\delta}^{\beta} \cdots \, d\lambda \right] \right|$$
$$\leqslant \frac{2M\sigma}{\pi} \int_{\delta}^{\infty} \frac{d\lambda}{\lambda^2+\sigma^2} = \frac{2M}{\pi} \left(\frac{\pi}{2} - \arctan \frac{\delta}{\sigma} \right)$$

and

$$\left| \frac{\sigma}{\pi} \int_{\gamma-\delta}^{\gamma+\delta} \frac{\varphi(\lambda,\sigma)}{(\lambda-\gamma)^2+\sigma^2}\, d\lambda - \varphi(\gamma,0)\frac{2}{\pi} \arctan \frac{\delta}{\sigma} \right|$$
$$= \left| \frac{\sigma}{\pi} \int_{\gamma-\delta}^{\gamma+\delta} \frac{\varphi(\lambda,\sigma)-\varphi(\gamma,0)}{(\lambda-\gamma)^2+\sigma^2}\, d\lambda \right| < \frac{\varepsilon}{2} \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2+\sigma^2} = \frac{\varepsilon}{2}.$$

These estimates imply that

$$|h(\gamma,\sigma)-\varphi(\gamma,0)|$$
$$< \frac{2M}{\pi}\left(\frac{\pi}{2} - \arctan\frac{\delta}{\sigma}\right) + \frac{\varepsilon}{2} + |\varphi(\gamma,0)| \cdot \left(1 - \frac{2}{\pi}\arctan\frac{\delta}{\sigma}\right)$$
$$\leqslant \frac{\varepsilon}{2} + \frac{4M}{\pi}\left(\frac{\pi}{2} - \arctan\frac{\delta}{\sigma}\right).$$

Now choose $\sigma_0 \in (0,\delta)$ such that the last term is less than $\varepsilon/2$ for $0 \leqslant \sigma \leqslant \sigma_0$. Then we have $|h(\gamma,\sigma)-\varphi(\gamma,0)| < \varepsilon$ for $0 \leqslant \sigma \leqslant \sigma_0$ and $\gamma \in K$ so that (c) holds.

In order to apply Lemma 6.3 to the right-hand side of (6.21), we decompose the ball $|p| < m$ into

$$D_1(\rho) := \{ p \in R^3 : |p| < m, ||p| - \sqrt{\alpha}| < \rho, ||p| - \sqrt{\beta}| < \rho \}$$

and $D_2(\rho) := \{p : |p| < m\} - D_1(\rho)$. Lemmas 6.2 and 6.3 imply that

$$H_\pm(p,\sigma) := \frac{\sigma}{\pi}\int_\alpha^\beta \frac{(\Phi_{\lambda\pm i\sigma}f)(p)\cdot\overline{(\Phi_{\lambda\pm i\sigma}g)(p)}}{(\lambda-|p|^2)^2+\sigma^2}\,d\lambda \tag{6.23}$$

converges to $(\Phi_{|p|^2\pm i0}f)(p)\cdot\overline{(\Phi_{|p|^2\pm i0}g)(p)}$ if $|p|^2\in(\alpha,\beta)$ and to 0 if $|p|^2\notin[\alpha,\beta]$ as $\sigma\downarrow 0$, uniformly in $D_2(\rho)$. Note that $(\Phi_{|p|^2\pm i0}f)(p) = (\Phi_\pm f)(p)$ by (6.3), (6.12), and (6.13). Hence we obtain

$$\lim_{\sigma\downarrow 0}\int_{D_2(\rho)} H_\pm(p,\sigma)\,dp \tag{6.24}$$

$$= \int_{p\in D_2(\rho),\alpha<|p|^2<\beta} \Phi_\pm f\cdot\overline{\Phi_\pm g}\,dp \to \int_{\alpha<|p|^2<\beta} \Phi_\pm f\cdot\overline{\Phi_\pm g}\,dp \qquad \text{as} \quad \rho\to 0$$

since $\{p : \alpha<|p|^2<\beta\}\subset\{p : |p|<m\}$ by the choice of m. Since $H_\pm(p,\sigma)$ is bounded uniformly with respect to σ by (6.18) and Lemma 6.3(a) and the volume of $D_1(\rho)$ converges to 0 as $\rho\to 0$, we have

$$\lim_{\sigma\downarrow 0}\int_{D_1(\rho)} H_\pm(p,\sigma)\,dp\to 0 \qquad \text{as} \quad \rho\to 0. \tag{6.25}$$

By combining (6.21) and (6.23)–(6.25), we obtain

$$((P_\beta - P_\alpha)f, g) = \int_{\alpha<|p|^2<\beta} \Phi_\pm f\cdot\overline{\Phi_\pm g}\,dp \tag{6.26}$$

for $f, g\in \mathbf{C}_0^\infty(\Omega)$ and $0<\alpha<\beta<\infty$. If $f=g$, (6.26) yields

$$\|P_\beta f\|^2 - \|P_\alpha f\|^2 = \int_{\alpha<|p|^2<\beta} |\Phi_\pm f|^2\,dp. \tag{6.27}$$

Since $P_\alpha f\to P_{+0}f$ as $\alpha\downarrow 0$ and $P_\beta f\to f$ as $\beta\to\infty$ in $\mathbf{L}_2(\Omega)$, we get

$$\|f\|^2 - \|P_{+0}f\|^2 = \int_{R^3} |\Phi_\pm f|^2\,dp \qquad \text{for} \quad f\in\mathbf{C}_0^\infty(\Omega). \tag{6.28}$$

In particular, the improper integral on the right-hand side converges. This implies that $\Phi_\pm f\in \mathbf{C}^\infty(R^3-\{0\})\cap\mathbf{L}_2(R^3)$ and that (6.5) holds. By letting $\alpha\downarrow 0$ and $\beta\to\infty$ in (6.26), we obtain

$$(f-P_{+0}f, g) = \int_{R^3} \Phi_\pm f\cdot\Phi_\pm g\,dp \qquad \text{for} \quad f, g\in\mathbf{C}_0^\infty(\Omega). \tag{6.29}$$

Note that the convergence of the integral in (6.28) implies that also the improper integral in (6.29) converges.

By (6.5), $\Phi_\pm$ is a bounded linear operator from $\mathbf{C}_0^\infty(\Omega)$ into $\mathbf{L}_2(R^3)$. Since $\mathbf{C}_0^\infty(\Omega)$ is dense in $\mathbf{L}_2(\Omega)$, $\Phi_\pm$ can be uniquely extended to a bounded linear operator from $\mathbf{L}_2(\Omega)$ into $\mathbf{L}_2(R^3)$, and the identity (6.5) can be carried over to the whole space $\mathbf{L}_2(\Omega)$:

$$\|\Phi_\pm F\|^2_{\mathbf{L}_2(R^3)} = \|F\|^2_{\mathbf{L}_2(\Omega)} - \|P_{+0}F\|^2_{\mathbf{L}_2(\Omega)} \qquad \text{for} \quad F \in \mathbf{L}_2(\Omega). \quad (6.30)$$

This remark completes the definition of the generalized Fourier transforms Φ_+ and Φ_- in $\mathbf{L}_2(\Omega)$.

Since $P_{+0}F = F$ for $F \in N(A)$ and $P_{+0}F = 0$ for $F \in N(A)^\perp$, (6.30) yields

$$\Phi_\pm F = 0 \qquad \text{for} \quad F \in N(A) \quad (6.31)$$

and

$$\|\Phi_\pm F\|^2_{\mathbf{L}_2(R^3)} = \|F\|^2_{\mathbf{L}_2(\Omega)} \qquad \text{for} \quad F \in N(A)^\perp. \quad (6.32)$$

By applying the identity

$$(F, G) = \tfrac{1}{4}(\|F+G\|^2 - \|F-G\|^2 + i\,\|F+iG\|^2 - i\,\|F-iG\|^2), \quad (6.33)$$

we conclude from (6.30) that

$$(\Phi_\pm F, \Phi_\pm G)_{\mathbf{L}_2(R^3)} = (F, G)_{\mathbf{L}_2(\Omega)} - (P_{+0}F, P_{+0}G)_{\mathbf{L}_2(\Omega)} \qquad \text{for } F, G \in \mathbf{L}_2(\Omega). \quad (6.34)$$

Since $\Phi_\pm$ is a bounded linear operator from $\mathbf{L}_2(\Omega)$ into $\mathbf{L}_2(R^3)$, there exists a uniquely determined (*adjoint*) bounded linear operator $\Phi_\pm^*$ from $\mathbf{L}_2(R^3)$ into $\mathbf{L}_2(\Omega)$ such that

$$(\Phi_\pm F, G)_{\mathbf{L}_2(R^3)} = (F, \Phi_\pm^* G)_{\mathbf{L}_2(\Omega)} \quad (6.35)$$

for every $F \in \mathbf{L}_2(\Omega)$ and every $G \in \mathbf{L}_2(R^3)$. Now assume that $F \in N(A)$. It follows from (6.35) and (6.31) that

$$(F, \Phi_\pm^* G)_{\mathbf{L}_2(\Omega)} = (\Phi_\pm F, G)_{\mathbf{L}_2(R^3)} = 0$$

for every $G \in \mathbf{L}_2(R^3)$ so that

$$\Phi_\pm^* G \in N(A)^\perp \qquad \text{for} \quad G \in \mathbf{L}_2(R^3). \quad (6.36)$$

Note that $(P_{+0}F, P_{+0}G) = (F, P_{+0}G)$ for $F, G \in \mathbf{L}_2(\Omega)$ since P_{+0} is a projection operator. Therefore it follows from (6.35) and (6.34) that

$$(F, \Phi_\pm^* \Phi_\pm G)_{\mathbf{L}_2(\Omega)} = (\Phi_\pm F, \Phi_\pm G)_{\mathbf{L}_2(R^3)} = (F, G - P_{+0}G)_{\mathbf{L}_2(\Omega)}$$

for $F, G \in \mathbf{L}_2(\Omega)$, and hence

$$\Phi_{\pm}^{*}\Phi_{\pm} = I - P_{+0}. \tag{6.37}$$

Now assume that $f \in \mathbf{C}_0^\infty(\Omega)$ and $g \in \mathbf{C}_0^\infty(R^3 - \{0\})$. By (6.35) and (6.3), we have

$$\begin{aligned}(f, \Phi_{\pm}^{*} g)_{\mathbf{L}_2(\Omega)} &= (\Phi_{\pm} f, g)_{\mathbf{L}_2(R^3)} \\ &= \int_{R^3} \overline{g(p)} \cdot \left[\sum_{j=1}^{3} e_j \int_{\Omega} f(x) \cdot \overline{w_{\pm}(x, p; e_j)}\, dx\right] dp \\ &= \int_{\Omega} f(x) \cdot \left[\sum_{j=1}^{3} \int_{R^3} \overline{(e_j \cdot g(p))}\, \overline{w_{\pm}(x, p; e_j)}\, dp\right] dx\end{aligned}$$

and hence

$$(\Phi_{\pm}^{*} g)(x) = \sum_{j=1}^{3} \int_{R^3} g_j(p)\, w_{\pm}(x, p; e_j)\, dp \tag{6.38}$$

for $g = (g_1, g_2, g_3) \in \mathbf{C}_0^\infty(R^3 - \{0\})$. In particular, it follows from Lemma 6.1 that $\Phi_{\pm}^{*} g \in \mathbf{C}^\infty(\Omega)$ for $g \in \mathbf{C}_0^\infty(R^3 - \{0\})$. Since $\mathbf{C}_0^\infty(R^3 - \{0\})$ is dense in $\mathbf{L}_2(R^3)$, the adjoint operator $\Phi_{\pm}^{*}$ is uniquely characterized by (6.38). Note that $w_{\pm}(x, p; a)$ depends linearly on a by Lemma 6.1. Hence (6.38) can be written in the form

$$(\Phi_{\pm}^{*} g)(x) = \int_{R^3} w_{\pm}(x, p; g(p))\, dp \qquad \text{for} \quad g \in \mathbf{C}_0^\infty(R^3 - \{0\}). \tag{6.39}$$

7. Remarks on the Functional Calculus

By applying the generalized Fourier transforms Φ_+ and Φ_-, the functional calculus for the selfadjoint operator A in the Hilbert space $\mathbf{L}_2(\Omega)$ can be related to multiplication operators in the transformed space $\mathbf{L}_2(R^3)$ (compare [3, Theorem 3.2; 16, Theorem 6.15] for related results for the Schrödinger equation and in the scalar case). We prove:

LEMMA 7.1. *Assume that ψ is a bounded, piecewise continuous complex-valued function defined on $[0, \infty)$ and set*

$$\psi_1(p) := \psi(|p|^2) \qquad \textit{for} \quad p \in R^3. \tag{7.1}$$

Then we have

$$\Phi_{\pm}\, \psi(A)\, F = \psi_1 \cdot \Phi_{\pm} F \qquad \textit{for} \quad F \in \mathbf{L}_2(\Omega). \tag{7.2}$$

Remark. According to the functional calculus for selfadjoint operators, $\psi(A)$ is a bounded linear operator from $\mathbf{L}_2(\Omega)$ into $\mathbf{L}_2(\Omega)$, defined by

$$\psi(A)\,F=\int_0^\infty \psi(\lambda)\,d(P_\lambda F). \tag{7.3}$$

It follows immediately from [15, Formula (7.10)] that

$$\|\psi(A)\,F\| \leqslant \|\psi\|_\infty\,\|F\|, \tag{7.4}$$

where $\|\psi\|_\infty := \operatorname{supr}\{|\psi(\lambda)|: \lambda\in[0,\infty)\}$. The product on the right-hand side in (7.2) can be defined by applying [12, Definition 5.1] to every component of the vector $\Phi_\pm F$. Lemma 7.1 can be extended to unbounded functions ψ if the domains of definition are suitably restricted. A special result in this direction will be given at the end of this section.

Proof of Lemma 7.1. At first we show that

$$(\Phi_\pm(P_\beta f-P_\alpha f),h)_{\mathbf{L}_2(R^3)}=\int_{\alpha<|p|^2<\beta}(\Phi_\pm f)(p)\,\overline{h(p)}\,dp \tag{7.5}$$

for $f\in \mathbf{C}_0^\infty(\Omega)$, $h\in \mathbf{C}_0^\infty(R^3)$ and $0<\alpha<\beta<\infty$. By (6.26), we have

$$((P_\beta-P_\alpha)f,f)_{\mathbf{L}_2(\Omega)}=\int_{\alpha<|p|^2<\beta}|\Phi_\pm f|^2\,dp \tag{7.6}$$

for $f\in \mathbf{C}_0^\infty(\Omega)$. In order to extend (7.6) to arbitrary elements $F\in \mathbf{L}_2(\Omega)$, we set, for $G=(G_1,G_2,G_3)\in \mathbf{L}_2(R^3)$,

$$G^r := (G_1^r, G_2^r, G_3^r), \tag{7.7}$$

where G_i^r denotes the restriction of the functional $G_i\in L_2(R^3)$ to $C_0^\infty(\Omega_{\alpha,\beta})$ with

$$\Omega_{\alpha,\beta} := \{p\in R^3: \alpha<|p|^2<\beta\}. \tag{7.8}$$

It follows from (7.6) that

$$((P_\beta-P_\alpha)\,F,F)_{\mathbf{L}_2(\Omega)}=\|(\Phi_\pm F)^r\|^2_{\Omega_{\alpha,\beta}} \tag{7.9}$$

for $F \in \mathbf{L}_2(L)$. In fact, (7.9) coincides with (7.6) if $F = f \in \mathbf{C}_0^\infty(\Omega)$, since $\Phi_\pm f \in C^\infty(R^3 - \{0\})$ so that

$$\|(\Phi_\pm f)^r\|^2_{\Omega_{\alpha,\beta}} = \int_{\Omega_{\alpha,\beta}} |\Phi_\pm f|^2 \, dp$$

by [12, Lemma 2.5]. Hence (7.9) follows from (7.6) by the remark that $\mathbf{C}_0^\infty(\Omega)$ is dense in $\mathbf{L}_2(\Omega)$ and the operators $P_\beta - P_\alpha$, $\Phi_\pm$ and $G \to G^r$ are bounded. Now we apply (7.9) to

$$F := (P_\beta - P_\alpha) f - f \tag{7.10}$$

with $f \in \mathbf{C}_0^\infty(\Omega)$. Note that

$$(P_\beta - P_\alpha) F = 0 \qquad \text{for} \quad 0 < \alpha < \beta < \infty \tag{7.11}$$

since $P_\lambda P_\mu = P_\mu P_\lambda = P_{\min(\lambda,\mu)}$. Therefore (7.9) yields $(\Phi_\pm F)^r = 0$ and hence, by (7.10),

$$[\Phi_\pm(P_\beta f - P_\alpha f - f)]^r = 0. \tag{7.12}$$

This implies that

$$(\Phi_\pm(P_\beta - P_\alpha) f, h)_{\mathbf{L}_2(R^3)} = (\Phi_\pm f, h)_{\mathbf{L}_2(R^3)} \tag{7.13}$$

for $h \in \mathbf{C}_0^\infty(\Omega_{\alpha,\beta})$. In particular, (7.5) holds for $h \in \mathbf{C}_0^\infty(\Omega_{\alpha,\beta})$.

Now set $F := (P_\beta - P_\alpha) f$. By applying (6.30) and observing that $P_{+0}(P_\beta - P_\alpha) f = 0$, we obtain

$$\|\Phi_\pm(P_\beta - P_\alpha) f\|^2_{\mathbf{L}_2(R^3)} = \|(P_\beta - P_\alpha) f\|^2_{\mathbf{L}_2(\Omega)} = ((P_\beta - P_\alpha)^2 f, (P_\beta - P_\alpha) f)_{\mathbf{L}_2(\Omega)}$$

and hence, by (7.9),

$$\|\Phi_\pm(P_\beta - P_\alpha) f\|_{\mathbf{L}_2(R^3)} = \|[\Phi_\pm(P_\beta - P_\alpha) f]^r\|_{\mathbf{L}_2(\Omega_{\alpha,\beta})}. \tag{7.14}$$

Set $G := \Phi_\pm(P_\beta - P_\alpha) f$ and choose a sequence $\{g_k\}$ in $\mathbf{C}_0^\infty(R^3)$ such that $\|G - g_k\| \to 0$ as $k \to \infty$. Then we have by [12, Lemma 2.5] and (7.14)

$$\int_{\Omega_{\alpha,\beta}} |g_k|^2 \, dp = \|g_k^r\|^2_{\mathbf{L}_2(\Omega_{\alpha,\beta})} \to \|G^r\|^2_{\mathbf{L}_2(\Omega(\alpha,\beta))} = \|G\|^2_{\mathbf{L}_2(R^3)}$$

and

$$\int_{R^3} |g_k|^2 \, dp \to \|G\|^2_{\mathbf{L}_2(R^3)}.$$

These relations imply

$$\int_{R^3-\Omega_{\alpha,\beta}} |g_k|^2\,dp \to 0 \qquad \text{as} \quad k \to \infty$$

and hence

$$(\Phi_\pm(P_\beta - P_\alpha)f, h)_{\mathbf{L}_2(R^3)} = \lim_{k\to\infty} \int g_k \cdot \bar{h}\,dp = 0 \tag{7.15}$$

for $h \in \mathbf{C}_0^\infty(R^3 - \Omega_{\alpha,\beta})$, so that (7.5) holds also for $h \in \mathbf{C}_0^\infty(R^3 - \Omega_{\alpha,\beta})$. Since $\mathbf{C}_0^\infty(\Omega_{\alpha,\beta} \cup (R^3 - \Omega_{\alpha,\beta}))$ is dense in $\mathbf{C}_0^\infty(R^3)$ with respect to the $\mathbf{L}_2$-norm, we conclude from (7.13) and (7.15) that (7.5) holds for every $h \in \mathbf{C}_0^\infty(R^3)$.

Now we choose a $\gamma > 0$ and a subdivision $0 =: \lambda_0 < \lambda_1 < \cdots < \lambda_m := \gamma$ of the interval $[0, \gamma]$. Since $P_0 = 0$, it follows from (7.5) that

$$\begin{aligned}&\left(\Phi_\pm \left[\sum_{k=1}^m \psi(\lambda_k)(P_{\lambda_k} - P_{\lambda_{k-1}})\right] f, h\right)_{\mathbf{L}_2(R^3)} \\ &\quad = \psi(\lambda_1)(\Phi_\pm P_{\lambda_1} f, h)_{\mathbf{L}_2(R^3)} + \sum_{k=2}^m \psi(\lambda_k) \int_{\lambda_{k-1} < |p|^2 < \lambda_k} \Phi_\pm f \cdot \bar{h}\,dp\end{aligned}$$

for $f \in \mathbf{C}_0^\infty(\Omega)$ and $h \in \mathbf{C}_0^\infty(R^3 - \{0\})$. Consider a sequence $\{Z_n\}$ of such subdivisions of $[0, \gamma]$ with $\max(\lambda_k - \lambda_{k-1}) \to 0$ as $n \to \infty$. Note that $\Phi_\pm$ is bounded and $\Phi_\pm P_{\lambda_1} f \to \Phi_\pm P_{+0} f = 0$ as $\lambda_1 \downarrow 0$ by (6.31). Hence, by letting $n \to \infty$, we obtain,

$$\begin{aligned}&\left(\Phi_\pm \left[\int_0^\gamma \psi(\lambda)\,d(P_\lambda f)\right], h\right)_{\mathbf{L}_2(R^3)} \\ &\quad = \int_{|p|^2<\gamma} \psi(|p|^2)(\Phi_\pm f)(p) \cdot \overline{h(p)}\,dp.\end{aligned} \tag{7.16}$$

Since ψ is bounded, we can perform the limit $\gamma \to \infty$ in (7.16) and get, by observing (7.1) and (7.3),

$$(\Phi_\pm \psi(A) f, h)_{\mathbf{L}_2(R^3)} = (\psi_1 \cdot \Phi_\pm f, h)_{\mathbf{L}_2(R^3)} \tag{7.17}$$

for $f \in \mathbf{C}_0^\infty(\Omega)$ and $h \in \mathbf{C}_0^\infty(R^3 - \{0\})$, and hence

$$\Phi_\pm \psi(A) f = \psi_1 \cdot \Phi_\pm f \qquad \text{for} \quad f \in \mathbf{C}_0^\infty(\Omega), \tag{7.18}$$

since $\mathbf{C}_0^\infty(R^3 - \{0\})$ is dense in $\mathbf{L}_2(R^3)$. Now (7.2) follows from (7.18) by the remark that $\psi(A)$, $\Phi_\pm$ and the multiplication with ψ_1 are bounded operators with respect to the $\mathbf{L}_2$-norms. This completes the proof of Lemma 7.1.

We conclude this section with the verification of

$$[\Phi_\pm(Af)](p) = |p|^2 (\Phi_\pm f)(p) \qquad \text{for } f \in \mathbf{C}_0^\infty(\Omega) \text{ and } p \in R^3 - \{0\}. \tag{7.19}$$

Note that $Af = -\Delta f \in \mathbf{C}_0^\infty(\Omega)$ so that $\Phi_\pm(Af) \in C^\infty(R^3 - \{0\})$. Formula (7.19) follows from (6.3), by using Green's formula and observing that $(\Delta + |p|^2)\, w_\pm(\cdot, p; a) = 0$:

$$\begin{aligned}[\Phi_\pm(Af)](p) &= -\sum_{j=1}^{3} e_j \int_\Omega \Delta f(x) \cdot \overline{w_\pm(x, p; e_j)}\, dx \\ &= -\sum_{j=1}^{3} e_j \int_\Omega f(x) \cdot \Delta_x \overline{w_\pm(x, p; e_j)}\, dx = |p|^2 (\Phi_\pm f)(p).\end{aligned}$$

8. Plane Wave Expansions

This section is devoted to the proof of the relation

$$\Phi_\pm \Phi_\pm^* = I. \tag{8.1}$$

The formulas (6.37) and (8.1) can be interpreted as an expansion theorem for vector fields $F \in N(A)^\perp$. In fact, since $P_{+0}F = 0$ for $F \in N(A)$, (6.37) says that every $F \in N(A)^\perp$ can be represented in the form

$$F = \Phi_\pm^* G \tag{8.2}$$

with a suitable $G \in \mathbf{L}_2(R^3)$. Formula (8.1) implies that G is uniquely determined by F and given by

$$G = \Phi_\pm F. \tag{8.3}$$

Since $\mathbf{C}_0^\infty(R^3 - \{0\})$ is dense in $\mathbf{L}_2(R^3)$, there exists a sequence $\{g_n\}$ in $\mathbf{C}_0^\infty(R^3 - \{0\})$ such that $\|G - g_n\| \to 0$. Since $\Phi_\pm^*$ is bounded, it follows from (6.39) and (8.2) that every $F \in N(A)^\perp$ is the $\mathbf{L}_2$-limit of the sequence

$$(\Phi_\pm^* g_n)(x) = \int_{R^3} w_\pm(x, p; g_n(p))\, dp. \tag{8.4}$$

In this sense, formula (8.2) can be interpreted as a plane wave expansion for fields $F \in N(A)^\perp$.

In order to prove (8.1), we consider an arbitrary $G \in \mathbf{L}_2(R^3)$ and set

$$H := \Phi_\pm \Phi_\pm^* G - G. \tag{8.5}$$

We have to show that $H=0$. Since $\Phi_{\pm}^{*}\Phi_{\pm}=I-P_{+0}$ by (6.37) and $\Phi_{\pm}^{*}G \in N(A)^{\perp}$ by (6.36), we obtain

$$\Phi_{\pm}^{*}H=(I-P_{+0})\,\Phi_{\pm}^{*}G-\Phi_{\pm}^{*}G=-P_{+0}\Phi_{\pm}^{*}G=0.$$

Since $\Phi_{\pm}^{*}H=0$, it follows from Lemma 7.1 that

$$\Phi_{\pm}^{*}(\psi_1\cdot H)=0 \tag{8.6}$$

for every bounded, piecewise continuous complex-valued function ψ, where ψ_1 is defined by (7.1). In fact, by observing [12, Lemma 5.3], we have for every $f\in \mathbf{C}_0^{\infty}(\Omega)$

$$\begin{aligned}(\Phi_{\pm}^{*}(\psi_1\cdot H),f)_{\mathbf{L}_2(\Omega)}&=(\psi_1\cdot H,\Phi_{\pm}f)_{\mathbf{L}_2(R^3)}=(H,\bar{\psi}_1\cdot\Phi_{\pm}f)_{\mathbf{L}_2(R^3)}\\&=(H,\Phi_{\pm}(\bar{\psi}(A)f))_{\mathbf{L}_2(R^3)}=(\Phi_{\pm}^{*}H,\bar{\psi}(A)f)=0,\end{aligned}$$

so that (8.6) holds.

Now we set as in [18, Sect. 6] for $0<\alpha<\beta<\infty$

$$\begin{aligned}\psi(\lambda)&:=e^{-it\sqrt{\lambda}}, &&\text{for}\quad \alpha^2\leqslant\lambda\leqslant\beta^2,\\&:=0, &&\text{for}\quad \lambda<\alpha^2\quad\text{and}\quad\lambda>\beta^2.\end{aligned}\tag{8.7}$$

By (7.1) we have

$$\begin{aligned}\psi_1(p)&=e^{-it|p|}, &&\text{for}\quad \alpha\leqslant|p|\leqslant\beta,\\&=0, &&\text{elsewhere}.\end{aligned}\tag{8.8}$$

Note that, for every given $h\in\mathbf{C}_0^{\infty}(R^3)$, there exists a sequence $\{g_n\}$ in $\mathbf{C}_0^{\infty}(R^3-\{0\})$ such that $\int|\psi_1 h-g_n|^2\,dp\to 0$ as $n\to\infty$. Hence we obtain, by applying (6.38) to $g=g_n$ and letting $n\to\infty$,

$$\Phi_{\pm}^{*}(\psi_1\cdot h)=\sum_{j=1}^{3}\int_{\alpha<|p|<\beta}e^{-it|p|}h_j(p)\,w_{\pm}(\cdot,p;e_j)\,dp \tag{8.9}$$

for $h=(h_1,h_2,h_3)\in\mathbf{C}_0^{\infty}(R^3)$. In particular, we have $\Phi_{\pm}^{*}(\psi_1\cdot h)\in\mathbf{C}(\bar{\Omega})$. Set $H=(H_1,H_2,H_3)$ and denote the restriction of the functional H_j to $C_0^{\infty}(\alpha<|p|<\beta)$ by H_j^r. By choosing a sequence $\{h_n\}$ in $\mathbf{C}_0^{\infty}(R^3)$ with $\|H-h_n\|\to 0$ as $n\to\infty$, we conclude from (8.6) and (8.9) that

$$\sum_{j=1}^{3}(e^{-it|p|}H_j^r,\overline{(w_{\pm})_k}\,(x,\cdot\,;e_j))_{L_2(\alpha<|p|<\beta)}=0 \tag{8.10}$$

for $x\in\bar{\Omega}$ and $k=1,2,3$ with $(w_{\pm})_k=e_k w_{\pm}$. Here $e^{-it|p|}H_j^r$ denotes the product of the function $g(p)=e^{-it|p|}$ and the functional

$H_j^r \in L_2(\alpha < |p| < \beta)$ (compare [12, Definition 5.1]). By (6.6), Eq. (8.10) can be written in the form

$$j(x)\, u_0(x, t) + u_\pm(x, t) = 0 \tag{8.11}$$

with

$$u_0(x, t) := \sum_{j=1}^{3} (e^{-it|p|} H_j^r, (\bar{w}_0)_k\, (x, \cdot\,; e_j))_{L_2(\alpha < |p| < \beta)} \tag{8.12}$$

and

$$u_\pm(x, t) := \sum_{j=1}^{3} (e^{-it|p|} H_j^r, (\bar{v}_\pm)_k\, (x, \cdot\,; e_j))_{L_2(\alpha < |p| < \beta)}. \tag{8.13}$$

We shall show below that

$$\int_{R^3} |u_0(x, t)|^2\, dx = \| H_k^r \|_{L_2(\alpha < |p| < \beta)} \tag{8.14}$$

for every real t and

$$\int_{R^3} |u_0(x, t)|^2\, dx \to 0 \qquad \text{as} \quad t \to \infty \tag{8.15}$$

for $k = 1, 2, 3$. Relations (8.14) and (8.15) imply that $H_k^r = 0$, and hence $H_k \varphi = 0$ for every $\varphi \in C_0^\infty(\alpha < |p| < \beta)$. Since α, β are arbitrary numbers with $0 < \alpha < \beta < \infty$, we have $H_k \varphi = 0$ for every $\varphi \in C_0^\infty(R^3 - \{0\})$, and hence $H = (H_1, H_2, H_3) = 0$, since $C_0^\infty(R^3 - \{0\})$ is dense in $C_0^\infty(R^3)$ with respect to the L_2-norm. By (8.5), $H = 0$ is equivalent to (8.1). Thus the proof of (8.1) is reduced to the verification of (8.14) and (8.15).

Verification of (8.14). Choose, for fixed k, a sequence $\{h_n\}$ in $C_0^\infty(\alpha < |p| < \beta)$ such that $\| H_k^r - h_n \| \to 0$ as $n \to \infty$ in $L_2(\alpha < |p| < \beta)$. Since $(w_0)_k\, (x, p; e_j) = (2\pi)^{-3/2}\, \delta_{jk} e^{ix \cdot p}$ by (6.1), formula (8.12) can be rewritten as

$$u_0(x, t) = \frac{1}{(2\pi)^{3/2}} \lim_{n \to \infty} \int_{\alpha < |p| < \beta} h_n(p)\, e^{i(x \cdot p - t|p|)}\, dp. \tag{8.16}$$

By Schwarz's inequality, the convergence is uniform with respect to x and t in $R^3 \times R$. Set

$$k_n(p, t) := e^{-it|p|} h_n(p). \tag{8.17}$$

Note that $k_n(\cdot, t) \in C_0^\infty(\alpha < |p| < \beta)$ and $|k_n(p, t)| = |h_n(p)|$. By (8.16) and (8.17), we have

$$u_0(x, t) = \lim_{n \to \infty} \hat{k}_n(-x, t), \tag{8.18}$$

where $\hat{k}_n$ denotes the classical Fourier transform of k_n with respect to the first variable. By applying Parseval's equation, we obtain

$$\int_{R^3} |\hat{k}_n(-x, t)|^2\, dx = \int_{R^3} |k_n(p, t)|^2\, dp = \int_{R^3} |h_n(p)|^2\, dp,$$

and hence

$$\int_{R^3} |\hat{k}_n(-x, t)|^2\, dx \to \|H_k^r\|^2_{L_2(\alpha < |p| < \beta)} \qquad \text{as} \quad n \to \infty. \tag{8.19}$$

Since the convergence in (8.16) and (8.18) is uniform, we get, by observing (8.19),

$$\begin{aligned} \int_{|x| < \rho} |u_0(x, t)|^2\, dx &= \lim_{n \to \infty} \int_{|x| < \rho} |\hat{k}_n(-x, t)|^2\, dx \\ &\leqslant \overline{\lim_{n \to \infty}} \int_{R^3} |\hat{k}_n(-x, t)|^2\, dx = \|H_k^r\|^2 \end{aligned}$$

for every $\rho > 0$. This estimate shows that the improper integral

$$\int_{R^3} |u_0(x, t)|^2\, dx$$

exists for every real t. Another application of Parseval's equation yields

$$\begin{aligned} \int_{|x| < \rho} |u_0(x, t) - \hat{k}_n(-x, t)|^2\, dx &= \lim_{m \to \infty} \int_{|x| < \rho} |\hat{k}_m(-x, t) - \hat{k}_n(-x, t)|^2\, dx \\ &\leqslant \overline{\lim_{m \to \infty}} \int_{R^3} |\hat{k}_m(-x, t) - \hat{k}_n(-x, t)|^2\, dx \\ &= \overline{\lim_{m \to \infty}} \int_{R^3} |h_m - h_n|^2\, dp = \|H_k^r - h_n\|^2 \end{aligned}$$

for every $\rho > 0$, and hence

$$\int_{R^3} |u_0(x, t) - \hat{k}_n(-x, t)|^2\, dx \leqslant \|H_k^r - h_n\|^2 \to 0 \qquad \text{as} \quad n \to \infty. \tag{8.20}$$

By applying the triangle inequality in $L_2(R^3)$, it follows from (8.20) that

$$\int_{R^3} |\hat{k}_n(-x, t)|^2 \, dx \to \int_{R^3} |u(x, t)|^2 \, dx \qquad \text{as} \quad n \to \infty. \tag{8.21}$$

By comparing (8.19) and (8.21), we obtain (8.14).

Verification of (8.15). We shall use the following elementary estimate:

LEMMA 8.1. *Assume that $\Delta v + \kappa^2 v = 0$ for $|x| \geqslant r_1$, with real $\kappa \neq 0$, and that $v = O(r^{-1})$ and $(\partial/\partial r - i\kappa) v = o(r^{-1})$ as $r = |x| \to \infty$. Then we have*

$$v(x) = \vartheta(x_0) \frac{e^{i\kappa|x|}}{|x|} + O(|x|^{-2}) \qquad \text{as} \quad |x| \to \infty \tag{8.22}$$

with $x_0 = x/|x|$ and

$$\vartheta(x_0) = \frac{1}{4\pi} \int_{|y| = r_1} \left[\frac{\partial}{\partial n} u(y) + \frac{i\kappa}{r} x_0 \cdot y u(y) \right] e^{-i\kappa x_0 \cdot y} \, dS_y. \tag{8.23}$$

The proof follows from the representation

$$v(x) = \frac{1}{4\pi} \int_{|y| = r_1} \left[\frac{\partial}{\partial n} u(y) \frac{e^{i\kappa|x-y|}}{|x-y|} - u(y) \frac{\partial}{\partial n_y} \frac{e^{i\kappa|x-y|}}{|x-y|} \right] dS_y \tag{8.24}$$

for $|x| > r_1$ (with $\partial/\partial n_y = (y/|y|) \cdot \nabla_y$) and the estimates

$$\begin{aligned} \frac{e^{i\kappa|x-y|}}{|x-y|} &= \frac{1}{|x|} \exp \left\{ i\kappa |x| \left(1 - 2 \frac{x \cdot y}{|x|^2}\right)^{1/2} \right\} + O(|x|^{-2}) \\ &= \frac{1}{|x|} e^{i\kappa|x| - i\kappa x_0 \cdot y} + O(|x|^{-2}) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial n_y} \frac{e^{i\kappa|x-y|}}{|x-y|} &= \frac{y}{|y|} \cdot \frac{y-x}{|y-x|} \left(i\kappa \frac{e^{i\kappa|x-y|}}{|x-y|} - \frac{e^{i\kappa|x-y|}}{|x-y|^2} \right) \\ &= -\frac{i\kappa x_0 \cdot y}{r} \frac{e^{i\kappa|x-y|}}{|x-y|} + O(|x|^{-2}) \\ &= -\frac{i\kappa x_0 \cdot y}{r} \frac{1}{|x|} e^{i\kappa|x| - i\kappa x_0 \cdot y} + O(|x|^{-2}) \end{aligned}$$

as $|x| \to \infty$. Note that both estimates hold uniformly with respect to y on the sphere $|y| = r_1$.

Now we apply Lemma 8.1 to the function $(v_\pm)_k = e_k \cdot v_\pm(\cdot, p; e_j)$ introduced in (6.6). Since $j(x) = 1$ for $|x| > r_0 + 1$, $(v_\pm)_k$ satisfies the assumptions of Lemma 8.1 with $r = r_0 + 1$ and $\kappa = \pm |p|$. Hence there exist functions $\vartheta^{jk}_\pm(x_0, p)$ and $q^{jk}_\pm(x, p)$ such that

$$(v_\pm)_k\,(x, p; e_j) = \vartheta^{jk}_\pm(x_0, p)\frac{e^{\pm i|p|\,|x|}}{|x|} + q^{jk}_\pm(x, p) \tag{8.25}$$

and

$$q^{jk}_\pm(x, p) = O(|x|^{-2}) \qquad \text{as} \quad |x| \to \infty. \tag{8.26}$$

The function $\vartheta^{jk}_\pm$ is given by (8.23) with $u = e_k \cdot v_\pm(\cdot, p; e_j)$, $\kappa = \pm |p|$, and $r = r_0 + 1$. In particular, it follows from (6.6), Lemma 6.1, and (8.23) that $\vartheta^{jk}_\pm(x_0, p)$ and $q^{jk}_\pm(x, p)$ depend continuously on both variables for $|x_0| = 1$, $x \in \bar{\Omega}$ and $p \in R^3 - \{0\}$. Furthermore, the proof of Lemma 8.1 shows that the estimate (8.26) holds uniformly for $\alpha \leqslant |p| \leqslant \beta$.

Now choose sequences $\{h_{nj}\}$ in $C_0^\infty(\alpha < |p| < \beta)$ such that $\|H^r_j - h_{nj}\| \to 0$ as $n \to \infty$ in $L_2(\alpha < |p| < \beta)$. It follows from (8.13) and (8.25) that

$$u_\pm(x, t) = u^1_\pm(x, t) + u^2_\pm(x, t) \tag{8.27}$$

for $(x, t) \in \bar{\Omega} + R$ with

$$u^1_\pm(x, t) = \frac{1}{|x|}\lim_{n\to\infty}\sum_{j=1}^{3}\int_{\alpha<|p|<\beta} e^{i(\pm|x| - t)|p|}\vartheta^{jk}_\pm(x_0, p)\, h_{nj}(p)\, dp \tag{8.28}$$

and

$$u^2_\pm(x, t) = \lim_{n\to\infty}\sum_{j=1}^{3}\int_{\alpha<|p|<\beta} e^{-it|p|}q^{jk}_\pm(x, p)\, h_{nj}(p)\, dp. \tag{8.29}$$

We have by (8.28), for fixed k,

$$u^1_\pm(x, t) = \frac{1}{|x|}\lim_{n\to\infty}\sum_{j=1}^{3} f^{nj}_\pm(\pm|x| - t, x_0) \tag{8.30}$$

with

$$f^{nj}_\pm(\tau, x_0) := \int_{\alpha<|p|<\beta} e^{i\tau|p|}\vartheta^{jk}_\pm(x_0, p)\, h_{nj}(p)\, dp. \tag{8.31}$$

Since $h_{nj} \in C_0^\infty(\alpha < |p| < \beta)$, we obtain, by setting $p = \rho p_0$ in (8.31) with $\rho = |p|$ and $|p_0| = 1$,

$$f^{nj}_\pm(\tau, x_0) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} e^{i\tau\rho} g^{nj}_\pm(\rho, x_0)\, d\rho \tag{8.32}$$

with

$$g_{\pm}^{nj}(\rho, x_0) = (2\pi)^{3/2} \rho^2 \int_{|p_0|=1} \vartheta_{\pm}^{jk}(x_0, \rho p_0)\, h_{nj}(\rho p_0)\, dS_{p_0}. \tag{8.33}$$

Note that $g_{\pm}^{nj}(\cdot, x_0) \in C_0^{\infty}(-\infty, \infty)$ and $\operatorname{supp} g_{\pm}^{nj}(\cdot, x_0) \in (\alpha, \beta)$. By (8.32), $f_{\pm}^{nj}$ is the Fourier transform of $g_{\pm}^{nj}$ with respect to the first variable. Hence, Parseval's equation implies that

$$\begin{aligned}
&\int_{-\infty}^{\infty} |f_{\pm}^{nj}(\tau, x_0)|^2\, d\tau \\
&= \int_{\alpha}^{\beta} |g_{\pm}^{nj}(\rho, x_0)|^2\, d\rho \\
&= (2\pi)^3 \int_{\rho=\alpha}^{\beta} \rho^4 \left| \int_{|p_0|=1} \vartheta_{\pm}^{jk}(x_0, \rho p_0)\, h_{nj}(\rho p_0)\, dS_{p_0} \right|^2 d\rho \\
&\leqslant (2\pi)^3 \int_{\rho=\alpha}^{\beta} \left[\int_{|p_0|=1} \rho^2 |\vartheta_{\pm}^{jk}(x_0, \rho p_0)|^2\, dS_{p_0} \right] \\
&\quad \cdot \left[\int_{|p_0|=1} \rho^2 |h_{nj}(\rho p_0)|^2\, dS_{p_0} \right] d\rho \\
&\leqslant C \int_{\alpha<|p|<\beta} |h_{nj}(p)|^2\, dp
\end{aligned}$$

with a suitable constant $C > 0$, since $\vartheta_{\pm}^{jk}(x_0, \rho p_0)$ is bounded for $|x_0| = 1$, $|p_0| = 1$ and $\alpha \leqslant \rho \leqslant \beta$. Set

$$f_{\pm}^{j}(\tau, x_0) := \lim_{n\to\infty} f_{\pm}^{nj}(\tau, x_0). \tag{8.34}$$

It follows from (8.31) and Schwarz's inequality that the limit in (8.34) is uniform with respect to τ and x_0 for $\tau \in R$ and $|x_0| = 1$. The above estimate implies that

$$\begin{aligned}
\int_{\tau_1}^{\tau_2} |f_{\pm}^{j}(\tau, x_0)|^2\, d\tau &= \lim_{n\to\infty} \int_{\tau_1}^{\tau_2} |f_{\pm}^{nj}(\tau, x_0)|^2 d\tau \\
&\leqslant C \varlimsup_{n\to\infty} \int_{\alpha<|p|<\beta} |h_{nj}(p)|^2\, dp = C\, \|H_j^r\|^2_{L_2(\alpha<|p|<\beta)}
\end{aligned}$$

for all real τ_1, τ_2. This yields

$$\int_{-\infty}^{\infty} |f_{\pm}^{j}(\tau, x_0)|^2\, d\tau \leqslant C\, \|H_j^r\|^2_{L_2(\alpha<|p|<\beta)}. \tag{8.35}$$

In particular, the improper integral on the left-hand side converges. Furthermore, since the convergence in (8.34) is uniform, we have by (8.30) for $\gamma > 0$ and $t \in R$

$$\int_{|x|<\gamma} |u_{\pm}^1(x,t)|^2\,dx = \int_{|x|<\gamma} \frac{1}{|x|^2} \left| \sum_{j=1}^{3} f_{\pm}^j(\pm|x|-t, x_0) \right|^2 dx$$

$$\leqslant 3 \sum_{j=1}^{3} \int_{|x_0|=1} \left[\int_0^{\gamma} |f_{\pm}^j(\pm r - t, x_0)|^2\,dr \right] dS_{x_0},$$

and hence, by substituting $\tau = \pm r - t$, $dr = \pm d\tau$ in the inner integral,

$$\int_{|x|<\gamma} |u_{\pm}^1(x,t)|^2\,dx \leqslant \pm 3 \sum_{j=1}^{3} \int_{|x_0|=1} \left[\int_{-t}^{\pm\gamma - t} |f_{\pm}^j(\tau, x_0)|^2\,d\tau \right] dS_{x_0}. \quad (8.36)$$

By observing (8.35), we obtain

$$\int_{|x|<\gamma} |u_{\pm}^1(x,t)|^2\,dx \leqslant 12\pi C \sum_{j=1}^{3} \|H_j^r\|^2 = 12\pi C \|H^r\|^2,$$

and hence

$$\int_{R^3} |u_{\pm}^1(x,t)|^2\,dx \leqslant 12\pi C \|H^r\|_{L_2(\alpha < |p| < \beta)} \quad (8.37)$$

for every real t. In particular, the improper integral on the left-hand side converges. Furthermore, since the integrals on the left-hand sides in (8.35) and (8.37) converge, it follows from (8.36), by letting $\gamma \to \infty$, that

$$\int_{R^3} |u_{\pm}^1(x,t)|^2\,dx \leqslant \pm 3 \sum_{j=1}^{3} \int_{|x_0|=1} \left[\int_{-t}^{\pm\infty} |f_{\pm}^j(\tau, x_0)|^2\,d\tau \right] dS_{x_0}. \quad (8.38)$$

By (8.35) the function

$$\psi(r) := \sum_{j=1}^{3} \int_{|x_0|=1} \left[\int_0^r |f_{\pm}^j(\tau, x_0)|^2\,d\tau \right] dS_{x_0}$$

is monotone and bounded by $12\pi C \|H^r\|^2$ in $(-\infty, \infty)$. Hence $\psi(r)$ converges to a finite limit $\alpha_{\pm}$ as $r \to \pm\infty$. By (8.38) we have

$$\int_{R^3} |u_{\pm}^1(x,t)|^2\,dx \leqslant \pm 3(\alpha_{\pm} - \psi(-t)),$$

and hence

$$\int_{R^3} |u_{\pm}^1(x,t)|^2\,dx \to 0 \qquad \text{as} \quad t \to \mp\infty. \quad (8.39)$$

Now we turn to the discussion of $u^2_{\pm}$. Since (8.26) holds uniformly for $\alpha < |p| < \beta$, there exists a $C_1 > 0$ such that

$$|q^{jk}_{\pm}(x, p)| \leqslant C_1 |x|^{-2} \quad \text{for} \quad x \in R^3 \quad \text{and} \quad \alpha < |p| < \beta.$$

Thus (8.29) implies that

$$\begin{aligned} |u^2_{\pm}(x, t)| &\leqslant \varlimsup_{n\to\infty} \frac{C_1}{|x|^2} \sum_{j=1}^{3} \int_{\alpha<|p|<\beta} |h_{nj}(p)|\, dp \\ &\leqslant \frac{4\pi}{3} \beta^3 \frac{C_1}{|x|^2} \sum_{j=1}^{3} \varlimsup_{n\to\infty} \int_{\alpha<|p|<\beta} |h_{nj}(p)|^2\, dp = \frac{C_2}{|x|^2} \end{aligned}$$

with $C_2 = (4\pi/3)\beta^3 C_1 \|H^r\|^2$. Since

$$\int_{|x|>r} \frac{dx}{|x|^4} = \frac{4\pi}{r},$$

we obtain, with $C := 4\pi C_2^2$,

$$\int_{|x|>r} |u^2_{\pm}(x, t)|^2\, dx \leqslant \frac{C}{r} \quad \text{for} \quad t \in R \quad \text{and} \quad r > r_0. \tag{8.40}$$

Let $\varepsilon > 0$ be given. By (8.39), there exists a $t_0 > 0$ such that

$$\int_{R^3} |u^1_+(x, t)|^2\, dx < \frac{\varepsilon}{8} \quad \text{for} \quad t < -t_0$$

and

$$\int_{R^3} |u^1_-(x, t)|^2\, dx < \frac{\varepsilon}{8} \quad \text{for} \quad t > t_0. \tag{8.41}$$

Furthermore, by (8.40), there exists a $R > r_0 + 1$ such that

$$\int_{|x|>R} |u^2_{\pm}(x, t)|^2\, dx < \frac{\varepsilon}{8} \quad \text{for} \quad t \in R. \tag{8.42}$$

It follows from (8.27), (8.41), and (8.42) that

$$\int_{|x|>R} |u_+(x, t)|^2\, dx < \frac{\varepsilon}{2} \quad \text{for} \quad t < -t_0 \tag{8.43}$$

and

$$\int_{|x|>R} |u_-(x, t)|^2\, dx < \frac{\varepsilon}{2} \quad \text{for} \quad t > t_0. \tag{8.44}$$

Since $j(x) = 1$ for $|x| > R > r_0 + 1$, we conclude from (8.11) that

$$\int_{|x|>R} |u_0(x,t)|^2\, dx < \frac{\varepsilon}{2} \qquad \text{for} \quad |t| > t_0. \tag{8.45}$$

In order to complete the proof of (8.15), we have to show that there exists a $t_1 > t_0$ such that

$$\int_{|x|<R} |u_0(x,t)|^2\, dx < \frac{\varepsilon}{2} \qquad \text{for} \quad t > t_1. \tag{8.46}$$

Since the convergence in (8.16) is uniform, there exists a $h \in C_0^\infty(\alpha < |p| < \beta)$ with

$$\int_{|x|<R} \left| u_0(x,t) - \frac{1}{(2\pi)^{3/2}} \int_{\alpha<|p|<\beta} h(p)\, e^{i(x\cdot p - t|p|)}\, dp \right|^2 dx < \varepsilon/4$$

for every t. Note that

$$\int_{|x|<R} \left| \int_{\alpha<|p|<\beta} h(p)\, e^{i(x\cdot p - t|p|)}\, dp \right|^2 dx$$

$$= \int_{|x|<R} \left| \int_{|p_0|=1} \left[\int_{\rho=\alpha}^{\beta} h(\rho p_0)\, e^{i\rho x\cdot p_0} \rho^2 e^{-it\rho}\, d\rho \right] dp_0 \right|^2 dx.$$

By performing an integration by parts in the inner integral (with $du = e^{-it\rho}\, d\rho$), it follows that the last expression converges to 0 as $t \to \infty$. In particular, by the triangle inequality in $L_2(|x| < R)$, there exists a $t_1 > t_0$ such that (8.46) holds. Formulas (8.45) and (8.46) yield

$$\int_{R^3} |u_0(x,t)|^2\, dx < \varepsilon \qquad \text{for} \quad t > t_0.$$

This concludes the proof of (8.15), and hence of (8.1), by the remarks after (8.15).

We collect the main results on the generalized Fourier transforms Φ_+ and Φ_- in the following theorem:

THEOREM 8.1. *Assume that Ω is the exterior of n disjoint bodies with boundaries $S_1, \ldots, S_n \in C^6$. Define $\Phi_+ f$ and $\Phi_- f$, for $f \in \mathbf{C}_0^\infty(\Omega)$, by* (6.3). *Then we have $\Phi_\pm f \in \mathbf{C}^\infty(R^3 - \{0\}) \cap \mathbf{L}_2(R^3)$ and*

$$\|\Phi_\pm f\|^2_{\mathbf{L}_2(R^3)} = \|f\|^2_{\mathbf{L}_2(\Omega)} - \|P_{+0} f\|^2_{\mathbf{L}_2(\Omega)}, \tag{8.47}$$

where P_{+0} denotes the projection of $\mathbf{L}_2(\Omega)$ onto the null space $N(A)$ of A characterized in Theorem 2.1. *By* (8.47), *Φ_+ and Φ_- can be extended, by* *continuity, to bounded linear operators on* $\mathbf{L}_2(\Omega)$. *Let* $\Phi_\pm^*: \mathbf{L}_2(R^3) \to \mathbf{L}_2(\Omega)$ *be the adjoint operator of* $\Phi_\pm$. *Then we have*

$$\Phi_\pm N(A) = \{0\}, \; \Phi_\pm N(A)^\perp = \mathbf{L}_2(R^2), \tag{8.48}$$

$$\Phi_\pm^* \mathbf{L}_2(R^3) = N(A)^\perp, \tag{8.49}$$

$$\Phi_\pm^* \Phi_\pm = I - P_{+0}, \tag{8.50}$$

$$\Phi_\pm \Phi_\pm^* = I. \tag{8.51}$$

In particular, the restriction of $\Phi_\pm$ *to* $N(A)^\perp$ *is a unitary operator from* $N(A)^\perp$ *onto* $\mathbf{L}_2(R^3)$. *If* $g \in \mathbf{C}_0^\infty(R^3 - \{0\})$, *then* $\Phi_\pm^* g$ *is given by* (6.38) *or* (6.39), *and we have* $\Phi_\pm^* g \in \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^\infty(\Omega)$. *The same results (with* P'_{+0} *instead of* P_{+0} *and* A' *instead of* A*) hold for the generalized Fourier transforms* Φ'_+ *and* Φ'_- *in the magnetic case.*

9. Orthogonal Decompositions

The generalized Fourier transforms lead to simple orthogonal decompositions of the Hilbert space $\mathbf{L}_2(\Omega)$ into closed subspaces, consisting of irrotational or solenoidal vector fields, respectively. We begin with the verification of some properties of the operators $\Phi_\pm^*$ and $(\Phi'_\pm)^*$. Recall that, by Lemma 6.1, $\Phi_\pm^* g \in \mathbf{C}^\infty(\Omega)$ and $(\Phi'_\pm)^* g \in \mathbf{C}^\infty(\Omega)$ if $g \in \mathbf{C}_0^\infty(R^3 - \{0\})$.

Lemma 9.1. *Assume that* $g \in C_0^\infty(R^3 - \{0\})$. *Then we have:*

(a) *If* $g(p) \perp p$ *for every* $p \in R^3 - \{0\}$, *then* $\nabla \cdot \Phi_\pm^* g = \nabla \cdot (\Phi'_\pm)^* g = 0$ *in* Ω;

(b) *if* $g(p) \parallel p$ *for every* $p \in R^3 - \{0\}$, *then* $\nabla \times \Phi_\pm^* g = \nabla \times (\Phi'_\pm)^* g = 0$ *in* Ω.

Proof. Recall that $w_\pm(\cdot, p; a) = w_0(\cdot, p; a) + E_\pm$ and $w'_\pm(\cdot, p; a) = w_0(\cdot, p; a) + H_\pm$, where $E_\pm$ and $H_\pm$ belong to $\mathbf{C}^2(\bar{\Omega}) \cap \mathbf{C}^\infty(\Omega)$ and are solutions of the boundary value problems

$$\begin{aligned} (\Delta + |p|^2) E_\pm &= 0 && \text{in} \quad \Omega, \\ n \times E_\pm &= -n \times w_0(\cdot, p; a), && \\ \nabla \cdot E_\pm &= -\nabla \cdot w_0(\cdot, p; a) && \text{on} \quad \partial\Omega, \\ E_\pm &= O(r^{-1}), && \\ \left(\frac{\partial}{\partial r} \mp i|p|\right) E_\pm &= o(r^{-1}) && \text{as} \quad r = |x| \to \infty \end{aligned} \tag{9.1}$$

and

$$\begin{aligned} (\Delta + |p|^2) H_\pm &= 0 && \text{in} \quad \Omega, \\ n \times (\nabla \times H_\pm) &= -n \times [\nabla \times w_0(\cdot, p; a)], \\ n \cdot H_\pm &= -n \cdot w_0(\cdot, p; a) && \text{on} \quad \partial\Omega, \\ H_\pm &= O(r^{-1}), \\ \left(\frac{\partial}{\partial r} \mp i|p|\right) H_\pm &= o(r^{-1}) && \text{as} \quad r = |x| \to \infty, \end{aligned} \tag{9.2}$$

respectively. By [11, formula (2.3)] and (5.2), also the derivatives of $E_\pm$ and $H_\pm$ satisfy the radiation condition (1.4) with $\kappa = \pm |p|$.

Verification of (a). Note that $\nabla_x \cdot w_0(x, p; a) = 0$ if $p \perp a$ by (6.1), since $\nabla_x \cdot (a e^{ix \cdot p}) = ia \cdot p e^{ix \cdot p}$. Hence (9.1) implies for $p \perp a$ that $\varphi := \nabla \cdot E_\pm$ is a solution of the scalar Dirichlet problem $(\Delta + \kappa^2)\varphi = 0$ in Ω, $\varphi = 0$ on $\partial\Omega$, and $\varphi = O(r^{-1})$, $(\partial/\partial r - i\kappa)\varphi = o(r^{-1})$ as $r \to |x| \to \infty$, with $\kappa = \pm |p|$. By the well-known uniqueness theorem for this problem (compare, for example, [9, Satz 2]), we obtain $\nabla \cdot E_\pm = \varphi = 0$ in Ω, and hence

$$\nabla_x \cdot w_\pm(x, p; a) = 0 \qquad \text{if} \quad p \perp a. \tag{9.3}$$

Furthermore, if $p \perp a$, it follows from (9.2) and $\nabla \cdot w_0(\cdot, p; a) = 0$ that

$$\begin{aligned} \frac{\partial}{\partial n} \nabla \cdot H_\pm &= n \cdot \nabla(\nabla \cdot H_\pm) = n \cdot [\nabla \times (\nabla \times H_\pm) + \Delta H_\pm] \\ &= -\nabla_0 \cdot [n \times (\nabla \times H_\pm)] - |p|^2 n \cdot H_\pm \\ &= \nabla_0 \cdot [n \times (\nabla \times w_0)] + |p|^2 n \cdot w_0 \\ &= -n \cdot [\nabla \times (\nabla \times w_0) + \Delta w_0] = -n \cdot \nabla(\nabla \cdot w_0) = 0 \end{aligned}$$

on $\partial\Omega$, where $\nabla_0 \cdot a$ denotes the surface divergence of the tangential field a. Hence $\psi := \nabla \cdot H_\pm$ is a solution of the scalar Neumann problem $(\Delta + \kappa^2)\psi = 0$ in Ω, $(\partial/\partial n)\psi = 0$ on $\partial\Omega$, and $\psi = O(r^{-1})$, $(\partial/\partial r - i\kappa)\psi = o(r^{-1})$ as $r = |x| \to \infty$, with $\kappa = \pm |p|$. This implies that $\nabla \cdot H_\pm = \psi = 0$ in Ω, and hence

$$\nabla_x \cdot w'_\pm(x, p; a) = 0 \qquad \text{if} \quad p \perp a. \tag{9.4}$$

Now Lemma 9.1(a) follows from (6.39), (9.3), (9.4), and the formula

$$((\Phi'_\pm)^* g)(x) = \int_{R^3} w'_\pm(x, p; g(p))\, dp \qquad \text{for} \quad g \in \mathbf{C}_0^\infty(R^3 - \{0\}), \tag{9.5}$$

which can be proved in the same way as (6.39).

Verification of (b). Since $\nabla_x \times (ae^{ix\cdot p}) = ip \times ae^{ix\cdot p} = 0$ if $p \| a$, we have $\nabla_x \times w_0(x, p; a) = 0$. Set $A := \nabla \times E_\pm$. It follows from (9.1) that, for $p \| a$,

$$\begin{aligned} n \cdot A &= n \cdot (\nabla \times E_\pm) = -\nabla_0 \cdot (n \times E_\pm) = \nabla_0 \cdot (n \times w_0) \\ &= -n \cdot (\nabla \times w_0) = 0 \end{aligned}$$

and

$$\begin{aligned} n \times (\nabla \times A) &= n \times [\nabla \times (\nabla \times E_\pm)] = n \times [\nabla(\nabla \cdot E_\pm)] - \Delta E_\pm] \\ &= n \times [\nabla_0(\nabla \cdot E_\pm)] + |p|^2\, n \times E_\pm \\ &= -n \times [\nabla_0(\nabla \cdot w_0)] - |p|^2\, n \times w_0 \\ &= -n \times [\nabla(\nabla \cdot w_0) - \Delta w_0] = -n \times [\nabla \times (\nabla \times w_0)] = 0. \end{aligned}$$

Hence A is a solution of the boundary value problem (A′), formulated at the beginning of Section 5, with $F = 0$ and $\kappa = \pm |p|$. Thus the uniqueness part of Lemma 5.1 implies that $\nabla \times E_\pm = A = 0$ in Ω, and hence

$$\nabla_x \times w_\pm(x, p; a) = 0 \qquad \text{if} \quad p \| a. \tag{9.6}$$

Now set $B := \nabla \times H_\pm$. If $p \| a$, B is a solution of the boundary value problem (A), formulated at the beginning of Section 4, with $F = 0$ and $\kappa = \pm |p|$, since $\nabla \cdot B = 0$ in Ω and $n \times B = n \times (\nabla \times H_\pm) = -n \times (\nabla \times w_0)$ by (9.2). Thus the uniqueness part of Lemma 4.2 implies that $\nabla \times H_\pm = B = 0$ in Ω, and hence

$$\nabla_x \times w'_\pm(x, p; a) = 0 \qquad \text{if} \quad p \| a. \tag{9.7}$$

Lemma 9.1(b) follows from (6.39), (9.5), (9.6), and (9.7).

Set

$$g^1(p) := \frac{p \cdot g(p)}{|p|^2}\, p, \qquad g^2(p) := g(p) - g^1(p) = \frac{1}{|p|^2}\, p \times [g(p) \times p] \tag{9.8}$$

for $g \in \mathbf{C}_0^\infty(R^3 - \{0\})$. Note that g^1, $g^2 \in \mathbf{C}_0^\infty(R^3 - \{0\})$, and $g^1(p) \| p$ and $g^2(p) \perp p$ for every $p \in R - \{0\}$. Hence Lemma 9.1 implies that

$$\nabla \times (\Phi_\pm^* g^1) = 0 \qquad \text{and} \qquad \nabla \cdot (\Phi_\pm^* g^2) = 0 \qquad \text{in} \quad \Omega \tag{9.9}$$

for $g \in C_0^\infty(R^3 - \{0\})$. The operators $g \to g^1$ and $g \to g^2$ from $\mathbf{C}_0^\infty(R^3 - \{0\})$ into $\mathbf{C}_0^\infty(R^3 - \{0\})$ are bounded with respect to the $\mathbf{L}_2$-norm. Hence these operators can be uniquely extended to bounded operators $G \to G^1$ and $G \to G^2$ from $\mathbf{L}_2(R^2)$ into $\mathbf{L}_2(R^3)$. The relations (9.9) remain valid for $G \in \mathbf{L}_2(R^2)$ if the differential operators $\nabla \times$ and $\nabla \cdot$ are interpreted in the sense of distributions. In fact, choose a sequence $\{g_n\}$ in $\mathbf{C}_0^\infty(R^3 - \{0\})$ such that $\|G-g_n\|\to 0$ in $\mathbf{L}_2(R^3)$. Since the operators $G\to G^1$, $G\to G^2$ and $\Phi_\pm^*$ are bounded, we have

$$\|\Phi_\pm^* G^1-\Phi_\pm^* g_n^1\|_{\mathbf{L}_2(\Omega)}\to 0 \qquad \text{and} \qquad \|\Phi_\pm^* G^2-\Phi_\pm^* g_n^2\|_{\mathbf{L}_2(\Omega)}\to 0. \quad (9.10)$$

Denote the ith component of the vector $\Phi_\pm^* G^1$ by $(\Phi_\pm^* G^1)_i$. Since $\nabla\times(\Phi_\pm^* g_n^1)=0$ by (9.9), we obtain for $\varphi\in C_0^\infty(\Omega)$ and i, $k=1,2,3$, by applying the definition of distribution derivatives, the limit relation (9.10), and the integral theorem of Gauss,

$$\begin{aligned}
&[\partial_i(\Phi_\pm^* G^1)_k-\partial_k(\Phi_\pm^* G^1)_i]\,\varphi\\
&\quad=-(\Phi_\pm^* G^1)_k\,(\partial_i\varphi)+(\Phi_\pm^* G^1)_i\,(\partial_k\varphi)\\
&\quad=\lim_{n\to\infty}\int_\Omega[-(\Phi_\pm^* g_n^1)_k\,\partial_i\varphi+(\Phi_\pm^* g_n^1)_i\,\partial_k\varphi]\,dx\\
&\quad=\lim_{n\to\infty}\int_\Omega[\partial_i(\Phi_\pm^* g_n^1)_k-\partial_k(\Phi_\pm^* g_n^1)_i]\,\varphi\,dx=0,
\end{aligned}$$

and hence $\nabla\times(\Phi_\pm^* G^1)=0$. A similar argument implies that $\nabla\cdot(\Phi_\pm^* G^2)=0$.

Now we show that

$$(\Phi_\pm^* G^1,\Phi_\pm^* H^2)_{\mathbf{L}_2(\Omega)}=0 \qquad \text{for} \quad G,H\in\mathbf{L}_2(R^3). \quad (9.11)$$

Choose sequences $\{g_n\}$ and $\{h_n\}$ in $\mathbf{C}_0^\infty(R^3-\{0\})$ such that $\|G-g_n\|\to 0$ and $\|H-h_n\|\to 0$ in $\mathbf{L}_2(R^3)$. It follows from (6.34), (8.1), and (6.36) that

$$\begin{aligned}
&(\Phi_\pm^* G^1,\Phi_\pm^* H^2)_{\mathbf{L}_2(\Omega)}\\
&\quad=(\Phi_\pm\Phi_\pm^* G^1,\Phi_\pm\Phi_\pm^* H^2)_{\mathbf{L}_2(R^3)}+(P_{+0}\Phi_\pm^* G^1,P_{+0}\Phi_\pm^* G^2)_{\mathbf{L}_2(\Omega)}\\
&\quad=(G^1,H^2)_{\mathbf{L}_2(R^3)}=\lim_{n\to\infty}(g_n^1,h_n^2)_{\mathbf{L}_2(R^3)}\\
&\quad=\lim_{n\to\infty}\int g_n^1\overline{h_n^2}\,dp=0.
\end{aligned}$$

since $g_n^1(p)\perp h_n^2(p)$ for $p\in R^3-\{0\}$ by (9.8). This completes the verification of (9.11).

Our results on the operators $G\to G^i$ $(i=1,2)$ lead to an orthogonal decomposition of the Hilbert space $\mathbf{L}_2(\Omega)$. Set

$$P_\pm^i F:=\Phi_\pm^*[(\Phi_\pm F)^i] \qquad \text{for} \quad F\in\mathbf{L}_2(\Omega) \qquad (i=1,2). \quad (9.12)$$

Note that

$$G=G^1+G^2 \qquad \text{for} \quad G\in\mathbf{L}_2(R^3), \quad (9.13)$$

$$(G^i)^k=\delta_{ik}G^i \qquad \text{for} \quad G\in\mathbf{L}_2(R^3) \quad \text{and} \quad i,k=1,2 \quad (9.14)$$

and

$$(G^i, H) = (G, H^i) \qquad \text{for} \quad G, H \in \mathbf{L}_2(R^3) \quad \text{and} \quad i = 1, 2. \tag{9.15}$$

By (9.8), these relations are obvious if $G, H \in C_0^\infty(R^3 - \{0\})$. In the general case, they follow by approximating G and H by sequences $\{g_n\}$ and $\{h_n\}$ in $\mathbf{C}_0^\infty(R^3 - \{0\})$ with respect to the $\mathbf{L}_2$-norm. By applying (8.1), (9.14), and (9.15), we obtain for $F, G \in \mathbf{L}_2(\Omega)$

$$\begin{aligned} P_\pm^i P_\pm^k F &= \Phi_\pm^* \{[\Phi_\pm \Phi_\pm^* (\Phi_\pm F)^k]^i\} = \Phi_\pm^* \{[(\Phi_\pm F)^k]^i\} \\ &= \delta_{ik} \Phi_\pm^* [(\Phi_\pm F)^i] = \delta_{ik} P_\pm^i F \end{aligned}$$

and

$$\begin{aligned} (P_\pm^i F, G)_{\mathbf{L}_2(\Omega)} &= (\Phi_\pm^* [(\Phi_\pm F)^i], G)_{\mathbf{L}_2(\Omega)} = ((\Phi_\pm F)^i, \Phi_\pm G)_{\mathbf{L}_2(R^3)} \\ &= (\Phi_\pm F, (\Phi_\pm G)^i)_{\mathbf{L}_2(R^3)} = (F, \Phi_\pm^* [(\Phi_\pm G)^i])_{\mathbf{L}_2(\Omega)} \\ &= (F, P_\pm^i G)_{\mathbf{L}_2(\Omega)}, \end{aligned}$$

and hence

$$P_\pm^i P_\pm^k = \delta_{ik} P_\pm^i \qquad \text{for} \quad i, k = 1, 2 \tag{9.16}$$

and

$$(P_\pm^i)^* = P_\pm^i \qquad \text{for} \quad i = 1, 2. \tag{9.17}$$

Furthermore, we have by (9.13) and (6.37)

$$P_\pm^1 F + P_\pm^2 F = \Phi_\pm^* [(\Phi_\pm F)^1 + (\Phi_\pm F)^2] = \Phi_\pm^* \Phi_\pm F = (I - P_{+0}) F,$$

and hence

$$P_{+0} + P_\pm^1 + P_\pm^2 = I. \tag{9.18}$$

It follows from (6.36) and (9.12) that

$$P_{+0} P_\pm^i = 0 \qquad \text{for} \quad i = 1, 2. \tag{9.19}$$

Relations (9.16)–(9.19) show that $P_\pm^i$ is an orthogonal projection of the Hilbert space $\mathbf{L}_2(\Omega)$ and that the projections P_{+0}, $P_\pm^1$, $P_\pm^2$ yield an orthogonal decomposition of $\mathbf{L}_2(\Omega)$ into the closed subspaces $N(A)$, $M_\pm^1$ and $M_\pm^2$ with

$$M_\pm^i := \{P_\pm^i F : F \in \mathbf{L}_2(\Omega)\} \qquad (i = 1, 2). \tag{9.20}$$

In order to characterize the ranges $M_\pm^1$ of the projections $P_\pm^i$ $(i = 1, 2)$, we introduce the linear spaces

$$\begin{aligned} M_1 &:= \{f \in \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^1(\Omega) \cap N(A)^\perp : \nabla \times f = 0 \text{ in } \Omega, n \times f = 0 \text{ on } \partial\Omega, \text{ and} \\ &\qquad f = O(r^{-2}) \text{ as } r = |x| \to \infty\}, \\ M_2 &:= \{f \in \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^1(\Omega) \cap N(A)^\perp : \nabla \cdot f = 0 \text{ in } \Omega \text{ and} \\ &\qquad f = O(r^{-2}) \text{ as } r = |x| \to \infty\}. \end{aligned}$$

Furthermore, we denote the completion of M_i with respect to the $\mathbf{L}_2$-norm with $\bar{M}_i$. We shall prove:

LEMMA 9.2. *The ranges $M^i_\pm$ of the projections $P^i_\pm$, introduced in* (9.12), *are given by*

$$M^1_+ = M^1_- = \bar{M}_1 \qquad \text{and} \qquad M^2_+ = M^2_- = \bar{M}_2.$$

In particular, we have $P^i_+ = P^i_-$ for $i = 1, 2$.

Proof. By using (6.38), (6.6), (6.1), and (8.25) and substituting $p = \rho p_0$ with $p_0 = p/|p|$ in the second term, we obtain for $g = (g_1, g_2, g_3) \in \mathbf{C}_0^\infty(R^3 - \{0\})$ and $|x| > r_0 + 1$

$$(\Phi^*_\pm g)(x) = \frac{1}{(2\pi)^{3/2}} \int_{R^3} e^{ip\cdot x} g(p)\, dp + \frac{1}{|x|} \sum_{j,k=1}^{3} e_k \int_{\rho=\alpha}^{\beta} e^{\pm i\rho|x|}$$
$$\times \left[\iint_{|p_0|=1} \vartheta^{jk}_\pm(x_0, \rho p_0)\, g_j(\rho p_0)\, \rho^2\, dS_{p_0}\right] d\rho$$
$$+ \sum_{j,k=1}^{3} e_k \int_{R^3} q^{jk}_\pm(x, p)\, g_j(p)\, dp,$$

where α, β are positive numbers with $\operatorname{supp} g \subset \{x: \alpha < |x| < \beta\}$. It follows from (8.23), with $u = (v_\pm)_k(\cdot, p; e_j)$ and $\kappa = \pm|p|$, and Lemma 6.1 that the inner integral in the second term has continuous derivatives with respect to ρ, which are uniformly bounded for $|x_0| = 1$ and $\alpha \leqslant \rho \leqslant \beta$. Hence an integration by parts implies that the second term can be estimated by $C_1 |x|^{-2}$ for $|x| > r$ with a suitable constant C_1. Similar estimates can be obtained for the first term, by integrating by parts, and for the third term, by using (8.26). This yields

$$\Phi^*_\pm g = O(r^{-2}) \qquad \text{as} \quad r = |x| \to \infty \qquad \text{for} \quad g \in \mathbf{C}_0^\infty(R^3 - \{0\}). \tag{9.21}$$

It follows from (6.39), Lemma 9.1, (9.21), and $n \times w_\pm = 0$ on $\partial\Omega$ that

$$\Phi^*_\pm(g^1) \in M_1 \qquad \text{and} \qquad \Phi^*_\pm(g^2) \in M_2 \qquad \text{for} \quad g \in \mathbf{C}_0^\infty(R^3 - \{0\}). \tag{9.22}$$

Since $\mathbf{C}_0^\infty(R^3 - \{0\})$ is dense in $\mathbf{L}_2(R^3)$ and since the operators $G \to G^i$ and $\Phi^*_\pm$ are bounded, (9.22) implies that

$$\Phi^*_\pm(G^1) \in \bar{M}_1 \qquad \text{and} \qquad \Phi^*_\pm(G^2) \in \bar{M}_2 \qquad \text{for} \quad G \in \mathbf{L}_2(R^3). \tag{9.23}$$

In particular, we have by (9.12)

$$P^1_\pm F \in \bar{M}_1 \qquad \text{and} \qquad P^2_\pm F \in \bar{M}_2 \qquad \text{for} \quad F \in \mathbf{L}_2(\Omega), \tag{9.23}$$

and hence $M^i_+ \subset \bar{M}_i$ and $M^i_- \subset \bar{M}_i$ for $i = 1, 2$.

Now we show that the linear spaces $\bar{M}_1$ and $\bar{M}_2$ are orthogonal:

$$(E, H)_{L_2(\Omega)} = 0 \qquad \text{for} \quad E \in \bar{M}_1, \ H \in \bar{M}_2. \tag{9.24}$$

Since M_1 and M_2 are dense in $\bar{M}_1$ and $\bar{M}_2$, it is sufficient to verify (9.24) for $E \in M_1$ and $H \in M_2$. Consider the fields $E_1, ..., E_n$ introduced in (2.1). As in the proof of Lemma 2.4, we can find numbers $c_1, ..., c_n$ such that the field

$$H' := H - \sum_{k=1}^{n} c_k E_k, \tag{9.25}$$

with given $H \in M_2$, satisfies the conditions

$$\int_{S_i} n \cdot H' \, dS = 0 \qquad (i = 1, ..., n). \tag{9.26}$$

Since $\nabla \cdot H' = 0$ in Ω and $H' = O(|x|^{-2})$ as $|x| \to \infty$, it follows from (9.26) that there exists a field $G \in \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^1(\Omega)$ such that

$$\nabla \times G = H' \qquad \text{in} \quad \Omega \tag{9.27}$$

and

$$G = O(|x|^{-1}) \qquad \text{as} \quad |x| \to \infty. \tag{9.28}$$

A construction of a field G with these properties is described in [13, p. 381–382, 5. Schritt; 14, p. 105] and uses [14, Lemma 8.1]. Now consider a field $E \in M_1$. Since $E_1, ..., E_n \in N(A)$ and $E \in N(A)^\perp$, we obtain by (9.25), (9.27), (9.28) and the properties of the space M_1

$$\int_\Omega E \cdot \bar{H} \, dx = \int_\Omega E \cdot \bar{H}' \, dx = \int_\Omega E \cdot (\nabla \times \bar{G}) \, dx$$

$$= \int_{\partial\Omega} (n \times E) \cdot \bar{G} \, dS + \int_\Omega (\nabla \times E) \cdot \bar{G} \, dx = 0$$

so that (9.24) holds for $E \in M_1$ and $H \in M_2$, and hence also for $E \in \bar{M}_1$ and $H \in \bar{M}_2$.

Since $P_{+0} F = 0$ for $F \in N(A)^\perp$, (9.18) implies

$$F = P^1_\pm F + P^2_\pm F \qquad \text{for} \quad F \in N(A)^\perp. \tag{9.29}$$

Formula (9.29), in connection with (9.23) and $\bar{M}_1 \perp \bar{M}_2$, shows that

$$N(A)^\perp = \bar{M}_1 \oplus \bar{M}_2 \tag{9.30}$$

and

$$\operatorname{range}(P^1_\pm) = \bar{M}_1, \qquad \operatorname{range}(P^2_\pm) = \bar{M}_2. \tag{9.31}$$

This completes the proof of Lemma 9.2.

In a similar way, by setting

$$\tilde{P}^i_{\pm} F := (\Phi'_{\pm})^* \, [(\Phi'_{\pm} F)^i] \tag{9.32}$$

and

$$\tilde{M}^i_{\pm} := \{\tilde{P}^i_{\pm} F : F \in \mathbf{L}_2(\Omega)\} \tag{9.33}$$

(i = 1, 2), we obtain a second orthogonal decomposition

$$\mathbf{L}_2(\Omega) = N(A') \oplus \tilde{M}^1_{\pm} \oplus \tilde{M}^2_{\pm} \tag{9.34}$$

of $\mathbf{L}_2(\Omega)$ into closed subspaces. Set

$$\begin{aligned} M'_1 := \{ & f \in \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^1(\Omega) \cap N(A')^{\perp} : \nabla \times f = 0 \text{ in } \Omega \\ & \text{and } f = O(r^{-2}) \text{ as } r = |x| \to \infty \}, \\ M'_2 := \{ & f \in \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^1(\Omega) \cap N(A')^{\perp} : \nabla \cdot f = 0 \text{ in } \Omega, \\ & n \cdot f = 0 \text{ on } \partial\Omega \text{ and } f = O(r^{-2}) \text{ as } r = |x| \to \infty \}. \end{aligned}$$

We shall prove in analogy to Lemma 9.2:

LEMMA 9.3. *The ranges $\tilde{M}_{\pm i}$ of the projections $\tilde{P}^i_{\pm}$ are given by*

$$\tilde{M}^1_{+} = \tilde{M}^1_{+} = \bar{M}'_1 \qquad \text{and} \qquad \tilde{M}^2_{+} = \tilde{M}^2_{-} = \bar{M}'_2.$$

In particular, we have $\tilde{P}^i_{+} = \tilde{P}^i_{-}$ for $i = 1, 2$.

Proof. The same argument as in the proof of (9.21) yields

$$(\Phi'_{\pm})^* g = O(r^{-2}) \qquad \text{as} \quad r = |x| \to \infty \quad \text{for} \quad g \in \mathbf{C}_0^{\infty}(R^3 - \{0\}). \tag{9.35}$$

It follows from (9.5), Lemma 9.1, (9.35), and $n \cdot w'_{\pm} = 0$ on $\partial\Omega$ that

$$(\Phi'_{\pm})^* (g^1) \in M'_1 \quad \text{and} \quad (\Phi'_{\pm})^* (g^2) \in M'_2 \quad \text{for } g \in \mathbf{C}_0^{\infty}(R^3 - \{0\}), \tag{9.36}$$

and hence

$$(\Phi'_{\pm})^* (G^1) \in \bar{M}'_1 \quad \text{and} \quad (\Phi'_{\pm})^* (G^2) \in \bar{M}'_2 \quad \text{for} \quad G \in \mathbf{L}_2(R^3). \tag{9.37}$$

In particular, we have by (9.32)

$$\tilde{P}^1_{\pm} F \in \bar{M}'_1 \quad \text{and} \quad \tilde{P}^2_{\pm} F \in \bar{M}'_2 \quad \text{for} \quad F \in \mathbf{L}_2(\Omega). \tag{9.38}$$

In order to show that $\bar{M}_1'$ and $\bar{M}_2'$ are orthogonal, we consider fields $E \in M_1'$ and $H \in M_2'$. Let $H_1, \ldots, H_p$ be the fields defined by (3.2) and (3.4) and set

$$E' := E - \sum_{j=1}^{p} a_j H_j \qquad \text{with} \qquad a_j := \int_{C_j} E \cdot t\, ds. \tag{9.39}$$

Note that $\nabla \times E' = 0$ in Ω and $E = O(|x|^{-2})$ as $|x| \to \infty$. It follows as in Section 3 (compare (3.10)) that

$$\int_C E' \cdot t\, ds = 0 \tag{9.40}$$

for every closed curve C in $\bar{\Omega}$. Hence, by (3.11), we can find a function $\psi \in C^1(\bar{\Omega})$ such that

$$\nabla \psi = E' \qquad \text{in} \quad \Omega \tag{9.41}$$

and

$$\psi = O(|x|^{-1}) \qquad \text{as} \quad |x| \to \infty. \tag{9.42}$$

Since $H_1, \ldots, H_p \in N(A')$ and $H \in N(A')^{\perp}$, we obtain by (9.39), (9.41), (9.42), and the properties of the space M_2'

$$\int_\Omega E \cdot \bar{H}\, dx = \int_\Omega E' \cdot \bar{H}\, dx = \int_\Omega \nabla\psi \cdot \bar{H}\, dx$$

$$= -\int_{\partial\Omega} \psi n \cdot \bar{H}\, dS - \int_\Omega \psi \nabla \cdot \bar{H}\, dx = 0.$$

This implies that $(E, H) = 0$ for $E \in M_1'$ and $H \in M_2'$, and hence also for $E \in \bar{M}_1'$ and $H \in \bar{M}_2'$, so that the spaces $\bar{M}_1'$ and $\bar{M}_2'$ are orthogonal.

In analogy to (9.29), we have

$$F = \tilde{P}_\pm^1 F + \tilde{P}_\pm^2 F \qquad \text{for} \quad F \in N(A')^{\perp}. \tag{9.43}$$

Formula (9.43), in connection with (9.38) and $\bar{M}_1' \perp \bar{M}_2'$, yields

$$N(A')^{\perp} = \bar{M}_1' \oplus \bar{M}_2' \tag{9.44}$$

and

$$\text{range}(\tilde{P}_\pm^1) = \bar{M}_1', \qquad \text{range}(\tilde{P}_\pm^2) = \bar{M}_2'. \tag{9.45}$$

This completes the proof of Lemma 9.3.

We collect the main results of this section in

THEOREM 9.1. *Assume that Ω is the exterior of n disjoint bodies with boundaries $S_1,\ldots, S_n \in C^6$ and consider the linear subspaces M_1, M_2, M_1', M_2'* of $\mathbf{L}_2(\Omega)$ *introduced above. Then the following two orthogonal decompositions of* $\mathbf{L}_2(\Omega)$ *hold:*

$$\mathbf{L}_2(\Omega) = N(A) \oplus \bar{M}_1 \oplus \bar{M}_2, \tag{9.46}$$

$$\mathbf{L}_2(\Omega) = N(A') \oplus \bar{M}_1' \oplus \bar{M}_2'. \tag{9.47}$$

The corresponding projections are P_{+0}, P^1_+, P^2_+ in the first case and P'_{+0}, $\tilde{P}^1_+$, $\tilde{P}^2_+$ in the second case, where $P^i_\pm$ and $\tilde{P}^i_\pm$ are defined by (9.12) *and* (9.32), *respectively, and $G \to G^i$ is the continuous extension of the mapping* (9.8) *to* $\mathbf{L}_2(R^3)$. *Furthermore, the identities $P^i_+ = P^i_-$ and $\tilde{P}^i_+ = \tilde{P}^i_-$ hold for $i = 1, 2$. In addition, we have, in the sense of distributions,*

$$\nabla \times F = 0 \qquad in \quad N(A) \oplus \bar{M}_1 \qquad and\ in \quad N(A') \oplus \bar{M}_1' \tag{9.48}$$

and

$$\nabla \cdot F = 0 \qquad in \quad N(A) \oplus \bar{M}_2 \qquad and\ in \quad N(A') \oplus \bar{M}_2'. \tag{9.49}$$

By the definition of the spaces M_1 and M_2', the properties $F \in \bar{M}_1$ and $G \in \bar{M}_2'$ contain, in addition to (9.48) and (9.49), weak versions of the classical boundary conditions $n \times F = 0$ and $n \cdot G = 0$, respectively. The orthogonal decompositions of $\mathbf{L}_2(\Omega)$ into linear subspaces of irrotational and solenoidal fields, described in Theorem 9.1, are closely related to the decompositions studied by Weyl in his famous paper [17]. By Theorem 9.1, these decompositions correspond, via the generalized Fourier transforms ϕ_+ and ϕ'_+ from $\mathbf{L}_2(\Omega)$ into $\mathbf{L}_2(R^3)$, to the decomposition of the elements of $\mathbf{L}_2(R^3)$ into radial components and fields which are orthogonal to the radial directions.

10. REGULARITY CONSIDERATIONS

This section is devoted to the proofs of Lemmas 2.1 and 3.1. We shall use the notations introduced in [16, Sect. 3]. Let Ω and E satisfy the assumptions of Lemma 2.1. Since $E \in \mathbf{C}(\Omega)$, it is sufficient to study the behavior of E near an arbitrary boundary point x_0. Choose δ and ζ as in [16, Sect. 3] and consider the fields E_1 and E_1^+ defined in [16, Lemmas 3.1 and 3.2], respectively. Our first aim is to show that $E_1^+ \in \mathbf{V}_2$.

It is convenient to set $E_1^+(u) = 0$ for $u \in \bar{R}_3^+ - Z(\delta)$, where $R_3^+ := \{u \in R^3: u_3 > 0\}$ so that $E_1^+ \in \mathbf{C}(R_3^+)$. Choose $h \in C_0^\infty(R^2)$ such that

$$\operatorname{supp} h = \{u' : |u'| \leqslant 1\}, \qquad h \geqslant 0, \quad \int h\, du' = 1 \tag{10.1}$$

with $u' = (u_1, u_2)$, $|u'| = (u_1^2 + u_2^2)^{1/2}$ and set

$$F_k(u) := \int_{|v'|<1} E_1^+(u' + v'/k, u_3)\, h(v')\, dv'. \tag{10.2}$$

Note that $F_k \to E_1^+$ uniformly in $\overline{Z(\delta)}$. By substituting $u' + v'/k = w'$, $v' = k(w' - u')$, $dv' = k^2\, dw'$, we obtain

$$\partial_i F_k \in \mathbf{C}(\overline{Z(\delta)}) \qquad \text{for} \quad i = 1, 2. \tag{10.3}$$

Since $n \times E = 0$ on $\partial\Omega$, we have $E_{1i}^+ := t_i \cdot SE_1 = 0$ for $u_3 = 0$ and $i = 1, 2$, and hence

$$F_{k1} = F_{k2} = 0 \qquad \text{for} \quad u_3 = 0 \tag{10.4}$$

(with $F_k = (F_{k1}, F_{k2}, F_{k3})$. Furthermore, $\nabla \times E \in \mathbf{C}(\bar{\Omega})$ implies that $S(\nabla \times E_1) \in \mathbf{C}(\overline{Z(\delta)})$, and hence by [16, Eq. (3.30)]

$$\partial_i E_{1j}^+ - \partial_j E_{1i}^+ \in C(\overline{Z(\delta)}) \qquad \text{for} \quad i, j = 1, 2, 3. \tag{10.5}$$

By differentiating (10.2), we obtain

$$\begin{aligned} &\partial_i F_{kj}(u) - \partial_j F_{ki}(u) \\ &\quad = \int_{|v'|<1} (\partial_i E_{1j}^+ - \partial_j E_{1i}^+)(u' + v'/k, u_3)\, h(v')\, dv' \end{aligned} \tag{10.6}$$

for $u_3 > 0$ and $i, j = 1, 2, 3$. By (10.5), $\partial_i F_{kj} - \partial_j F_{ki}$ can be continuously extended onto $\overline{Z(\delta)}$, and (10.6) holds also for $u_3 = 0$. In particular, we have

$$\partial_i F_{kj} - \partial_j F_{ki} \in C(\overline{Z(\delta)}) \qquad \text{for} \quad i, j = 1, 2, 3. \tag{10.7}$$

Furthermore, it follows from (10.6) that

$$\partial_i F_{kj} - \partial_j F_{ki} \to \partial_i E_{1j}^+ - \partial_j E_{1i}^+ \qquad \text{as} \quad k \to \infty \text{ uniformly in } \overline{Z(\delta)}. \tag{10.8}$$

Since $\nabla \cdot E \in C(\bar{\Omega})$, we have by [16, Eq. (3.31)]

$$S(\nabla \cdot E_1) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u_i} (\sqrt{g}\, g^{ij} E_{1j}^+) \in C(\overline{Z(\delta)})$$

and hence, since $E_1^+ \in \mathbf{C}(\overline{Z(\delta)})$,

$$g^{ij} \partial_i E_{1j}^+ \in C(\overline{Z(\delta)}). \tag{10.9}$$

By differentiating (10.2), we obtain for $u_3 > 0$

$$g^{ij}\partial_i F_{kj} = S_k^1 + S_k^2 \tag{10.10}$$

with

$$S_k^1(u) := \int_{|v'|<1} (g^{ij}\partial_i E_{1j}^+)(u' + v'/k, u_3)\, h(v')\, dv' \tag{10.11}$$

and

$$S_k^2(u) := \int_{|v'|<1} [g^{ij}(u) - g^{ij}(u' + v'/k, u_3)] \\ \times h(v')(\partial_i E_{1j}^+)(u' + v'/k, u_3)\, dv'. \tag{10.12}$$

We conclude from (10.9) as above that

$$S_k^1 \in C(\overline{Z(\delta)}) \tag{10.13}$$

and

$$S_k^1 \to g^{ij}\partial_i E_{1j}^+ \quad \text{as} \quad k \to \infty \quad \text{uniformly in} \quad \overline{Z(\delta)}. \tag{10.14}$$

Since $g^{33} = 1$ and $g^{i3} = g^{3i} = 0$ for $i = 1, 2$, only the terms with $i, j = 1, 2$ give a contribution to S_k^2 in (10.12). By applying the integral theorem of Gauss, we obtain for $u_3 > 0$ and $i = 1, 2$ (with $v' = (v_1, v_2)$)

$$S_k^2(u) = \int_{|v'|<1} [\cdots]\, h(v')\, k \frac{\partial}{\partial v_i} [E_{1j}^+(u' + v'/k, u_3) - E_{1j}^+(u)]\, dv'$$

$$= -\int_{|v'|<1} [E_{1j}^+(u' + v'/k, u_3) - E_{1j}^+(u)]\, T^j(u, v', k)\, dv'$$

with

$$T^j(u, v', k) = k \frac{\partial}{\partial v_i} \{[g^{ij}(u) - g^{ij}(u' + v'/k, u_3)]\, h(v')\}$$

$$= -h(v')(\partial_i g^{ij})(u' + v'/k, u_3) \\ + k[g^{ij}(u) - g^{ij}(u' + v'/k, u_3)]\, \partial_i h(v').$$

By applying the mean value theorem to the last term, it follows that $T^j(u, v', k)$, as a function of $u \in \overline{Z(\delta)}$ and v', is bounded uniformly with respect to k. Hence the last representation for S_k^2 yields

$$S_k^2 \to 0 \quad \text{as} \quad k \to \infty \quad \text{uniformly in} \quad \overline{Z(\delta)}. \tag{10.15}$$

This, together with (10.10), (10.13), and (10.14), implies that

$$g^{ij}\partial_i F_{kj} \in C(\overline{Z(\delta)}) \tag{10.16}$$

and

$$g^{ij}\partial_i F_{kj} \to g^{ij}\partial_i E^+_{1j} \quad \text{as} \quad k \to \infty \quad \text{uniformly in} \quad \overline{Z(\delta)}. \tag{10.17}$$

It follows from (10.3) and (10.7) that $\partial_i F_{kj} \in C(\overline{Z(\delta)})$ for $(i,j) \neq (3,3)$. Since $g^{33} = 1$, we conclude from (10.16) that also $\partial_3 F_{k3} \in C(\overline{Z(\delta)})$. Thus we obtain

$$F_k \in \mathbf{C}^1(\overline{Z(\delta)}). \tag{10.18}$$

In particular, F_{k3} can be approximated by a sequence in $C^\infty(\overline{Z(\delta)})$ with respect to the 1-norm by [12], Theorem 10.2. By the choice of ζ (compare [16, Eq. (3.10)], we have $\operatorname{supp} E^+_1 \subset \overline{Z(2\delta/3)}$, and hence $\operatorname{supp} F_k \subset \overline{Z(5\delta/6)}$ if $1/k < \delta/6$ or $k > k_0 := [6/\delta]$. Choose δ' with $5\delta/6 < \delta' < \delta$. It follows from (10.4) and (10.18) by the argument after [16, Eq. (5.18)] that there exists a sequence $\{G_k\}$ such that

$$\operatorname{supp} G_k \subset \overline{Z(\delta')}, \qquad G_k \in C_0^\infty(Z(\delta)) \times C_0^\infty(Z(\delta)) \times C^\infty(\overline{Z(\delta)}) \tag{10.19}$$

and

$$\|F_k - G_k\|_{1,Z(\delta)} < 1/2k \tag{10.20}$$

for every $k > k_0$. By (10.19) and [16, Lemma 4.4] we have $G_k \in \mathbf{V}_2$. Since $\mathbf{S}_2$ is dense in $\mathbf{V}_2$ with respect to the 1-norm, there exists a sequence $\{S_k\}$ in $\mathbf{S}_2$ such that $\|G_k - S_k\|_{1,Z(\delta)} < 1/2k$ and hence, by (10.20),

$$\|F_k - S_k\|_{1,Z(\delta)} < 1/k \tag{10.21}$$

for $k > k_0$. By observing (10.2), (10.8), (10.17), and (10.21), we obtain the following limit relations in $\mathbf{L}_2(Z(\delta))$:

$$\begin{gathered} \|S_k - E^+_1\|_{Z(\delta)} \to 0, \\ \|(\partial_i S_{kj} - \partial_j S_{ki}) - (\partial_i E^+_{1j} - \partial_j E^+_{1i})\|_{Z(\delta)} \to 0 \qquad (i,j = 1,2,3), \\ \|g^{ij}\partial_i S_{kj} - g^{ij}\partial_i E^+_{1j}\|_{Z(\delta)} \to 0 \qquad \text{as} \quad k \to \infty. \end{gathered} \tag{10.22}$$

In order to deduce from (10.22) that $\{S_k\}$ converges to E^+_1 with respect to the 1-norm, we use the coerciveness properties obtained in [15, Sect. 5]. Consider the bilinear forms B, B_0, and B^+ introduced in [16, formulas (3.1), (3.5), and (3.35)–(3.36)]. By [16, Eq. (3.37)] there exists a $c_3 > 0$ such that

$$B^+(G,G) \geqslant c_3 \|G\|^2_{1,Z(\delta)} \qquad \text{for every} \quad G \in \mathbf{S}_2. \tag{10.23}$$

Note that $(t_1 \cdot SG^-, t_2 \cdot SG^-, t_3 \cdot SG^-) = (G^-)^+ = G = (G_1, G_2, G_3)$ for $G \in \mathbf{S}_2$ by [14, Lemma 3.2]. Hence it follows from [16, (3.30)–(3.31)] that

$$S(\nabla \times G^-) = \frac{1}{\sqrt{g}} \begin{vmatrix} t_1 & t_2 & t_3 \\ \partial_1 & \partial_2 & \partial_3 \\ G_1 & G_2 & G_3 \end{vmatrix} \tag{10.24}$$

and

$$S(\nabla \cdot G^-) = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g}\, g^{ij} G_j) \tag{10.25}$$

for $G \in \mathbf{S}_2$. Furthermore, by [16, Eq. (3.19)] we have

$$SG^- = g^{ij} G_i t_j \qquad \text{for} \quad G \in \mathbf{S}_2. \tag{10.26}$$

These relations, together with [16, Eq. (3.28)], imply that there exist numbers c_4, $c_5 > 0$ such that, for every $G \in \mathbf{S}_2$,

$$\begin{aligned} B^+(G, G) = B_0(G^-, G^-) &= \|\nabla \times G^-\|^2 + \|\nabla \cdot G^-\|^2 + c_2 \|G^-\|^2 \\ &= (S(\nabla \times G^-), \Delta^- S(\nabla \times G^-)) + (S(\nabla \cdot G^-), \Delta^- S(\nabla \cdot G^-)) \\ &\quad + c_2(SG^-, \Delta^- SG^-) \\ &\leqslant c_4(\|S(\nabla \times G^-)\|^2 + \|S(\nabla \cdot G^-)\|^2 + \|SG^-\|^2) \\ &\leqslant c_5(\|G\|^2 + \sum_{i,j=1}^{3} \|\partial_i G_j - \partial_j G_i\|^2 + \|g^{ij}\partial_i G_j\|^2). \end{aligned}$$

By combining this estimate with (10.23), we obtain

$$\|G\|_1^2 \leqslant c \left(\|G\|^2 + \sum_{i,j=1}^{3} \|\partial_i G_j - \partial_j G_i\|^2 + \|g^{ij}\partial_i G_j\|^2\right) \tag{10.27}$$

for every $G \in \mathbf{S}_2$, where $c := c_3^{-1} c_5$. By applying (10.27) to $G = S_k - S_n$, (10.22) yields $\|S_k - S_n\|_1 \to 0$ as $k, n \to \infty$. Since $\mathbf{V}_2$ is complete with respect to the 1-norm, there exists a $V \in \mathbf{V}_2$ such $\|S_k - V\|_1 \to 0$ as $k \to \infty$. It follows from the first relation in (10.22) that $V = E_1^+$. Hence we obtain $\|S_k - E_1^+\|_1 \to 0$ as $k \to \infty$. This shows that $E_1^+ \in \mathbf{V}_2$.

Our next aim is to verify that

$$B_0(E_1, G) = (F_1, G)_{L_2(\Omega(x_0, \delta))} \qquad \text{for every} \quad G \in \mathbf{V}_1, \tag{10.28}$$

where

$$F_1 := \zeta F + (c_2 + \lambda)\, \zeta E - 2 \sum_{i=1}^{3} \frac{\partial \zeta}{\partial x_i} \frac{\partial E}{\partial x_i} - (\Delta \zeta)\, E. \tag{10.29}$$

Since $\mathbf{S}_1$ is dense in $\mathbf{V}_1$ respect to the 1-norm, we can assume that $G \in \mathbf{S}_1$. Set for $0 < \varepsilon < \delta$

$$\Omega_\varepsilon := \{x = x(u): u_1^2 + u_2^2 < \delta^2, \varepsilon < u_3 < \delta\}.$$

The integral theorem of Gauss yields

$$\begin{aligned}\int_{\Omega_\varepsilon} & [(\nabla \times E_1) \cdot (\nabla \times \bar{G}) + (\nabla \cdot E_1)(\nabla \cdot \bar{G})]\, dx \\ &= \int_{\Omega_\varepsilon} \nabla \cdot [(\nabla \cdot E_1)\, \bar{G} - (\nabla \times E_1) \times \bar{G}]\, dx - \int_{\Omega_\varepsilon} \bar{G} \cdot \Delta E_1\, dx \\ &= \int_{\partial\Omega_\varepsilon} [(\nabla \cdot E_1)(n \cdot \bar{G}) + (\nabla \times E_1) \cdot (n \times \bar{G})\, dS - \int_{\Omega_\varepsilon} \bar{G} \cdot \Delta E_1\, dx.\end{aligned}$$

Since $\nabla \times E_1$ and $\nabla \cdot E_1$ are continuous in $\overline{\Omega(x_0, \delta)}$ and since $\nabla \cdot E_1$ and $n \times G$ vanish on $\partial\Omega(x_0, \delta)$, we obtain, by letting $\varepsilon \downarrow 0$,

$$B(E_1, G) = -\lim_{\varepsilon \downarrow 0} \int_{\Omega_\varepsilon} \bar{G} \cdot \Delta E_1\, dx.$$

Since $\Delta E + \lambda E = -F$, (10.29) implies that $\Delta E_1 = \Delta(\zeta E) = -F_1 + c_2 E_1$, and hence, by the definition of B_0 ([16, Eq. (3.5)]),

$$B_0(E_1, G) = -\lim_{\varepsilon \downarrow 0} \int_{\Omega_\varepsilon} F_1 \cdot \bar{G}\, dx. \tag{10.30}$$

Since $E_1^+ \in \mathbf{V}_2$, as shown above, we have $E_1 \in \mathbf{V}_1 \subset H_1(\Omega(x_0, \delta))$ by [16, Lemma 3.2] and hence $F_1 \in \mathbf{C}(\Omega) \cap \mathbf{L}_2(\Omega)$ by (10.29). The argument used in the proof of Lemma 2.2 shows that the improper integral $\int_{\Omega(x_0, \delta)} |F_1|^2\, dx$ exists and that

$$\|F_1\|_{\Omega(x_0,\delta)} = \left[\int_{\Omega(x_0,\delta)} |F_1|^2\, dx\right]^{1/2}.$$

Since $G \in \mathbf{C}(\overline{\Omega(x_0, \delta)})$, the same statement holds if F_1 is replaced by $F_1 + \alpha G$ with complex α. This implies, since

$$(F_1, G) = \tfrac{1}{4}(\|F_1 + G\|^2 - \|F_1 - G\|^2 + i\,\|F_1 + iG\|^2 - i\,\|F_1 - iG\|^2),$$

that the right-hand side in (10.30) coincides with the inner product (F_1, G) in $\mathbf{L}_2(\Omega(x_0, \delta))$. This remark concludes the proof of (10.28).

Now we show by induction with respect to j that

$$E_1 \in \mathbf{H}_j(\Omega(x_0, \delta)) \tag{B$_j$}$$

for $1 \leqslant j \leqslant k+2$. (B_1) holds since $E_1 \in \mathbf{V}_1 \subset H_1(\Omega(x_0, \delta))$. Assume that (B_j) holds, where $1 \leqslant j \leqslant k+1$. The argument leading to [16, Eq. (3.43)] implies that

$$B^+(E_1^+, G) = (J, G) \qquad \text{for every} \quad G \in \mathbf{V}_2, \tag{10.31}$$

where

$$J = (J_1, J_2, J_3) \qquad \text{with} \qquad J_i = \sqrt{g}\, g^{ij} t_j \cdot SF_1. \tag{10.32}$$

Since $F_1 \in \mathbf{H}_{j-1}(\Omega(x_0, \delta))$ by (10.29) and (B_j), we have $J \in \mathbf{H}_{j-1}(Z(\delta))$. Hence it follows from (10.31) by [16, Lemma 5.1] that

$$E_1^+ \in \mathbf{H}_{j+1}(Z(\delta')) \qquad \text{for every} \quad \delta' < \delta. \tag{10.33}$$

Since supp $E_1^+ \subset \overline{Z(2\delta/3)}$, we have even

$$E_1^+ \in \mathbf{H}_{j+1}(Z(\delta)). \tag{10.34}$$

In fact, choose δ' with $2\delta/3 < \delta' < \delta$ and $\zeta \in C^\infty(\overline{Z(\delta)})$ such that $\zeta = 1$ in $Z(2\delta/3)$ and supp $\zeta \subset \overline{Z(\delta')}$. Since $E_1^+ = \zeta E_1^+$, (10.34) follows from [16, (5.7)]. The relation (10.34) implies (B_{j+1}), thus concluding the induction argument. For $j = k+2$ we obtain $E_1 \in \mathbf{H}_{k+2}(\Omega(x_0, \delta))$ and hence, by Sobolev's imbedding theorem, $E_1 \in \mathbf{C}^k(\overline{\Omega(x_0, \delta)})$. Because $E = E_1$ in $\overline{\Omega(x_0, \delta/3)}$ by the choice of ζ, this implies $E \in \mathbf{C}^k(\overline{\Omega(x_0, \delta/3)})$ and hence $E \in \mathbf{C}^k(\bar{\Omega})$, since $E \in \mathbf{C}^k(\Omega)$ and x_0 is an arbitrary boundary point. This concludes the proof of Lemma 2.1.

Lemma 3.1 can be proved in a similar way with obvious modifications. In order to prove that $H_1^+ \in \mathbf{V}_2'$, replace E_1^+ by H_1^+ in the definition (10.2) of F_k so that $F_{k3} = 0$ for $u_3 = 0$ (instead of (10.4)). Choose the sequence $\{G_k\}$ in (10.19)–(10.20) such that $G_{k1}, G_{k2} \in C^\infty(\overline{Z(\delta)})$ and $G_{k3} \in C_0^\infty(Z(\delta))$. It follows from [16, Lemma 4.5] that $G_k \in \mathbf{V}_2'$. Finally, note that

$$\lim_{\varepsilon \downarrow 0} \int_{\partial\Omega_\varepsilon} [(\nabla \cdot H_1)(n \cdot \bar{G}) + (\nabla \times H_1) \cdot (n \times \bar{G})]\, dS = 0$$

for $G \in \mathbf{S}'$ since $n \cdot G = 0$ and $n \times (\nabla \times H_1) = 0$ on $\partial\Omega(x_0, \delta)$, so that the same argument as in the proof of (10.28) can be used to verify that $B_0(H_1, G) = (F_1, G)$ for $G \in \mathbf{V}_1'$.

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