Thus by (25.20) there exists a $C > 1$ independent of N such that

$$| f^N(z) | \leqq CM^N e^{(a+b)r}$$

or

$$| f(z) | \leqq CMe^{(a+b)r/N}.$$

Since N can be chosen arbitrarily large, it follows that

$$| f(z) | \leqq CM.$$

But this means $f(z)$ must be a constant. This completes the proof.

CHAPTER VIII

AN INEQUALITY AND FUNCTIONS OF ZERO TYPE

26. Generalization of a problem of Pólya. In this chapter we shall prove an inequality and show how it leads to a considerable extension of results of Pólya on functions of zero type. The inequality is

THEOREM XLIII. *Let $M(x)$ be a positive even function monotone decreasing for increasing $| x |$. Let $M(x) \to \infty$ as $x \to 0$. Let $f(z)$ be analytic for ($| x | \leqq a$, $| y | \leqq b$) and let*

$$(26.01) \qquad | f(x + iy) | \leqq M(x), \qquad | x | \leqq a, | y | \leqq b.$$

If

$$(26.02) \qquad \int_0^1 \log \log M(x)\, dx < \infty,$$

then there exists a constant C, depending only on $M(x)$ and δ such that

$$(26.03) \qquad | f(x + iy) | \leqq C, \qquad | x | \leqq a, | y | \leqq b(1 - \delta).$$

We shall show in Chapter IX that Theorem XLIII is a best possible result.[1]

The problem of Pólya which we shall generalize here is the following: *Let $G(z)$ be an entire function such that*

$$(26.04) \qquad \limsup_{|z| \to \infty} \frac{\log | G(z) |}{| z |} \leqq 0.$$

If as $n \to \infty$

$$(26.05) \qquad G(\pm n) = O(1),$$

then $G(z)$ is a constant.[2]

Roughly what we shall show is that instead of $G(\pm n) = O(1)$ it suffices for

$$G(\pm\lambda_n) = O(1)$$

where $\{\lambda_n\}$ satisfies the requirement

$$\lambda_n - n = O\left(\frac{n}{(\log n)^{1+\delta}}\right), \qquad n \to \infty, \delta > 0.$$

This restriction on $\{\lambda_n\}$ is only a little more stringent than

[1] The results of this and the next chapter are sketched in Abstract 472, Bulletin of the American Mathematical Society, vol. 44 (1938), p. 789; and in Abstract 141, loc. cit., vol. 45 (1939), p. 236.

[2] This problem was set by Pólya, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 40 (1931), Problem 105. Many solutions have been given.

(25.11) $$\psi(re^{i(\alpha-\delta)}) = O(\exp\{-(a^2+b^2)^{1/2}r\cos\delta + (k+\epsilon)r\} + \exp\{\gamma r - pr\sin\delta\}).$$

Let us choose $\epsilon = \frac{1}{2}[(a^2+b^2)^{1/2} - k]$. Then since $(a^2+b^2)^{1/2} > k$ we can choose $\delta > 0$ so small that

$$(a^2+b^2)^{1/2}\cos\delta - k - \epsilon > p\sin\delta.$$

Thus (25.11) becomes

$$\psi(re^{i(\alpha-\delta)}) = O(e^{\gamma r - pr\sin\delta}).$$

We now introduce a small $\eta > 0$ and take $\gamma = \eta\sin\delta$. Then

(25.12) $$\psi(re^{i(\alpha-\delta)}) = O(e^{-(p-\eta)\sin\delta}).$$

Similarly

(25.13) $$\psi(re^{i(-\pi+\alpha+\delta)}) = O(e^{-(p-\eta)\sin\delta}).$$

From (25.08), $\psi(z)$ is analytic and of exponential type for $-\pi+\alpha < \operatorname{am} z < \alpha$. But by (25.12) and (25.13),

(25.14) $$\psi(z)e^{ize - i\alpha(p-\eta)}$$

is bounded along the lines $\operatorname{am} z = \alpha - \delta$ and $\operatorname{am} z = -\pi+\alpha+\delta$. Thus by Theorem C of Phragmén-Lindelöf, (25.14) is bounded in the entire sector $-\pi+\alpha+\delta \leqq \operatorname{am} z \leqq \alpha-\delta$. In particular then it is bounded for $\operatorname{am} z = 0$. Thus

$$\psi(x) = Oe^{-(p-\eta)x\sin\alpha}, \qquad x \to \infty.$$

Recalling $p = (a^2+b^2)^{1/2}(b/a)$ and (25.05), this gives

$$\psi(x)F_1(x)F_2(x) = O(\exp\{\{b+\epsilon+\eta-(a^2+b^2)^{1/2}b\sin\alpha/a\}x\}), \qquad x\to\infty.$$

Since $(a^2+b^2)^{1/2}\sin\alpha = a$, this becomes

(25.15) $$\psi(x)F_1(x)F_2(x) = O(e^{(\epsilon+\eta)x}), \qquad x\to\infty.$$

But by (25.08)

$$f(x) = \psi(x)F_1(x)F_2(x) + F_2(x)e^{-(b-\gamma)-ib^2x/a}\sum_1^\infty \frac{f(z_n)F_1(x)e^{z_n(b-\gamma+ib^2/a)}}{F_1'(z_n)F_2(z_n)(x-z_n)}$$
$$+ F_1(x)F_2(x)e^{-i(a-\gamma)x-a^2x/b}\sum_1^\infty \frac{f(-iw_n)e^{w_n(a-\gamma-ia^2/b)}}{F_1(-iw_n)F_2'(-iw_n)(x+iw_n)}.$$

Using (25.15) and (25.05) and recalling that $a \geqq b$, this gives

$$f(x) = O(e^{(3\epsilon+\eta+\gamma)x}), \qquad x\to\infty.$$

But $\gamma = \eta\sin\delta$ and ϵ and η can be chosen arbitrarily close to zero. Thus re-

defining ϵ we obtain $f(x) = O(e^{\epsilon x})$, $(x\to\infty)$, for any x. But $f(-z)$ possesses the same properties as $f(z)$. Thus the result holds for negative x. That is

(25.16) $$f(x) = O(e^{\epsilon|x|}), \qquad |x|\to\infty$$

If

$$\limsup_{|y|\to\infty}\frac{\log|f(iy)|}{|y|} = c,$$

then it follows from (25.16) and Theorem C′ of Phragmén-Lindelöf that

$$f(z) = O(e^{cr|\sin\theta|+\epsilon r}).$$

But by Theorem XXXII since $D_2 > 0$,

$$c = \limsup_{n\to\infty}\frac{\log|f(\pm iw_n)|}{|w_n|}.$$

Thus $c = 0$ and

(25.17) $$f(z) = O(e^{\epsilon|z|}).$$

Let $z_{-n} = -z_n$ and $w_{-n} = -w_n$. For any $N > 0$ let

(25.18) $$H_N(z) = \frac{f^N(z)}{F_1(z)F_2(z)} - \sum_{-\infty}^{\infty}\frac{f^N(z_n)}{F_1'(z_n)F_2(z_n)(z-z_n)} - \sum_{-\infty}^{\infty}\frac{f^N(iw_n)}{F_1(iw_n)F_2'(iw_n)(z-iw_n)}$$

where the terms $n = 0$ are omitted from the sums. By (25.06) and (25.07) the series converge and $H_N(z)$ is of exponential type. Also by (25.06), (25.07) and (25.17)

$$H_N(\pm re^{\pm i\alpha}) = O(e^{\{N\epsilon-(a^2+b^2)^{1/2}\}r} + 1/r).$$

Since ϵ can be taken arbitrarily small, it follows that

(25.19) $$H_N(\pm re^{\pm i\alpha}) = o(1), \qquad r\to\infty.$$

Thus $H_N(z)$ is bounded on the lines $\operatorname{am} z = \pm\alpha, \pm(\pi-\alpha)$. Thus by Theorem C of Phragmén-Lindelöf, it is bounded in the entire plane and therefore a constant. But by (25.19) this constant must be zero. Thus (25.18) becomes

(25.20) $$f^N(z) = F_1(z)F_2(z)\left(\sum_{-\infty}^{\infty}\frac{f^N(z_n)}{F_1'(z_n)F_2(z_n)(z-z_n)} + \sum_{-\infty}^{\infty}\frac{f^N(iw_n)}{F_1(iw_n)F_2'(iw_n)(z-iw_n)}\right)$$

By (25.01) there exists an M such that

25.[6] An extension of a theorem of Iyer. Here we use the method of §23 on another problem.

THEOREM XLII.[7] *Let* $\{z_n\}$ *and* $\{w_n\}$ *be two sequences of complex numbers such that*

$$\lim_{n\to\infty} \frac{n}{z_n} = D_1 > 0, \qquad \lim_{n\to\infty} \frac{n}{w_n} = D_2 > 0,$$

and for some $d > 0$,

$$|z_n - z_m| \geqq |n - m|d, \qquad |w_n - w_m| \geqq |n - m|d.$$

Let $f(z)$ *be an entire function such that*

(25.01) $$f(\pm z_n) = O(1), \qquad f(\pm i w_n) = O(1), \qquad n \to \infty,$$

and

(25.02) $$\limsup_{|z|\to\infty} \frac{\log|f(z)|}{|z|} = k < \pi(D_1^2 + D_2^2)^{1/2}.$$

Then $f(z)$ *is a constant.*

That the theorem is best possible follows from considering $f(z) = \sin \pi D_1 z \sinh \pi D_2 z$ with $z_n = n/D_1$ and $w_n = n/D_2$.

Proof of Theorem XLII. Let

(25.03) $$a = \pi D_1, \qquad b = \pi D_2, \qquad \alpha = \tan^{-1} a/b,$$

and let

$$F_1(z) = \prod_1^\infty \left(1 - \frac{z^2}{z_n^2}\right), \qquad F_2(z) = \prod_1^\infty \left(1 + \frac{z^2}{w_n^2}\right).$$

Then $F_1(z)$ and $F_2(iz)$ satisfy the requirements of Theorem XXXI and thus we have

(25.04) $$\frac{1}{F_1'(\pm z_n)} = O(e^{\epsilon|z_n|}), \qquad \frac{1}{F_2'(\pm i w_n)} = O(e^{\epsilon|w_n|}),$$

[6] Cf. Levinson, *Integral functions bounded on sequences of points*, Duke Mathematical Journal, vol. 4 (1938), p. 170.

[7] Iyer proved this theorem for real sequences $\{z_n\}$ and $\{w_n\}$ with (25.02) replaced by the more restrictive condition

$$k < \pi \min(D_1, D_2)$$

or else with (25.01) replaced by

$$\lim_{n\to\infty} \frac{\log|f(\pm z_n)|}{|z_n|} = -\infty, \qquad \lim_{n\to\infty} \frac{\log|f(\pm i w_n)|}{|w_n|} = -\infty.$$

V. G. Iyer, *On the order and type of integral functions bounded at a sequence of points*, Annals of Mathematics, vol. 38 (1937), p. 311.

(25.05) $$F_1(re^{i\theta}) = O(\exp\{ar|\sin\theta| + \epsilon r\}),$$
$$F_2(re^{i\theta}) = O(\exp\{br|\cos\theta| + \epsilon r\}),$$

(25.06) $$\frac{1}{F_1(re^{i\theta})} = O(\exp\{-ar|\sin\theta| + \epsilon r\}), \qquad |z \pm z_n| \geqq \tfrac{1}{4}d,$$

(25.07) $$\frac{1}{F_2(re^{i\theta})} = O(\exp\{-br|\cos\theta| + \epsilon r\}), \qquad |z \pm i w_n| \geqq \tfrac{1}{4}d.$$

For $\gamma > 0$ let

(25.08) $$\psi(z) = \frac{f(z)}{F_1(z)F_2(z)} - \sum_1^\infty \frac{f(z_n)\exp\{(z - z_n)(-b - ib^2/a + \gamma)\}}{F_1'(z_n)F_2(z_n)(z - z_n)} - \sum_1^\infty \frac{f(-iw_n)\exp\{(z + iw_n)(-ia - a^2/b + i\gamma)\}}{F_1(-iw_n)F_2'(-iw_n)(z + iw_n)}.$$

The series converge by (25.01), (25.04), (25.06) and (25.07). For any $\epsilon > 0$ and small $\delta > 0$,

(25.09) $$\psi(re^{i(\alpha-\delta)}) = O(\exp\{-[a\sin(\alpha - \delta) + b\cos(\alpha - \delta) - k - \epsilon]r\} + \exp\{-(b - \gamma)r\cos(\alpha - \delta) + (b^2/a)r\sin(\alpha - \delta)\} + \exp\{(a - \gamma)r\sin(\alpha - \delta) - (a^2/b)r\cos(\alpha - \delta)\}).$$

Since $\tan\alpha = a/b$, simple calculations give

$$a\sin(\alpha - \delta) + b\cos(\alpha - \delta) = (a^2 + b^2)^{1/2}\cos\delta,$$

$$-b\cos(\alpha - \delta) + \frac{b^2}{a}\sin(\alpha - \delta) = -(a^2 + b^2)^{1/2}\frac{b}{a}\sin\delta,$$

$$a\sin(\alpha - \delta) - \frac{a^2}{b}\cos(\alpha - \delta) = -(a^2 + b^2)^{1/2}\frac{a}{b}\sin\delta.$$

Thus (25.09) becomes

(25.10) $$\psi(re^{i(\alpha-\delta)}) = O(\exp\{-[(a^2 + b^2)^{1/2}\cos\delta - k - \epsilon]r\} + \exp\{\gamma r - (a^2 + b^2)^{1/2}(b/a)r\sin\delta\} + \exp\{\gamma r - (a^2 + b^2)^{1/2}(a/b)r\sin\delta\}).$$

There is no essential difference between the cases $a > b$, $b > a$. Here we assume that $a \geqq b$. Then if

$$p = (a^2 + b^2)^{1/2} b/a,$$

it follows that

$$p \leqq (a^2 + b^2)^{1/2} a/b$$

and (25.10) becomes

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D, \qquad \lambda_{n+1} - \lambda_n \geqq d > 0.$$

Let $\Lambda(u)$ *be the number of* $\lambda_n < u$. *If*

$$\int_1^\infty \frac{\Lambda(u) - (b + \frac{1}{2}a)u}{u^2}\, du = \infty$$

and if for some positive $t(u)$ *such that*

$$\int_1^\infty \frac{t(u)}{u^2}\, du < \infty$$

the inequality

$$\Lambda(u) + t(u) \geqq (b + \tfrac{1}{2}a)u$$

holds, then

(24.62) $$\limsup_{n\to\infty} \frac{\log |\Phi(\lambda_n)|}{\lambda_n} \leqq \pi p$$

implies

(24.63) $$\Phi(re^{i\theta}) = O(\exp\{\pi r(p\cos\theta + b\,|\sin\theta| + \epsilon)\}), \qquad |\theta| \leqq \tfrac{1}{2}\pi,$$

for any $\epsilon > 0$.

Proof of Theorem XXXIX. As in the proof of Theorem XXXIV, we consider

$$\phi(z) = \frac{\Phi(z)}{\Gamma(1 + az)}.$$

Proceeding as in Theorem XXXIV, it follows from Theorem XXXVIII that

$$\lim_{x\to\infty} \frac{\log|\phi(x)|}{x} = -\infty.$$

Then $g(z)$ is defined as in (23.25) and

$$\psi(z) = \frac{\phi(z) - g(z)}{F(z)}.$$

However, instead of working with $\psi(z)$ we work with $\psi(z)K_1(z)$ where $K_1(z)$ is defined as in the proof of Theorem XXXVIII. Thus (23.27) is replaced by

$$|\psi(iy)K_1(iy)| \leqq M_1 + \sum_1^\infty \left| \frac{\phi(\lambda_n)K_1(\lambda_n)e^{A\lambda_n}}{F'(\lambda_n)} \right|.$$

Proceeding as in Theorem XXXIV with $\psi(z)K_1(z)$ instead of $\psi(z)$, there are no further difficulties.

Related to Theorem XXXIX in exactly the same way as Theorem XXXV is related to Theorem XXXIV is the following theorem.

THEOREM XL. *If, in Theorem* XXXIX, (24.62) *is replaced by*

$$\Phi(\lambda_n) = O(e^{-k\lambda_n \log \lambda_n}), \qquad n \to \infty,$$

with $k > 0$, *then* (24.63) *is replaced by*

$$\Phi(re^{i\theta}) = O(\exp\{(-k\log r\cos\theta + \epsilon\cos\theta + \pi b\,|\sin\theta|)r\}), \qquad |\theta| \leqq \tfrac{1}{2}\pi.$$

The proof of this theorem follows closely that of Theorem XXXIX in just the same way as the proof of Theorem XXXV follows closely that of Theorem XXXIV. A sharper result than Theorem XXXVI is the following theorem.

THEOREM XLI. *If, in Theorem* XXXIX, (24.62) *is replaced by*

(24.64) $$\lim_{n\to\infty} \frac{\log|\Phi(\lambda_n)| + 2b\lambda_n \log\lambda_n}{\lambda_n} = -\infty,$$

then $\Phi(z) \equiv 0$.

Proof of Theorem XLI. Applying (24.64) to $e^{Bz}\Phi(z)$, $(B > 0)$, it follows that

$$e^{B\lambda_n}\Phi(\lambda_n) = O(e^{-2b\lambda_n \log\lambda_n}).$$

Applying Theorem XL to $e^{Bz}\Phi(z)$, it follows that

(24.65) $|\Phi(z)| \leqq B_1 \exp\{(-2b\log r\cos\theta - B\cos\theta + \pi b\,|\sin\theta|)r\}$, $|\theta| \leqq \frac{1}{2}\pi$,

where B_1 is a constant depending on B. Applying Carleman's theorem, Theorem B, to $\Phi(z)$, we have, if $\Phi(z) \not\equiv 0$,

$$-A_1 \leqq \frac{1}{2\pi}\int_1^R \log|\Phi(iy)\Phi(-iy)|\left(\frac{1}{y^2} - \frac{1}{R^2}\right)dy + \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \log|\Phi(Re^{i\theta})|\cos\theta\, d\theta,$$

where A_1 depends only on $\Phi(z)$. Using (24.61), this becomes

$$-A_2 \leqq b\int_1^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right) y\,dy + \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \log|\Phi(re^{i\theta})|\cos\theta\, d\theta$$

where A_2 depends only on $\Phi(z)$. Applying (24.65) this becomes

$$-A_2 \leqq b\log R + \frac{-2b\log R - B + G}{\pi}\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$
$$+ b\int_{-\pi/2}^{\pi/2} |\sin\theta|\cos\theta\, d\theta + \frac{\log B_1}{\pi R}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta.$$

Or

$$-A_2 \leqq -\tfrac{1}{2}B + \tfrac{1}{2}\epsilon + b + \frac{2}{\pi R}\log B_1.$$

Letting $R \to \infty$ we see that by choosing

$$B > 2A_2 + 2b + \epsilon$$

we obtain a contradiction. Thus $\Phi(z) \equiv 0$.

Thus if we take $\theta(y) = \frac{1}{4}\zeta(|y|)$, then (24.54) is satisfied. Also

$$\int_1^A \frac{\theta(y)}{y^2}\,dy = \frac{1}{4}\int_1^A \frac{\zeta(y)}{y^2}\,dy \geqq \int_1^A \frac{d\zeta(y)}{y} - \frac{1}{4}\frac{\zeta(A)}{A}.$$

Since $n/z_n \to D$, letting $A \to \infty$ we have

$$\int_1^\infty \frac{\theta(y)}{y^2}\,dy \geqq \frac{1}{4}\int_1^\infty \frac{d\zeta(y)}{y} - \tfrac{1}{4}D = \tfrac{1}{4}\sum_1^\infty \frac{1}{|z_n|} - \tfrac{1}{4}D = \infty.$$

This proves (24.55) and completes the proof of the lemma.

Proof of Theorem XXXVII. The proof follows word for word the alternative proof of Theorem XXXIII except for the facts that (24.16) and (24.25) are both replaced by

$$(24.56) \qquad \frac{\phi(iy)}{F(iy)} = O(e^{-\theta(y)}), \qquad |y| \to \infty,$$

and that $H_1(u)$ is no longer analytic, which nullifies the argument from (24.25) to (24.26). Equation (24.56) follows from $\phi(iy) = O(1)$ and (24.54).

To replace the argument from (24.25) to (24.26), we recall that

$$H_1(u) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\phi(it)e^{-Bit}}{F(it)(1+it)}e^{iut}\,dt.$$

If we set

$$G(t) = \frac{1}{(2\pi)^{1/2}}\frac{\phi(it)e^{-Bit}}{F(it)(1+it)},$$

then $H_1(u)$ is the Fourier transform of $G(t)$ and by (24.56), $G(t)$ satisfies (24.27). Thus by the argument following (24.27), since by (24.21) $H(u) = 0$, $(u < 0)$, or in other words $H_1(u) = H_2(u)$, $(u < 0)$, and since $H_2(u)$ is analytic for $u < B - \epsilon$, it follows that

$$H_1(u) = H_2(u), \qquad u < B - \epsilon,$$

or in other words that $H(u) = 0$, $(u < B - \epsilon)$. This is (24.26), and the argument now again follows exactly the alternative proof of Theorem XXXIII.[5]

[5] In case $\phi(iy)$ is not bounded but satisfies

$$\int_{-\infty}^\infty \frac{\log^+|\phi(iy)|}{1+y^2}\,dy < \infty,$$

a function $K_1(z)$, analytic and bounded in the right half-plane, is introduced which satisfies $\log|K_1(iy)| = -\log^+|\phi(iy)|$. $K_1(z)$ is used in much the same way as a similar function also designated by $K_1(z)$ is used in the proof of Theorem XXXIII. $K_1(z) = e^{-u(x,y)-iv(x,y)}$ where

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{x\log^+|\phi(i\eta)|}{x^2+(y-\eta)^2}\,d\eta$$

and $v(x, y)$ is the conjugate function to $u(x, y)$.

We now turn to the other part of the proof and show that

$$(24.57) \qquad \sum_1^\infty \frac{1}{|z_n|} = \infty$$

is a necessary condition. Let us assume

$$(24.58) \qquad \sum_1^\infty \frac{1}{|z_n|} < \infty.$$

Then

$$\phi(z) = \prod_1^\infty \frac{(z-z_n)(z-\bar{z}_n)}{(z+z_n)(z+\bar{z}_n)} = \prod_1^\infty\left(1 - \frac{2z}{z_n+z}\right)\left(1 - \frac{2z}{\bar{z}_n+z}\right)$$

exists and is analytic in the right half-plane, since, for $x > 0$, (24.58) gives

$$\sum_1^\infty \left|\frac{2z}{z_n+z}\right| < \infty, \qquad \sum_1^\infty \left|\frac{2z}{\bar{z}_n+z}\right| < \infty.$$

Since for $x > 0$

$$\left|\frac{z-z_n}{z+z_n}\right| \leqq 1, \qquad \left|\frac{z-\bar{z}_n}{z+\bar{z}_n}\right| \leqq 1,$$

it follows that

$$(24.59) \qquad |\phi(z)| \leqq 1, \qquad |\theta| \leqq \tfrac{1}{2}\pi.$$

Thus $\phi(z)$ satisfies the requirements of Theorem XXXVII. But $\phi(z_n) = 0$, and thus,

$$(24.60) \qquad \lim_{n\to\infty} \frac{\log|\phi(z_n)|}{|z_n|} = -\infty.$$

If (24.58) were sufficient for Theorem XXXVII, (24.60) would imply

$$\lim_{x\to\infty} \frac{\log|\phi(x)|}{x} = -\infty.$$

But this and (24.59), by Theorem C of Phragmén-Lindelöf, imply that $\phi(z) \equiv 0$, which is impossible. Thus (24.58) is insufficient for Theorem XXXVII, or in other words (24.57) is necessary. This completes the proof of Theorem XXXVII.

Theorem XXXIV can be refined by the use of Theorem XXXVIII to give the following:

THEOREM XXXIX. *Let $\Phi(z)$ be an analytic function in the right half-plane $|\text{am } z| \leqq \frac{1}{2}\pi$ such that for any $\epsilon > 0$*

$$(24.61) \qquad \Phi(re^{i\theta}) = O(\exp\{(a\log r\cos\theta + \pi b|\sin\theta| + \epsilon\cos\theta)r\}), \qquad |\theta| \leqq \tfrac{1}{2}\pi,$$

where $a \geqq 0$, $b \geqq -\frac{1}{2}a$. Let $\{\lambda_n\}$ be a positive sequence satisfying

Using (24.39) this gives

(24.49) $$\psi(x) = O\left(e^{(B+\delta)x}\int_0^\infty |H(u)|e^{-ux}\,du\right), \qquad x \to \infty.$$

Using (24.47) in (24.48) we can close the path of integration to the right and obtain

$$H(u) = 0, \qquad u < 0,$$

On the other hand, using the definition of $\psi(z)$ in (24.48) gives

$$H(u) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{K_1(s)\phi(s)e^{-Bs}}{F(s)}e^{us}\,ds - \sum_1^\infty \frac{\phi(\lambda_n)e^{-\epsilon\lambda_n}}{F'(\lambda_n)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{K_1(s)e^{(u-B+\epsilon)s}}{s-\lambda_n}\,ds.$$

By (24.40) this becomes

$$H(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K_1(it)\phi(it)e^{-iBt}}{F(it)}e^{iut}\,dt - \sum_1^\infty \frac{K_1(\lambda_n)\phi(\lambda_n)e^{(u-B)\lambda_n}}{F'(\lambda_n)}$$

for $u < B - \epsilon - \delta$. Or

(24.50) $$H(u) = H_1(u) - H_2(u), \qquad u < B - 2\epsilon'$$

where

(24.51) $$H_1(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K_1(it)\phi(it)e^{-iBt}}{F(it)}e^{iut}\,dt,$$

and

(24.52) $$H_2(u) = \sum_1^\infty \frac{K_1(\lambda_n)\phi(\lambda_n)e^{(u-B)\lambda_n}}{F'(\lambda_n)}, \qquad u < B - 2\epsilon.$$

The series for $H_2(u)$ converges for $u < B - \delta$ and therefore certainly for $u < B - 2\epsilon$ and represents an analytic function. Thus $H_2(u)$ is analytic for $u < B - 2\epsilon$.

Since $\phi(it) = O(e^{\pi L|t|})$, using (24.31) and (24.38) gives

$$\frac{K_1(it)\phi(it)}{F(it)} = O\left(\frac{e^{-\theta(t)}}{t^4}\right), \qquad |t| \to \infty.$$

Thus if

$$G(t) = (2\pi)^{1/2}\frac{K_1(it)\phi(it)e^{-iBt}}{F(it)}$$

then $G(t)$ satisfies (24.27) and (24.28); and by (24.51), $H_1(u)$ is the Fourier transform of $G(t)$. Since we have shown $H(u) = 0$, $u < 0$, it follows that

$$H_1(u) = H_2(u), \qquad u < 0.$$

But $H_2(u)$ is analytic, $u < B - 2\epsilon$, and $H_1(u)$ is the transform of $G(t)$. Thus from the remarks following (24.28)

$$H_1(u) = H_2(u), \qquad u < B - 2\epsilon,$$

or $H(u) = 0$, $(u < B - 2\epsilon)$. Using this in (24.49) gives

$$\psi(x) = O\left(e^{(B+\delta)x}\int_{B-2\epsilon}^\infty |H(u)|e^{-ux}\,du\right), \qquad x \to \infty.$$

By (24.46) and (24.48), $H(u)$ is bounded. Thus

$$\psi(x) = O(e^{(\delta+2\epsilon)x}) = O(e^{3\epsilon x}), \qquad x \to \infty,$$

since $\epsilon > \delta$. But $\phi(x) = g(x) + F(x)\phi(x)$ and therefore $\phi(x) = O(e^{4\epsilon x})$. δ can be chosen arbitrarily small and therefore so can ϵ. Therefore (24.42) follows and the theorem is proved.

In much the same way the following theorem is proved.

Theorem XXXVIII-A. *Theorem* XXXVIII *remains true if* (24.06) *and* (24.08) *are both replaced by*

$$\phi(iy) = O(e^{\pi L|y| - \theta(y)}), \qquad |y| \to \infty,$$

for an even function $\theta(y)$ *increasing for* $y > 0$ *and satisfying*

$$\int_1^\infty \frac{\theta(y)}{y^2}\,dy = \infty.$$

We now turn to the proof of Theorem XXXVII.

Lemma 24.2. *Let*

(24.53) $$F(z) = \prod_1^\infty\left(1 - \frac{z^2}{z_n^2}\right)$$

where $\{z_n\}$ *satisfies the requirements of Theorem* XXXVII. *Then*

(24.54) $$\frac{1}{F(iy)} = O(e^{-\theta(y)}), \qquad |y| \to \infty,$$

where $\theta(y)$ *is even and increasing for* $y > 0$, *and*

(24.55) $$\int_1^\infty \frac{\theta(y)}{y^2}\,dy = \infty.$$

Proof of Lemma 24.2. If $z_n = r_n e^{i\theta_n}$ there is no loss of generality in assuming that $\theta_n \leqq \frac{1}{4}\pi$. We have, if $\zeta(u)$ is the number of $|z_n| < u$,

$$\log|F(iy)| = \sum_1^\infty \log\left(1 + \frac{2y^2}{r_n^2}\cos 2\theta_n + \frac{y^4}{r_n^4}\right)^{1/2}$$
$$\geqq \tfrac{1}{2}\sum_1^\infty \log\left(1 + \frac{y^4}{r_n^4}\right) = \frac{1}{2}\int_0^\infty \log\left(1 + \frac{y^4}{u^4}\right)d\zeta(u)$$
$$> \frac{\log 2}{2}\int_0^{|y|} d\zeta(u) > \tfrac{1}{4}\zeta(|y|).$$

Then by (24.36)

$$K_1(iy) = \frac{1}{2\pi}\int_{-\delta}^{\delta} \bar{k}(t)e^{iyt}\,dt \int_{-\infty}^{\infty} K(u)e^{-iut}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} K(u)\,du \int_{-\delta}^{\delta} \bar{k}(t)e^{i(y-u)t}\,dt$$

$$= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} K(u)\bar{K}(u-y)\,du.$$

Or for $y > 0$,

$$|K_1(iy)| \leqq \frac{1}{(2\pi)^{1/2}} \max_{-\infty<u<y/2} |\bar{K}(u-y)| \int_{-\infty}^{y/2} |K(u)|\,du$$

$$+ \max_{y/2<u<\infty} |K(u)| \int_{y/2}^{\infty} |\bar{K}(u-y)|\,du$$

$$\leqq \frac{2}{(2\pi)^{1/2}} \max_{|u|\geqq y/2} |K(u)| \int_{-\infty}^{\infty} |K(u)|\,du.$$

Using (24.35) this gives

(24.38) $$K_1(iy) = O\left(\frac{e^{-t_1(y)}}{y^4}\right), \qquad |y| \to \infty.$$

On the other hand by (24.37)

$$K_1(x) \geqq \frac{e^{-\delta x}}{(2\pi)^{1/2}}\int_{-\delta}^{\delta} |k(t)|^2\,dt, \qquad x > 0,$$

or

(24.39) $$\frac{1}{K_1(x)} = O(e^{\delta x}), \qquad x \to \infty.$$

Also from (24.37)

(24.40) $$K_1(z) = O(e^{\delta|z|}).$$

As in the alternative proof of Theorem XXXIII it suffices to show that

(24.41) $$\lim_{n\to\infty} \frac{\log|\phi(\lambda_n)|}{\lambda_n} \leqq 0$$

implies

(24.42) $$\limsup_{x\to\infty} \frac{\log|\phi(x)|}{x} \leqq 0.$$

We define

(24.43) $$g(z) = \sum_{1}^{\infty} \frac{\phi(\lambda_n)e^{-\epsilon\lambda_n}}{F'(\lambda_n)(z-\lambda_n)} e^{\epsilon z}F(z)$$

with $\epsilon > \delta$, and

(24.44) $$\psi(z) = \frac{\phi(z) - g(z)}{F(z)}.$$

Then as before $g(z)$ is an entire function and $\psi(z)$ is analytic and of exponential type for $|\operatorname{am} z| \leqq \frac{1}{2}\pi$. From (24.44)

(24.45) $$|\psi(iy)| \leqq \left|\frac{\phi(iy)}{F(iy)}\right| + \sum_{1}^{\infty} \left|\frac{\phi(z_n)e^{-\epsilon z_n}}{F'(z_n)(iy - z_n)}\right|.$$

By (24.40) $K_1(z)\psi(z)$ is also of exponential type for $|\operatorname{am} z| \leqq \frac{1}{2}\pi$. By (24.38) and (24.45)

$$K_1(iy)\psi(iy) = O\left(\frac{e^{-t_1(y)}|\phi(iy)|}{|F(iy)|\,y^4} + \frac{1}{|y|^5}\right), \qquad |y| \to \infty.$$

But $\phi(iy) = O(e^{\pi L|y|})$. Thus using (24.31),

(24.46) $$K_1(iy)\psi(iy) = O\left(\frac{e^{-\theta(y)}}{y^4} + \frac{1}{|y|^5}\right) = O\left(\frac{1}{y^4}\right).$$

It now follows from Theorem C′ of Phragmén-Lindelöf that for some B

$$K_1(z)\psi(z) = O\left(\frac{e^{Bx}}{|z|^4}\right), \qquad |\operatorname{am} z| \leqq \tfrac{1}{2}\pi,$$

or

(24.47) $$K_1(z)\psi(z)e^{-Bz} = O\left(\frac{1}{|z|^4}\right), \qquad |\operatorname{am} z| \leqq \tfrac{1}{2}\pi.$$

Using (24.47) it follows by Cauchy's integral theorem that for $x > 0$,

$$K_1(z)\psi(z)e^{-Bz} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{K_1(s)\psi(s)e^{-Bs}}{z - s}\,ds$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K_1(s)\psi(s)e^{-Bs}\,ds \int_0^{\infty} e^{-u(z-s)}\,du$$

$$= \frac{1}{2\pi i}\int_0^{\infty} e^{-uz}\,ds \int_{-i\infty}^{i\infty} K_1(s)\psi(s)e^{-Bs+us}\,ds.$$

Here we are following almost identically the alternative proof of Theorem XXXIII. We define

(24.48) $$H(u) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K_1(s)\psi(s)e^{-Bs+us}\,ds.$$

Then

$$K_1(z)\psi(z)e^{-Bz} = \int_0^{\infty} H(u)e^{-uz}\,du.$$

and if $\{\lambda_n\}$ *satisfies the requirements of Theorem* XXXVIII, *then there exists an even function* $\theta(u)$, *increasing for* $u > 0$, *such that*

(24.29) $$\int_1^\infty \frac{\theta(y)}{y^2}\,dy = \infty$$

and an even function $t_1(y)$, *increasing for* $y > 0$, *such that*

(24.30) $$\int_1^\infty \frac{t_1(y)}{y^2}\,dy < \infty,$$

and

(24.31) $$\frac{e^{\pi L|y|}}{F(iy)} = O(e^{-\theta(y)+t_1(y)}), \qquad |y| \to \infty.$$

Proof of Lemma 24.1. Clearly

$$\log|F(iy)| = \sum_1^\infty \log\left(1 + \frac{y^2}{\lambda_n^2}\right) = \int_0^\infty \log\left(1 + \frac{y^2}{u^2}\right) d\Lambda(u)$$

$$= 2\int_0^\infty \frac{\Lambda(u)}{u}\,\frac{y^2}{u^2+y^2}\,du.$$

Since

$$2\int_0^\infty \frac{y^2}{u^2+y^2}\,du = \pi\,|y|,$$

it follows that

$$\log|F(iy)| - \pi L\,|y| = 2\int_0^\infty \frac{\Lambda(u) - Lu}{u}\,\frac{y^2}{u^2+y^2}\,du.$$

Or

(24.32) $$\log|F(iy)| - \pi L\,|y| \geqq 2\int_1^\infty \frac{\Lambda(u) - Lu + t(u)}{u}\,\frac{y^2}{u^2+y^2}\,du - 2\int_1^\infty \frac{t(u)}{u}\,\frac{y^2}{u^2+y^2}\,du - 2L.$$

Let

(24.33) $$t_1(y) = 2\int_1^\infty \frac{t(u)}{u}\,\frac{y^2}{u^2+y^2}\,du.$$

Recalling $\Lambda(y) + t(y) > Ly$, (24.32) becomes

$$\log|F(iy)| - \pi L\,|y| \geqq 2\int_1^y \frac{\Lambda(u) - Lu + t(u)}{u}\,\frac{y^2}{u^2+y^2}\,du - t_1(y) - 2L$$

$$\geqq \int_1^y \frac{\Lambda(u) - Lu + t(u)}{u}\,du - t_1(y) - 2L.$$

If we set

(24.34) $$\theta(y) = \int_1^y \frac{\Lambda(u) - Lu + t(u)}{u}\,du, \qquad y > 1,$$

$\theta(-y) = \theta(y)$, then

$$\log|F(iy)| - \pi L\,|y| \geqq \theta(y) - t_1(y) - 2L,$$

which gives (24.31).

From (24.34), $\theta(y)$ is increasing and

$$\int_1^\infty \frac{\theta(y)}{y^2}\,dy = \int_1^\infty \frac{dy}{y^2}\int_1^y \frac{\Lambda(u) - Lu + t(u)}{u}\,du$$

$$\geqq \int_1^\infty \frac{dy}{y^2}\int_1^y \frac{\Lambda(u) - Lu}{u}\,du = \int_1^\infty \frac{\Lambda(u) - Lu}{u}\,du\int_u^\infty \frac{dy}{y^2}$$

$$= \int_1^\infty \frac{\Lambda(u) - Lu}{u^2}\,du = \infty.$$

This proves (24.29).

From (24.33)

$$t_1'(y) = 2\int_1^\infty \frac{t(u)}{u}\,\frac{2yu^2}{(u^2+y^2)^2}\,du > 0.$$

Thus $t_1(y)$ is increasing for $y > 0$. Also

$$\int_0^\infty \frac{t_1(y)}{y^2}\,dy = 2\int_0^\infty \frac{dy}{y^2}\int_1^\infty \frac{t(u)}{u}\,\frac{y^2}{u^2+y^2}\,du$$

$$= 2\int_1^\infty \frac{t(u)}{u}\,du\int_0^\infty \frac{dy}{u^2+y^2} = \pi\int_1^\infty \frac{t(u)}{u^2}\,du < \infty,$$

which completes the proof of the lemma.

Proof of Theorem XXXVIII. If $t_1(u)$ satisfies the requirements of Lemma 24.1, then by Theorem XXVI for any $\delta > 0$ there exists a function $K(w)$ such that

(24.35) $$K(u) = O\left(\frac{e^{-t_1(2u)}}{1+u^4}\right),$$

and the Fourier transform of $K(u)$, $k(x)$ vanishes outside of $(-\delta, \delta)$. Thus

(24.36) $$k(x) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^\infty K(u)e^{-iux}\,du.$$

Let

(24.37) $$K_1(z) = \frac{1}{(2\pi)^{1/2}}\int_{-\delta}^{\delta} k(t)\bar{k}(t)e^{zt}\,dt.$$

Using (24.18) and closing the path of integration to the right in (24.19), it is clear that

(24.21)
$$H(u) = 0, \qquad u < 0.$$

On the other hand, if we use (24.14) in (24.19) it becomes

$$H(u) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\phi(s)e^{-Bs}}{F(s)(1+s)} e^{us}\,ds - \sum_1^\infty \frac{\phi(z_n)e^{-\epsilon z_n}}{F'(z_n)} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{(u-B+\epsilon)s}}{(1+s)(s-z_n)}\,ds.$$

Or if $u < B - \epsilon$

$$H(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi(it)e^{-Bit}}{F(it)(1+it)} e^{iut}\,dt - \sum_1^\infty \frac{\phi(z_n)e^{(u-B)z_n}}{F'(z_n)(1+z_n)}.$$

That is, if

(24.22)
$$H_1(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi(it)e^{-Bit}}{F(it)(1+it)} e^{iut}\,dt$$

and

(24.23)
$$H_2(u) = \sum_1^\infty \frac{\phi(z_n)e^{(u+B)z_n}}{F'(z_n)(1+z_n)},$$

then

(24.24)
$$H(u) = H_1(u) - H_2(u), \qquad u < B - \epsilon.$$

The series for $H_2(u)$ converges for $u < B$ and represents an analytic function. This follows from (24.12) and (22.06). *Thus $H_2(u)$ is analytic for $u < B$.*

If $\delta = \frac{1}{2}\pi(D - L)$, then from (24.16)

(24.25)
$$\frac{\phi(iy)}{F(iy)} = O(e^{-\delta|y|}), \qquad |y| \to \infty.$$

If $w = u + iv$ then it follows from (24.22) and (24.25) that $H_1(w)$ is defined for $|v| < \delta$. The derivative $H_1'(w)$ also exists in this strip. *Thus $H_1(u)$ is analytic for $(-\infty < u < \infty)$.*

Since $H(u) = H_1(u) - H_2(u)$, it now follows that *$H(u)$ is analytic for $u < B - \epsilon$.* But by (24.21)

$$H(u) = 0, \qquad u < 0.$$

Since $H(u)$ is analytic, $u < B - \epsilon$, it follows that

(24.26)
$$H(u) = 0, \qquad u < B - \epsilon.$$

Thus (24.20) becomes

$$\frac{\psi(x)e^{-Bx}}{1+x} = \int_{B-\epsilon}^{\infty} H(u)e^{-ux}\,du.$$

By (24.18) and (24.19) $H(u)$ is bounded. Thus

$$\frac{\psi(x)e^{-Bx}}{1+x} = O(e^{-x(B-\epsilon)}), \qquad x \to \infty,$$

or

$$\psi(x) = O(e^{2\epsilon x}), \qquad x \to \infty.$$

Since $\phi(x) = g(x) + F(x)\psi(x)$, it follows that

$$\limsup_{x\to\infty} \frac{\log|\phi(x)|}{x} \leqq 2\epsilon.$$

Since ϵ can be taken arbitrarily small, we have completed the alternative proof of Theorem XXXIII.

The essential difference of this proof from those of §23 is that here we get $\psi(z)$ in terms of $H(u)$. In this section we are attempting to refine the theorems of §23 so that $D > L$ is no longer necessary. We shall now see how dropping $D > L$ affects the argument of the alternative proof just given. A perusal will show that the full force of $D > L$ is used only in the paragraph containing (24.25) where we prove that $H_1(u)$ is analytic, $(-\infty < u < \infty)$. Clearly if $D = L$, (24.25) will no longer hold and $H_1(u)$ need no longer be analytic. That $H_2(u)$ is analytic for $u < B$ regardless of how D compares with L follows from (24.23). Thus the question becomes this. Is there any weaker condition than analyticity that can be imposed on $H_1(u)$ that together with

$$H_1(u) = H_2(u), \qquad u < 0,$$

and $H_2(u)$ analytic for $u < B$, implies

$$H_1(u) = H_2(u), \qquad u < B?$$

That there are such conditions was proved in Chapter V. In particular Theorem XXIV will be of interest here. This theorem states that *if $G(t) \in L(-\infty, \infty)$ and if for $t \to \infty$*

(24.27)
$$G(t) = O(e^{-\theta(t)}),$$

where $\theta(t)$ is monotone increasing and

(24.28)
$$\int_1^\infty \frac{\theta(t)}{t^2}\,dt = \infty,$$

then if the Fourier transform of $G(t)$ coincides with an analytic function over some interval it coincides with the analytic function over its entire interval of analyticity.

In order to apply this theorem to $H_1(u)$, it follows from (24.22) that it must be shown that $\phi(it)/F(it)$ satisfies (24.27). This is the object of the following lemma.

Lemma 24.1. *If*

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

If $D > 0$ in (24.02), this theorem is an immediate consequence of Theorem XXXIII. However if $D = 0$ Theorem XXXIII no longer gives any results. Thus if $z_n = n \log (n + 1)$, (24.05) is satisfied and Theorem XXXVII applies whereas the theorems of §23 give no results.

Another theorem proved here is

THEOREM XXXVIII. *Let $\phi(z)$ be analytic and of exponential type in the half-plane* $| \operatorname{am} z | \leqq \frac{1}{2}\pi$. *Let*

(24.06) $$\phi(iy) = O(e^{\pi L|y|}).$$

Let $\{\lambda_n\}$ be an increasing positive sequence such that

(24.07) $$\lim_{n\to\infty} \frac{n}{\lambda_n} = D, \qquad \lambda_{n+1} - \lambda_n \geqq d > 0.$$

Let $\Lambda(u)$ be the number of $\lambda_n < u$. If

(24.08) $$\int_1^\infty \frac{\Lambda(y) - Ly}{y^2}\,dy = \infty$$

and if

(24.09) $$\Lambda(y) + t(y) > Ly$$

for some positive function $t(y)$ satisfying

(24.10) $$\int_1^\infty \frac{t(y)}{y^2}\,dy < \infty,$$

then

(24.11) $$\limsup_{n\to\infty} \frac{\log |\phi(\lambda_n)|}{\lambda_n} = \limsup_{n\to\infty} \frac{\log |\phi(x)|}{x}.$$

Thus if

$$\lim_{y\to\infty} \frac{\Lambda(y)}{y} = D$$

it is possible that $D = L$ and yet that (24.08) and therefore Theorem XXXVIII holds. In §23 it was necessary that $D > L$.

The method of this section can best be presented by giving an alternative proof of Theorem XXXIII of the preceding section.

Alternative proof of Theorem XXXIII. Clearly to prove this theorem it suffices to show that

(24.12) $$\limsup_{n\to\infty} \frac{\log |\phi(z_n)|}{|z_n|} \leqq 0$$

implies

$$\limsup_{x\to\infty} \frac{\log |\phi(x)|}{x} \leqq 0.$$

We again introduce

(24.13) $$g(z) = \sum_1^\infty \frac{\phi(z_n)e^{-\epsilon z_n}}{F'(z_n)(z - z_n)}\, e^{\epsilon z} F(z)$$

which by (24.12) exists for any $\epsilon > 0$. As in §23, $g(z_n) = \phi(z_n)$ and

(24.14) $$\psi(z) = \frac{\phi(z) - g(z)}{F(z)}$$

is analytic and of exponential type for $| \operatorname{am} z | \leqq \frac{1}{2}\pi$. From (24.14)

(24.15) $$|\psi(iy)| \leqq \left|\frac{\phi(iy)}{F(iy)}\right| + \sum_1^\infty \left| \frac{\phi(z_n)e^{-\epsilon z_n}}{F'(z_n)(iy - z_n)} \right|.$$

Since $D > L$, it follows from (22.05) and (23.06) that

(24.16) $$\frac{\phi(iy)}{F(iy)} = O(e^{-(1/2)\pi(D-L)|y|}), \qquad |y| \to \infty.$$

From the last two results

(24.17) $$\psi(iy) = O(1/|y|).$$

But $\psi(z)$ is of exponential type in the right half-plane. This and (24.17) by a theorem of Phragmén-Lindelöf, Theorem C′, give

$$\psi(re^{i\theta}) = O\left(\frac{1}{r}\, e^{Br\cos\theta}\right), \qquad |\theta| \leqq \tfrac{1}{2}\pi,$$

for some B. From this

(24.18) $$\frac{\psi(z)e^{-Bz}}{1+z} = O(1/|z|^2), \qquad |\operatorname{am} z| \leqq \tfrac{1}{2}\pi.$$

Using (24.18), it follows by the Cauchy integral theorem that for $x > 0$

$$\frac{\psi(z)e^{-Bz}}{1+z} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\psi(s)e^{-Bs}}{1+s}\,\frac{ds}{z-s}$$
$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\psi(s)e^{-Bs}}{1+s}\,ds \int_0^\infty e^{-u(z-s)}\,du$$
$$= \frac{1}{2\pi i}\int_0^\infty e^{-uz}\,du \int_{-i\infty}^{i\infty} \frac{\psi(s)e^{-Bs}}{1+s}\, e^{us}\,ds.$$

Or if

(24.19) $$H(u) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\psi(s)e^{-Bs}}{1+s}\, e^{us}\,du,$$

then for $x > 0$

(24.20) $$\frac{\psi(z)e^{-Bz}}{1+z} = \int_0^\infty H(u)e^{-uz}\,du.$$

(23.32) $$\limsup_{r\to\infty} \frac{\log |\Phi(re^{i\theta})|}{r} \leqq \pi b |\sin\theta| + C\cos\theta.$$

Theorem XXXIV now follows from Theorem XXXIII.

THEOREM XXXV. *If in Theorem XXXIV, (23.17) is replaced by*

(23.33) $$\Phi(z_n) = O(e^{-kz_n \log |z_n|})$$

where $k > 0$, then (23.18) is replaced by

(23.34) $$\Phi(re^{i\theta}) = O(\exp\{(-k\log r\cos\theta + \pi b|\sin\theta| + \epsilon)r\}), \quad (|\theta| \leqq \tfrac{1}{2}\pi).$$

Proof of Theorem XXXV. This proof is very much like the preceding one. Here again we set

$$\phi(z) = \frac{\Phi(z)}{\Gamma(1+az)}$$

and proceed with the single change of

$$\phi(z_n) = O(\exp\{-(a+k)|z_n|\log|z_n|\cos\theta_n + (a+\epsilon)|z_n|\})$$

in place of (23.21) until we reach (23.30) which, because we now define A as $(a+k)\log x$, becomes

$$|\phi(x)| \leqq M_3 \exp\{-(a+k)x\log x + \epsilon x\}\left(1 + \sum_1^\infty \left|\frac{\phi(z_n)\exp\{(a+k)z_n\log x\}}{F'(z_n)}\right|\right), \quad x > 0.$$

Continuing from here as in the preceding proof leads to

$$\Phi(x) = O(\exp\{-kx\log x + Cx\})$$

instead of to the $\Phi(x) = O(e^{Cx})$ of Theorem XXXIV. This leads instead of to (23.32) to the fact that $\Phi(z)\Gamma(1+kz)$ is of exponential type. Theorem XXXIII can now be applied to $\Phi(z)\Gamma(1+kz)$ to give Theorem XXXV.

THEOREM XXXVI. *If $\Phi(z)$ satisfies the requirements* of Theorem XXXIV *with (23.17) replaced by*

(23.35) $$\Phi(z_n) = O(e^{-kz_n\log|z_n|}),$$

and if

(23.36) $$k > 2b,$$

then

(23.37) $$\Phi(z) \equiv 0.$$

Proof of Theorem XXXVI. $\Phi(z)$ clearly satisfies the requirements of Theorem XXXV. By applying Carleman's theorem, Theorem B, to $\Phi(z)$ we have, assuming $\Phi(z)$ is not identically zero,

$$-A \leqq \frac{1}{2\pi}\int_1^R \log^+|\Phi(iy)\Phi(-iy)|\left(\frac{1}{y^2} - \frac{1}{R^2}\right)dy + \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2}\log^+|\Phi(Re^{i\theta})|\cos\theta\,d\theta$$

where A is some constant. Using (23.34) and replacing A by another constant A_1, we get

$$-A_1 \leqq (b+\epsilon)\int_1^R\left(\frac{1}{y^2} - \frac{1}{R^2}\right)y\,dy - \frac{k\log R}{\pi}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

or

$$-A_1 \leqq (b+\epsilon)\log R - \tfrac{1}{2}k\log R.$$

Letting $R \to \infty$, it follows that

$$2(b+\epsilon) \geqq k.$$

Since ϵ is arbitrary, it follows that

$$2b \geqq k.$$

But this contradicts (23.26). Thus $\Phi(z) \equiv 0$.

24. A sharper set of theorems. In this section we shall prove sharper theorems than those of §23. An example of the results of this section is given by the following theorem.

THEOREM XXXVII. *Let $\phi(z)$ be analytic and of exponential type in the sector $|\text{am}\, z| \leqq \frac{1}{2}\pi$. Let*[4]

(24.01) $$\phi(iy) = O(1), \qquad |y| \to \infty.$$

Let $\{z_n\}$ be a sequence of complex numbers such that

(24.02) $$\lim_{n\to\infty}\frac{n}{z_n} = D \geqq 0$$

and such that for some $d > 0$

(24.03) $$|z_n - z_m| \geqq |n - m|\,d.$$

A necessary and sufficient condition that

(24.04) $$\limsup_{n\to\infty}\frac{\log|\phi(z_n)|}{|z_n|} = \limsup_{x\to\infty}\frac{\log|\phi(x)|}{x},$$

is that

(24.05) $$\sum_1^\infty \frac{1}{|z_n|} = \infty.$$

[4] The condition (24.01) can easily be replaced by

$$\int_{-\infty}^{\infty}\frac{\log^+|\phi(iy)|}{1+y^2}dy < \infty,$$

as will be pointed out in the proof of this theorem.

and (23.25) it follows as in the proof of Theorem XXXII that

(23.26) $$\psi(z) = O(e^{10(a+|b|+D)r}).$$

Along the imaginary axis (assuming $\Re z_n > 1$ as we may with no restriction)

$$|\psi(iy)| \leqq \max \left|\frac{\phi(iy)}{F(iy)}\right| + \sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|.$$

From (22.05), (23.22), and $D > b + \frac{1}{2}a$ it follows that there exists an $M_1 > 0$ independent of A such that

$$\left|\frac{\phi(iy)}{F(iy)}\right| \leqq M_1.$$

Thus

(23.27) $$|\psi(iy)| \leqq M_1 + \sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|.$$

Since by (23.25)

$$g(z) = O(\exp\{-(A-\epsilon)r\cos\theta + (\pi D + \epsilon)r\sin\theta\}), \qquad r\to\infty, |\theta| \leqq \tfrac{1}{2}\pi,$$

and by (23.24)

$$\phi(z) = O(e^{-Ar\cos\theta}), \qquad r\to\infty, |\theta| < \tfrac{1}{4}\pi,$$

it follows easily from the definition of $\psi(z)$ that

(23.28) $$\psi(x) = O(e^{-(A-\epsilon)x}), \qquad x\to\infty.$$

Thus using (23.26), (23.27), and (23.28)

$$\psi(z) = O(e^{-(A-\epsilon)r\cos\theta})$$

by Theorem C′ of Phragmén-Lindelöf. Thus $\psi(z)e^{(A-\epsilon)z}$ is bounded in the right half-plane. It follows from (23.27), applying the Theorem C of Phragmén-Lindelöf, that

$$|e^{(A-\epsilon)z}\psi(z)| \leqq M_1 + \sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|, \qquad |\text{am } z| \leqq \tfrac{1}{2}\pi.$$

Since the right side is independent of ϵ, ϵ can be taken as zero and thus

(23.29) $$|\psi(x)| \leqq e^{-Ax}\left(M_1 + \sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|\right).$$

Using M_2 and M_3 to represent constants independent of A, it follows from (23.25) that

$$|g(x)| \leqq M_2 e^{-(A-\epsilon)x}\sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|.$$

Since

$$\phi(x) = g(x) + \psi(x)F(x)$$

it follows from the above two inequalities and $F(x) = O(e^{\epsilon x})$ that

$$|\phi(x)| \leqq M_3 e^{-(A-\epsilon)x}\left(1 + \sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|\right), \qquad x > 0.$$

Or setting $A = a\log x$, this becomes

(23.30) $$|\phi(x)| \leqq M_3 e^{-ax\log x + \epsilon x}\left(1 + \sum_1^\infty \left|\frac{\phi(z_n)e^{az_n\log x}}{F'(z_n)}\right|\right), \qquad x > 0.$$

From (23.21) and (22.06),

(23.31) $$\sum_1^\infty \left|\frac{\phi(z_n)e^{az_n\log x}}{F'(z_n)}\right| = O\left(\sum_1^\infty \exp\{-a|z_n|\log|z_n/x|\cos\theta_n + (\pi p + a + \epsilon)|z_n|\}\right).$$

But for $x > 0$,

$$\max_{0\leqq u<\infty}(e^{-\lambda u\log u/x}) = e^{\lambda x/e}.$$

If $B > 1$ is so large that $\log B > 2(\pi p + a + \epsilon)/a$, then (23.21) becomes

$$\sum_1^\infty \left|\frac{\phi(z_n)e^{az_n\log x}}{F'(z_n)}\right| = O\Big(\exp\{(\pi p + a + \epsilon)Bx + ax/e\}\sum_{|z_n|\leqq Bx} 1$$
$$+ \sum_{|z_n|>Bx}\exp\{-a|z_n|\log B\cos\theta_n + (\pi p + a + \epsilon)|z_n|\}\Big)$$
$$= O(x\exp\{(\pi p + a + \epsilon)Bx + ax/e\}$$
$$+ \sum_{|z_n|>Bx}\exp\{-|z_n|(2\cos\theta_n - 1)(\pi p + a + \epsilon)\})$$
$$= O(\exp\{(\pi p + 2a + \epsilon)Bx\}).$$

Using this in (23.30),

$$\phi(x) = O(\exp\{-ax\log x + Cx\})$$

where $C = (\pi p + 2a + \epsilon)B + \epsilon$. Using this in (23.19),

$$\Phi(x) = O(\Gamma(1 + ax)\exp\{-ax\log x + Cx\}).$$

Using Stirling's formula this becomes

$$\Phi(x) = O(e^{Cx}).$$

Using (23.15)

$$\Phi(iy) = O(e^{(\pi b+\epsilon)|y|}).$$

Using these last two results and (23.15) in Theorem C′ of Phragmén-Lindelöf, it follows that

From (23.11), (23.01) and (22.05) it follows that

$$\psi(re^{\pm i\alpha}) = O(\exp\{r(a\cos\alpha + b\sin\alpha - \pi D\sin\alpha + \epsilon)\} + \exp\{cr\cos\alpha\})$$

for any $\epsilon > 0$. Or setting $(\pi D - b)\tan\alpha = \gamma$,

(23.13) $$\psi(re^{\pm i\alpha}) = O(e^{r\cos\alpha(a-\gamma+\epsilon\sec\alpha)} + e^{cr\cos\alpha}).$$

Since $\pi D > b$, $\gamma > 0$. If we take $\epsilon < \frac{1}{2}\gamma\cos\alpha$, then (23.13) becomes

$$\psi(re^{\pm i\alpha}) = O(e^{pr\cos\alpha}), \qquad p = \max(a - \tfrac{1}{2}\gamma, c).$$

In other words $\psi(z)e^{-pz}$ is bounded for $\operatorname{am} z = \pm\alpha$. By the Phragmén-Lindelöf theorem, Theorem C, this and (23.12) imply that $\psi(z)e^{-pz}$ is bounded in the whole sector $(-\alpha, \alpha)$. Thus, in particular,

(23.14) $$\psi(x) = O(e^{px}), \qquad p = \max(a - \tfrac{1}{2}\gamma, c).$$

But from (23.10)

$$\phi(x) = \psi(x)F(x) + g(x).$$

Using (23.14), (22.04), and (23.09), this gives

$$\limsup_{x\to\infty} \frac{\log|\phi(x)|}{x} \leqq \max(a - \tfrac{1}{2}\gamma, c).$$

This contradicts our initial assumption (23.08) and proves the theorem.

Proof of Theorem XXXIII. Let us set

$$\limsup_{x\to\infty} \frac{\log|\phi(x)|}{x} = a.$$

Then by (23.06) and the Phragmén-Lindelöf theorem, Theorem C′,

$$\limsup_{r\to\infty} \frac{\log|\phi(re^{i\theta})|}{r} \leqq a\cos\theta + \pi L|\sin\theta|, \qquad |\theta| \leqq \tfrac{1}{2}\pi,$$

in the right half-plane. Theorem XXXIII now follows from Theorem XXXII.

THEOREM XXXIV. *Let $\Phi(z)$ be an analytic function in the right half-plane, $|\operatorname{am} z| \leqq \frac{1}{2}\pi$, such that*

(23.15) $$\Phi(re^{i\theta}) = O(\exp\{(a\log r\cos\theta + \pi b|\sin\theta| + \epsilon)r\}), \qquad |\theta| \leqq \tfrac{1}{2}\pi,$$

where $a \geqq 0$, $b \geqq -\frac{1}{2}a$, and ϵ is an arbitrary positive quantity. If $\{z_n\}$ satisfies (23.02) and (23.03) and if

(23.16) $$D > b + \tfrac{1}{2}a,$$

then

(23.17) $$\limsup_{n\to\infty} \frac{\log|\Phi(z_n)|}{|z_n|} \leqq \pi p$$

implies

(23.18) $$\Phi(re^{i\theta}) = O(\exp\{\pi(p\cos\theta + b|\sin\theta| + \epsilon)r\}), \qquad |\theta| \leqq \tfrac{1}{2}\pi.$$

To show that (23.16) is critical we consider $\Phi(z) = \Gamma(1 + z)\sin\frac{1}{2}\pi z$ for $z_n = 2n$. Then $D = \frac{1}{2}$, and by Stirling's formula for $\Gamma(1 + z)$ for complex z, Theorem F,

$$\Phi(re^{i\theta}) = O(\exp\{r\log r\cos\theta + \epsilon r\}), \qquad |\theta| \leqq \tfrac{1}{2}\pi.$$

Thus $b = 0$, $a = 1$ and therefore here $D = b + \frac{1}{2}a$. But here $p = -\infty$, and therefore the theorem does not hold.

Proof of Theorem XXXIV. We shall assume that $a > 0$, for if $a = 0$ we have Theorem XXXIII. Let

(23.19) $$\phi(z) = \frac{\Phi(z)}{\Gamma(1 + az)}.$$

From Stirling's formula it follows easily that for $|\theta| \leqq \frac{1}{2}\pi$ and large r

$$\log|\Gamma(1 + are^{i\theta})| = ar\log r\cos\theta - ar\theta\sin\theta - ar\cos\theta + \tfrac{1}{2}\log r + O(1).$$

Thus from (23.15)

(23.20) $$\phi(re^{i\theta}) = O(\exp\{(\pi b|\sin\theta| + a\theta\sin\theta + a\cos\theta + \epsilon)r\}), \quad |\theta| \leqq \tfrac{1}{2}\pi,$$

and from (23.17)

(23.21) $$\phi(z_n) = O(\exp\{(-a\log|z_n|\cos\theta_n + \pi p + a + \epsilon)|z_n|\}).$$

From (23.20)

(23.22) $$\limsup_{r\to\infty} \frac{\log|\phi(re^{i\theta})|}{r} \leqq \pi(b + \tfrac{1}{2}a)r|\sin\theta| + ar\cos\theta,$$

and from (23.21)

(23.23) $$\limsup_{n\to\infty} \frac{\log|\phi(z_n)|}{|z_n|} = -\infty.$$

Using $D > b + \frac{1}{2}a$ and the above two results in Theorem XXXIII gives

(23.24) $$\limsup_{x\to\infty} \frac{\log|\phi(x)|}{x} = -\infty.$$

We define

(23.25) $$g(z) = \sum_1^\infty \frac{\phi(z_n)e^{Az_n}}{F'(z_n)(z - z_n)} e^{-Az}F(z)$$

where A is any real number. That $g(z)$ exists follows from (23.23). We also consider

$$\psi(z) = \frac{\phi(z) - g(z)}{F(z)}.$$

Since $g(z_n) = \phi(z_n)$, $\psi(z)$ is analytic for $|\operatorname{am} z| \leqq \frac{1}{2}\pi$. From (22.05), (23.22),

CHAPTER VII

DETERMINATION OF THE RATE OF GROWTH OF ANALYTIC FUNCTIONS FROM THEIR GROWTH ON SEQUENCES OF POINTS[1]

23. Theorems of V. Bernstein. In this chapter we shall prove a set of theorems which, roughly stated, show that the rate of growth of an analytic function along a line can be determined by its rate of growth along a sufficiently dense sequence of points on the line. We shall deal exclusively with functions which are analytic and of order one in a sector. (Functions of other rates of growth can be transformed by mapping so as to fall under the scope of our theorems.)

The first theorem we shall prove here is the following.

THEOREM XXXII.[2] *Let $\phi(z)$ be analytic in some sector $|\operatorname{am} z| \leqq \alpha$. Suppose*

(23.01) $$\limsup_{r\to\infty} \frac{\log|\phi(re^{i\theta})|}{r} \leqq a\cos\theta + b|\sin\theta|, \qquad |\theta| \leqq \alpha.$$

Let $\{z_n\}$ be a sequence of complex numbers such that

(23.02) $$\lim_{n\to\infty} \frac{n}{z_n} = D,$$

where D is real, and such that for some $d > 0$

(23.03) $$|z_n - z_m| \geqq |n - m|\, d.$$

If

(23.04) $$\pi D > b,$$

then

(23.05) $$\limsup_{n\to\infty} \frac{\log|\phi(z_n)|}{|z_n|} = \limsup_{r\to\infty} \frac{\log|\phi(r)|}{r}.$$

A special case of Theorem XXXII is the following result.

THEOREM XXXIII. *Let $\phi(z)$ be analytic and of finite exponential type*[3] *in the*

[1] Cf. Levinson, *On the growth of analytic functions*, Transactions of the American Mathematical Society, vol. 43 (1938), p. 240.

[2] The theorems in §23 are due to V. Bernstein, *Séries de Dirichlet*, Paris, 1933, chap. 9. However Bernstein's proofs involved rather deep theorems in Dirichlet series which are entirely dispensed with here and replaced by methods of ordinary function theory. Also instead of the complex sequence $\{z_n\}$, used here, Bernstein's proofs are restricted to a real sequence.

[3] That is, $\phi(z) = O(e^{C|z|})$ for some C in the region under consideration.

half-plane $|\operatorname{am} z| \leqq \frac{1}{2}\pi$. Let

(23.06) $$\limsup_{|y|\to\infty} \frac{\log|\phi(iy)|}{y} = \pi L.$$

If $\{z_n\}$ satisfies (23.02) and (23.03), then

(23.07) $$D > L$$

implies (23.05).

It is easy to see by considering $\sin \pi z$ at the points $z_n = n$ that (23.07) is critical, for in this case $D = L = 1$ and it is clear that (23.05) is not true. Nevertheless we shall show in §24 that (23.07) can be replaced by a much weaker condition.

We now turn to the proof of these theorems.

Proof of Theorem XXXII. We observe that there is no loss of generality in taking $\alpha < \frac{1}{2}\pi$. Clearly, if (23.05) does not hold there exists a c such that

(23.08) $$\limsup_{n\to\infty} \frac{\log|\phi(z_n)|}{|z_n|} < c < a = \limsup_{x\to\infty} \frac{\log|\phi(x)|}{x}.$$

Let

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{z_n^2}\right).$$

Then it follows from (23.08) and (22.06) of Theorem XXXI that the series

(23.09) $$g(z) = \sum_1^\infty \frac{\phi(z_n)e^{-cz_n}}{F'(z_n)(z - z_n)} e^{cz} F(z)$$

converges and represents an entire function. Clearly

$$g(z_n) = \phi(z_n).$$

Therefore

(23.10) $$\psi(z) = \frac{\phi(z) - g(z)}{F(z)}$$

is analytic for $|\operatorname{am} z| \leqq \alpha$. Using (22.05), (23.01), and (23.09), it follows from

(23.11) $$\psi(z) = \frac{\phi(z)}{F(z)} - \sum_1^\infty \frac{\phi(z_n)e^{-cz_n}}{F'(z_n)(z - z_n)} e^{cz}$$

that

(23.12) $$\psi(z) = O(e^{(|c|+|a|+|b|)|z|})$$

for $|z - z_n| \geqq \frac{1}{4}d$. But $\psi(z)$ is analytic for $|\operatorname{am} z| \leqq \alpha$. Since (23.12) is true on the circles $|z - z_n| \geqq \frac{1}{4}d$, it must be true inside, since an analytic function takes its maximum value in a domain on the boundary. Thus (23.12) holds in the whole sector.

Again from (22.27)

$$\left|\frac{F(z)}{z-z_N}\right| \geqq \frac{1}{C_1}\exp\{-3B|z|\epsilon^{1/2} - 12\epsilon D|z|\}\,|\sin \pi Dz|. \tag{22.28}$$

In exactly the same way as (22.28) is obtained,

$$\left|\frac{F(z)}{z-z_N}\right| \geqq \frac{1}{C_1}\exp\{-3B|z|\epsilon^{1/2} - 12\epsilon D|z|\}\,|\cos \pi Dz|$$

can be obtained. Thus

$$\left|\frac{F(z)}{z-z_N}\right| \geqq \frac{1}{C_1}\exp\{-3B|z|\epsilon^{1/2} - 12\epsilon D|z|\}\max(|\cos \pi Dz|, |\sin \pi Dz|).$$

Redefining ϵ it follows from this for large $|z|$ that

$$\left|\frac{F(z)}{z-z_N}\right| \geqq \exp\{\pi Dr|\sin\theta| - \epsilon r\}. \tag{22.29}$$

Letting $z \to z_k$, z_N must eventually become z_k. Thus (22.29) gives

$$|F'(z_k)| \geqq e^{-\epsilon|z_k|},$$

which is (22.06). From (22.29) it also follows at once that

$$\frac{1}{F(z)} = O(\exp\{-\pi Dr|\sin\theta| + \epsilon r\}), \qquad |z - z_k| \geqq \tfrac{1}{8}c, \quad k = 1, 2, \dots,$$

which is (22.05). The results proved so far all hold for z in the right half-plane. But since $F(z)$ is even they must hold for all z. Thus for $D > 0$ we have demonstrated Theorem XXXI.

If $D = 0$ we already have (22.15) which states

$$|z - z_N|e^{-A|z|\delta^{1/2}} \leqq \prod_{\mathrm{II}}\left|1 - \frac{z^2}{z_n^2}\right| \leqq e^{A|z|\delta^{1/2}} \tag{22.30}$$

where $z_n \in \mathrm{II}$ if $\tfrac{1}{2}|z| < |z_n| < \tfrac{3}{2}|z|$. In I, $|z_n| \leqq \tfrac{1}{2}|z|$. Thus

$$1 \leqq \prod_{\mathrm{I}}\left|1 - \frac{z^2}{z_n^2}\right| \leqq \prod_{\mathrm{I}} 2\left|\frac{z^2}{z_n^2}\right|. \tag{22.31}$$

For large $|z_n|$, if $D = 0$, $|z_n| > n$. If K is the number of $z_n \in \mathrm{I}$, then

$$\prod_{\mathrm{I}} 2\left|\frac{z^2}{z_n^2}\right| = O\left(2^K\frac{|z|^{2K}}{(K!)^2}\right).$$

Using Stirling's formula for $K!$, this becomes

$$\prod_{\mathrm{I}} 2\left|\frac{z^2}{z_n^2}\right| = O\left(2^K\left|\frac{z}{K}\right|^{2K}e^{4K}\right) = O(\exp\{5K + 2K\log(|z|/K)\}). \tag{22.32}$$

But for large $|z|$ since $D = 0$, $k < \epsilon|z|$ for any $\epsilon > 0$. But since $x \log 1/x < x^{1/2}$,

$$\frac{K}{|z|}\log\frac{|z|}{K} \leqq \left(\frac{K}{|z|}\right)^{1/2} < \left(\frac{\epsilon|z|}{|z|}\right)^{1/2} = \epsilon^{1/2}.$$

Using this and $K < \epsilon|z|$ in (22.32),

$$\prod_{\mathrm{I}} 2\left|\frac{z^2}{z_n^2}\right| = O(e^{5\epsilon|z| + |z|\epsilon^{1/2}}).$$

Thus (22.31) becomes

$$1 \leqq \prod_{\mathrm{I}}\left|1 - \frac{z^2}{z_n^2}\right| = O(e^{5\epsilon|z| + |z|\epsilon^{1/2}}). \tag{22.33}$$

In III, $|z_n| \geqq \tfrac{3}{2}|z|$. Thus

$$\prod_{\mathrm{III}}\left(1 - \frac{r^2}{r_n^2}\right) \leqq \prod_{\mathrm{III}}\left|1 - \frac{z^2}{z_n^2}\right| \leqq \prod_{\mathrm{III}}\left(1 + \frac{r^2}{r_n^2}\right). \tag{22.34}$$

But

$$\prod_{\mathrm{III}}\left(1 + \frac{r^2}{r_n^2}\right) \leqq \exp\left\{r^2\sum_{\mathrm{III}}\frac{1}{r_n^2}\right\}.$$

Let $R(u)$ be the number of $r_n < u$. Then the above inequality becomes

$$\prod_{\mathrm{III}}\left(1 - \frac{r^2}{r_n^2}\right) \leqq \exp\left\{r^2\int_{3r/2}^{\infty}\frac{dR(u)}{u^2}\right\}$$

$$\leqq \exp\left\{r^2\int_{3r/2}^{\infty}\frac{2R(u)}{u^3}\,du\right\} \leqq \exp\left\{r\max_{u \geqq 3r/2}\frac{R(u)}{u}\right\} \leqq e^{\epsilon r}$$

for large r since $R(u)/u \to 0$ as $u \to \infty$. In much the same way

$$\prod_{\mathrm{III}}\left(1 - \frac{r^2}{r_n^2}\right) \geqq e^{-2\epsilon r}.$$

Thus (22.34) becomes

$$e^{-2\epsilon|z|} \leqq \prod_{\mathrm{III}}\left|1 - \frac{z^2}{z_n^2}\right| \leqq e^{\epsilon|z|}.$$

Combining this with (22.30) and (22.33) gives

$$|z - z_N|e^{-A|z|\delta^{1/2} - 2\epsilon|z|} \leqq |F(z)|$$

$$= O(\exp\{A|z|\delta^{1/2} + 5\epsilon|z| + |z|\epsilon^{1/2} + \epsilon|z|\})$$

Redefining ϵ in terms of ϵ and δ above, this gives

$$|z - z_N|e^{-\epsilon|z|} \leqq |F(z)| \leqq e^{\epsilon|z|}.$$

The conclusions of the theorem now follow easily for $D = 0$.

$$\left|\frac{n^2 - D^2 z^2}{z^2}\right| \geqq \tfrac{1}{2}D^2\epsilon. \tag{22.18}$$

Using (22.17) and (22.18),

$$1 - \epsilon \leqq \left|1 + \frac{z^2(z_n^2 D^2 - n^2)}{z_n^2(n^2 - D^2 z^2)}\right| \leqq 1 + \epsilon, \qquad n \geqq N_0, \tag{22.19}$$

for $|z_n| \leqq 2|z|$ and $z_n \in \text{I}$ or III. Since

$$1 + \frac{z^2(z_n^2 D^2 - n^2)}{z_n^2(n^2 - z^2 D^2)} = \frac{1 - z^2/z_n^2}{1 - z^2 D^2/n^2}, \tag{22.20}$$

(22.19) becomes

$$1 - \epsilon \leqq \left|\frac{1 - z^2/z_n^2}{1 - z^2 D^2/n^2}\right| \leqq 1 + \epsilon. \tag{22.21}$$

Clearly for large $|z|$ there exists a positive constant C_1, such that

$$\frac{1}{C_1} \leqq \prod_{1}^{N_0}\left|\frac{1 - z^2/z_n^2}{1 - z^2 D^2/n^2}\right| \leqq C_1.$$

The last two results give, since the number of z_n in I is less than $2|z|D$ for large $|z|$,

$$\begin{aligned}\frac{1}{C_1}(1 - \epsilon)^{2|z|D}\prod_{\text{I}}\left|1 - \frac{z^2 D^2}{n^2}\right| &\leqq \prod_{\text{I}}\left|1 - \frac{z^2}{z_n^2}\right| \\ &\leqq C_1(1 + \epsilon)^{2|z|D}\prod_{\text{I}}\left|1 - \frac{z^2 D^2}{n^2}\right|.\end{aligned} \tag{22.22}$$

For $|z_n| \geqq 2|z|$, by (22.17)

$$1 - \frac{1}{8}\frac{D^2\epsilon^2|z|^2}{n^2 - |z|^2 D^2} \leqq \left|1 + \frac{z^2(z_n^2 D^2 - n^2)}{z_n^2(n^2 - z^2 D^2)}\right| \leqq 1 + \frac{1}{8}\frac{D^2\epsilon^2|z|^2}{n^2 - |z|^2 D^2}.$$

Since $z_n \sim n/D$, for large $|z|$ this becomes

$$1 - \frac{\epsilon|z|^2 D^2}{n^2} \leqq \left|1 + \frac{z^2(z_n^2 D^2 - n^2)}{z_n^2(n^2 - z^2 D^2)}\right| \leqq 1 + \frac{\epsilon|z|^2 D^2}{n^2}.$$

Using this result with (22.20) and (22.21),

$$\begin{aligned}(1 - \epsilon)^{2|z|D}\prod_{|z_n| \geqq 2|z|}\left(1 - \frac{\epsilon|z|^2 D^2}{n^2}\right) &\leqq \prod_{\text{III}}\left|\frac{1 - z^2/z_n^2}{1 - D^2 z^2/n^2}\right| \\ &\leqq (1 + \epsilon)^{2|z|D}\prod_{|z_n| \geqq 2|z|}\left(1 + \frac{\epsilon|z|^2 D^2}{n^2}\right).\end{aligned} \tag{22.23}$$

Since for small x, $1 + x < e^x$ and $1 - x > e^{-2x}$, it follows that

$$\prod_{|z_n| \geqq 2|z|}\left(1 + \frac{\epsilon|z|^2 D^2}{n^2}\right) \leqq \exp\left\{\epsilon|z|^2 D^2 \sum_{n > |z|D}\frac{1}{n^2}\right\} \leqq e^{2\epsilon|z|D}$$

and

$$\prod_{|z_n| \geqq 2|z|}\left(1 - \frac{\epsilon|z|^2 D^2}{n^2}\right) \geqq \exp\left\{-2\epsilon|z|^2 D^2 \sum_{n > |z|D}\frac{1}{n^2}\right\} \geqq e^{-4\epsilon D|z|}.$$

Thus (22.23) becomes

$$e^{-8\epsilon D|z|} \leqq \prod_{\text{III}}\left|\frac{1 - z^2/z_n^2}{1 - z^2 D^2/n^2}\right| \leqq e^{4\epsilon D|z|}.$$

Combining this result with (22.22), (22.16), and (22.08),

$$\begin{aligned}\frac{|z - z_N|}{C_1}e^{-B|z|\epsilon^{1/2} - 12\epsilon|z|D}\prod_{\text{I, III}}\left|1 - \frac{z^2 D^2}{n^2}\right| \\ \leqq |F(z)| \leqq C_1 e^{B|z|\epsilon^{1/2} + 6\epsilon D|z|}\prod_{\text{I, III}}\left|1 - \frac{z^2 D^2}{n^2}\right|\end{aligned} \tag{22.24}$$

From the product formula for $\sin \pi z$,

$$\prod_{\text{I, III}}\left|1 - \frac{z^2 D^2}{n^2}\right| = \left|\frac{\sin \pi Dz}{\pi Dz}\right|\prod_{\text{II}}\frac{1}{|1 - z^2 D^2/n^2|}. \tag{22.25}$$

Let N_1 be a value of n such that $|Dz - n|$ is a minimum for $n = N_1$. Then proceeding just as in getting (22.16),

$$|zD - N_1|e^{-B|z|\epsilon^{1/2}} \leqq \prod_{\text{II}}\left|1 - \frac{z^2 D^2}{n^2}\right| \leqq e^{B|z|\epsilon^{1/2}}. \tag{22.26}$$

Using this result in (22.25) gives for large $|z|$

$$|\sin \pi Dz|e^{-2B|z|\epsilon^{1/2}} \leqq \prod_{\text{I, III}}\left|1 - \frac{z^2 D^2}{n^2}\right| \leqq \left|\frac{\sin \pi Dz}{zD - N_1}\right|e^{2B|z|\epsilon^{1/2}}.$$

The above result in (22.24) gives

$$\begin{aligned}\frac{1}{C_1}|z - z_N|\,|\sin \pi Dz|\,e^{-3B|z|\epsilon^{1/2} - 12\epsilon D|z|} \\ \leqq |F(z)| \leqq C_1\left|\frac{\sin \pi Dz}{zD - N_1}\right|e^{3B|z|\epsilon^{1/2} + 6\epsilon D|z|}\end{aligned} \tag{22.27}$$

For large $|z|$ if $z = re^{i\theta}$,

$$\left|\frac{\sin \pi Dz}{zD - N_1}\right| \leqq \exp\{\pi Dr|\sin\theta|\}.$$

Thus (22.27) gives

$$|F(z)| \leqq C_1 \exp\{3Br\epsilon^{1/2} + 6\epsilon Dr + \pi Dr|\sin\theta|\}.$$

Redefining ϵ this gives

$$F(z) = O(\exp\{\pi Dr|\sin\theta| + \epsilon r\})$$

which is (22.04).

(22.11) $$\left| \frac{1}{1 - z^2/z_n^2} \right| \leqq \frac{2 | z_n |}{| z_N - z_n |}.$$

By a prime we denote the omission from a product of the term $n = N$. By (22.11),

$$\prod_{\mathrm{II}}{}' \frac{1}{| 1 - z^2/z_n^2 |} \leqq \prod_{\mathrm{II}}{}' \frac{2 | z_n |}{| z_N - z_n |}.$$

If M represents the number of z_n in II, then since $| z_n - z_N | \geqq c | n - N |$,

(22.12) $$\prod_{\mathrm{II}}{}' \frac{1}{| 1 - z^2/z_n^2 |} \leqq 2^M \{(1 + \epsilon) | z |\}^M \prod_{\mathrm{II}}{}' \frac{1}{c | n - N |} \leqq \left(\frac{3}{c}\right)^M | z |^M \prod_{\mathrm{II}}{}' \frac{1}{| n - N |}.$$

Clearly at most two terms $| n - N |$ in the product on the right above can be equal. Thus if $[\frac{1}{2}(M - 1)]$ represents the largest integer less than or equal to $\frac{1}{2}(M - 1)$, then

$$\prod_{\mathrm{II}}{}' | n - N | \geqq ([\tfrac{1}{2}(M - 1)]!)^2.$$

Thus (22.12) becomes

$$\prod_{\mathrm{II}}{}' \frac{1}{| 1 - z^2/z_n^2 |} \leqq \left(\frac{3}{c}\right)^M \frac{| z |^M}{([\frac{1}{2}(M - 1)]!)^2}.$$

By Stirling's formula this becomes

$$\begin{aligned} \prod_{\mathrm{II}}{}' \frac{1}{| 1 - z^2/z_n^2 |} &\leqq \left(\frac{3}{c}\right)^M | z |^M e^{3M} e^{-M \log M} \\ &\leqq \left(\frac{3}{c}\right)^M e^{3M} \exp \{M \log | z | - M \log M\} \\ &= \left(\frac{3}{c}\right)^M e^{3M} \exp \left\{ | z | \left(\frac{M}{| z |} \log \frac{| z |}{M} \right) \right\}. \end{aligned}$$

But $M \leqq 2\epsilon | z | (D + \delta)$. Since $x \log 1/x < x^{1/2}$ for $x > 0$,

$$\frac{M}{| z |} \log \frac{| z |}{M} \leqq \left(\frac{M}{| z |}\right)^{1/2} < \frac{(2\epsilon | z | (D + \delta))^{1/2}}{| z |^{1/2}} = (2\epsilon(D + \delta))^{1/2}.$$

Thus

(22.13) $$\prod_{\mathrm{II}}{}' \frac{1}{| 1 - z^2/z_n^2 |} \leqq \left(\frac{3}{c}\right)^{2\epsilon | z | (D+\delta)} e^{10\epsilon | z | (D+\delta)} e^{| z | (2\epsilon(D+\delta))^{1/2}}.$$

Since z lies in the right half-plane,

$$\frac{1}{| 1 - z^2/z_N^2 |} = \frac{| z_N |}{| z - z_N | | 1 + z/z_N |} < \frac{2 | z_N |}{| z - z_N |}.$$

By the minimum property of z_N, for large $| z |$, $| z_N | < 2 | z |$. Therefore

$$\frac{1}{| 1 - z^2/z_N^2 |} \leqq \frac{4 | z |}{| z - z_N |}.$$

Also

$$\frac{4 | z |}{| z - z_N |} \geqq \frac{4 | z |}{| z | + | z_N |} \geqq \frac{4 | z |}{| z | + 2 | z |} > 1.$$

Thus regardless of whether or not $z_N \in$ II,

$$\prod_{\mathrm{II}} \frac{1}{| 1 - z^2/z_n^2 |} \leqq \frac{4 | z |}{| z - z_N |} \prod_{\mathrm{II}}{}' \frac{1}{| 1 - z^2/z_n^2 |}.$$

Using (23.13) it follows for large $| z |$ that

$$\prod_{\mathrm{II}} \frac{1}{| 1 - z^2/z_n^2 |} \leqq \left(\frac{3}{c}\right)^{2\epsilon | z | (D+\delta)} \frac{e^{10\epsilon | z | (D+\delta)} e^{| z | (3\epsilon(D+\delta))^{1/2}}}{| z - z_N |}.$$

It follows from this result and (22.09) that there exists an A independent $| z |$, δ, and ϵ such that for large $| z |$

(22.14) $$| z - z_N | e^{-A | z | \epsilon^{1/2} (D+\delta)^{1/2}} \leqq \prod_{\mathrm{II}} \left| 1 - \frac{z^2}{z_n^2} \right| \leqq e^{A | z | \epsilon^{1/2} (D+\delta)^{1/2}}.$$

If $D = 0$ let $\epsilon = \frac{1}{2}$. Then

(22.15) $$| z - z_N | e^{-A | z | \delta^{1/2}} \leqq \prod_{\mathrm{II}} \left| 1 - \frac{z^2}{z_n^2} \right| \leqq e^{A | z | \delta^{1/2}}.$$

If $D > 0$ let $\delta = 1$. Then if $B = A(D + 1)^{1/2}$

(22.16) $$| z - z_N | e^{-B | z | \epsilon^{1/2}} \leqq \prod_{\mathrm{II}} \left| 1 - \frac{z^2}{z_n^2} \right| \leqq e^{B | z | \epsilon^{1/2}}.$$

Until otherwise stated we shall assume $D > 0$ in what follows. For suf ciently large N_0, since $n/z_n \to D$,

(22.17) $$\left| \frac{D^2 z_n^2 - n^2}{z_n^2} \right| \leqq \tfrac{1}{8} D^2 \epsilon^2, \qquad n \geqq N$$

Let $z_n \in$ I or III, $(n \geqq N_0)$, and $| z_n | \leqq 2 | z |$. Clearly

$$| n^2 - z^2 D^2 | \geqq | D^2 z_n^2 - z^2 D^2 | - | n^2 - D^2 z_n^2 |.$$

Since $z_n \in$ I or III, $| | z_n | - | z | | \geqq \epsilon | z |$. Using this and (22.17) the abov inequality becomes

$$\begin{aligned} | n^2 - z^2 D^2 | &\geqq D^2 \epsilon | z | | z_n + z | - \tfrac{1}{8} D^2 \epsilon^2 | z_n |^2 \\ &\geqq D^2 \epsilon | z | | \tfrac{1}{2} z | - \tfrac{1}{2} D^2 \epsilon^2 | z |^2 = \tfrac{1}{2} D^2 \epsilon | z |^2 (1 - \epsilon). \end{aligned}$$

For sufficiently small ϵ this becomes

Using this and (21.05), then for some other A

$$|G(w)| \leqq Ae^{A|w|} \tag{21.12}$$

for $|w - \lambda_k| \geqq \frac{1}{4}c$, $(k > 0)$. But $G(w)$ is analytic in the whole plane. Thus the maximum value of $G(w)$ for $|w - \lambda_k| \leqq \frac{1}{4}c$ is taken on for $|w - \lambda_k| = \frac{1}{4}c$. Thus (21.12) holds for all w, and for some other A

$$e^{\delta w}G(w) = O(e^{A|w|}), \qquad |w| \to \infty.$$

The above result and (21.11), according to a theorem of Phragmén-Lindelöf (Theorem C), give

$$e^{\delta w}G(w) = O(1), \qquad |w| \to \infty,\ |\text{am}\, w| \leqq \gamma.$$

In particular then

$$G(u) = O(e^{-\delta u}), \qquad u \to \infty. \tag{21.13}$$

But $G(\lambda_n) = -a_nF'(\lambda_n)$ by (21.09), and therefore

$$a_n = O\left(\frac{e^{-\delta\lambda_n}}{|F'(\lambda_n)|}\right).$$

By (21.07) this becomes

$$a_n = O(e^{-\delta\lambda_n + \epsilon\lambda_n}).$$

Since ϵ is arbitrary,

$$a_n = O(e^{-\delta\lambda_n/2}).$$

But this would imply that the abscissa of convergence of $f(z)$ is at $-\frac{1}{2}\delta$ or to the left of $-\frac{1}{2}\delta$, which is contrary to our assumption that it is at zero. This completes the proof of the theorem.

22. A function with zeros having a density. Theorem XXX is obviously contained in

THEOREM XXXI. *Let $\{z_n\}$ be a sequence of complex numbers such that*

$$\lim_{n\to\infty} \frac{n}{z_n} = D \tag{22.01}$$

where D is real, and for some $c > 0$ let

$$|z_n - z_m| \geqq c|n - m|. \tag{22.02}$$

If

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{z_n^2}\right), \tag{22.03}$$

then, for any $\epsilon > 0$, as $r \to \infty$

$$F(re^{i\theta}) = O(\exp\{\pi Dr|\sin\theta| + \epsilon r\}), \tag{22.04}$$

$$\frac{1}{F(re^{i\theta})} = O(\exp\{-\pi Dr|\sin\theta| + \epsilon r\}), \qquad |re^{i\theta} - z_n| \geqq \tfrac{1}{4}c, \tag{22.05}$$

and, as $n \to \infty$,

$$\frac{1}{|F'(z_n)|} = O(e^{\epsilon|z_n|}). \tag{22.06}$$

Theorem XXXI is required in the next chapter. (22.02) can be considerably weakened but we shall not concern ourselves with this aspect here. Our aim is to use the results like Theorem XXXI to obtain a variety of gap and density theorems rather than carefully to investigate Theorem XXXI and similar results.

Proof of Theorem XXXI. Let us choose an $\epsilon > 0$. For fixed z and ϵ let $\{z_n\}$ be divided into three classes, I, II, and III defined as follows:

$$\begin{aligned} &z_n \in \text{I}, \quad |z_n| \leqq (1-\epsilon)|z|, \\ &z_n \in \text{II}, \quad (1-\epsilon)|z| < |z_n| < (1+\epsilon)|z|, \\ &z_n \in \text{III}, \quad |z_n| \geqq (1+\epsilon)|z|. \end{aligned} \tag{22.07}$$

Then

$$|F(z)| = \left(\prod_{\text{I}}\prod_{\text{II}}\prod_{\text{III}}\right)\left|1 - \frac{z^2}{z_n^2}\right|. \tag{22.08}$$

Since $n/z_n \to D$, the number of z_n in II for sufficiently large $|z|$ is less than $2\epsilon|z|(D + \delta)$ for $\delta > 0$. Thus

$$\prod_{\text{II}}\left|1 - \frac{z^2}{z_n^2}\right| \leqq \prod_{\text{II}} 3 \leqq 3^{2\epsilon|z|(D+\delta)} < e^{4\epsilon|z|(D+\delta)}. \tag{22.09}$$

From here on we assume that z is in the right half-plane. For sufficiently large n

$$\left|\frac{1}{1 - z^2/z_n^2}\right| = \frac{1}{|1 + z/z_n||1 - z/z_n|} \leqq \frac{1}{|1 - z/z_n|} = \left|\frac{z_n}{z - z_n}\right|. \tag{22.10}$$

Let a value of n which makes $|z - z_n|$ a minimum be N. Then

$$|z - z_n| \geqq |z - z_N|.$$

Thus

$$|z_N - z_n| \leqq |z - z_N| + |z_n - z| \leqq 2|z - z_n|$$

and therefore

$$\frac{1}{|z - z_n|} \leqq \frac{2}{|z_N - z_n|}.$$

Using this in (22.10) gives

and as[3] $n \to \infty$

(21.07) $$\frac{1}{F'(\lambda_n)} = O(e^{\epsilon\lambda_n}).$$

Theorem XXX will be a corollary of a more general theorem, whose proof will follow that of Theorem XXIX.

Proof of Theorem XXIX. With no loss of generality we can assume that the abscissa of convergence of $f(z)$ is $x = 0$. If there exists an interval of length greater than $2\pi D$ on $x = 0$ on which $f(z)$ has no singularity, we can also assume with no loss of generality that this interval is $-B \leqq y \leqq B$ where

$$B > \pi D.$$

If $f(z)$ contains no singularity on $x = 0$, $|y| \leqq B$, then there exists some $a > 0$ such that $f(z)$ is analytic for $x \geqq -a$, $|y| \leqq B$.

Let[4]

$$H(w) = \int_{-a}^{\infty} f(z)e^{wz}\, dz.$$

Then if $w = u + iv$, $H(w)$ is defined for $u < 0$. By the Cauchy integral theorem, we can change the path of integration to give

$$H(w) = \int_{-a}^{-a+iB} f(z)e^{wz}\, dz + \int_{-a+iB}^{\infty+iB} f(z)e^{wz}\, dz.$$

The path in the second integral is along the line $y = iB$ from $x = -a$ to $x = \infty$. Let $b > 0$. Then

$$H(w) = \int_{-a}^{-a+iB} f(z)e^{wz}\, dz + \int_{-a+iB}^{b+iB} f(z)e^{wz}\, dz + \int_{b+iB}^{\infty+iB} e^{wz}\left(\sum_1^{\infty} a_k e^{-\lambda_k z}\right) dz.$$

The condition $\lim n/\lambda_n = D$ causes the Dirichlet series for $f(z)$ to converge absolutely for $x > 0$ just as is the case with power series. Thus

(21.08) $$H(w) = \int_{-a}^{-a+iB} f(z)e^{wz}\, dz + \int_{-a+iB}^{b+iB} f(z)e^{wz}\, dz - \sum_1^{\infty} a_k \frac{e^{(w-\lambda_k)(b+iB)}}{w - \lambda_k}.$$

But now $H(w)$ is defined for all w. If $F(w)$ is defined as in Theorem XXX, then by (21.08)

$$G(w) = H(w)F(w)$$

is an entire function, and

(21.09) $$G(\lambda_n) = -a_n F'(\lambda_n).$$

By (21.08) if $w = \rho e^{i\gamma}$, $0 < \gamma < \frac{1}{2}\pi$, then

[3] It is possible to replace (21.03) in Theorems XXIX and XXX by less restrictive conditions. What is essential is that these conditions imply (21.07).

[4] The function $H(w)$ is used by V. Bernstein, *Séries de Dirichlet*, Paris, 1933, p. 111.

$$|H(\rho e^{i\gamma})| \leqq \exp\{-a\rho\cos\gamma\}\int_{-a}^{-a+iB} |f(z)\, dz|$$
$$+ \exp\{b\rho\cos\gamma - B\rho\sin\gamma\}\int_{-a+iB}^{b+iB} |f(z)\, dz|$$
$$+ \exp\{b\rho\cos\gamma - B\rho\sin\gamma\}\sum_1^{\infty}\frac{|a_k|\, e^{-b\lambda_k}}{b\sin\gamma}.$$

By choosing the path of integration as the reflection in the real axis of that used in (21.08), the above inequality holds for $\rho e^{-i\gamma}$. Thus

$$H(\rho e^{\pm i\gamma}) = O(\exp\{-a\rho\cos\gamma\} + \exp\{b\rho\cos\gamma - B\rho\sin\gamma\}).$$

Using this and (21.05),

$$G(\rho e^{\pm i\gamma}) = O(\exp\{-a\rho\cos\gamma + \pi D\rho\sin\gamma + \epsilon\rho\}$$
$$+ \exp\{b\rho\cos\gamma - (B - \pi D)\rho\sin\gamma + \epsilon\rho\}).$$

Since $B > \pi D$ we can take

$$b = \tfrac{1}{2}(B - \pi D)\tan\gamma.$$

Then $B - \pi D = 2b \operatorname{ctn}\gamma$ and

$$G(\rho e^{\pm i\gamma}) = O(\exp\{-a\rho\cos\gamma + \pi D\rho\sin\gamma + \epsilon\rho\} + \exp\{-b\rho\cos\gamma + \epsilon\rho\}).$$

Since ϵ is arbitrarily small,

$$G(\rho e^{\pm i\gamma}) = O(\exp\{-\tfrac{1}{2}a\rho\cos\gamma + \pi D\rho\sin\gamma\} + \exp\{-\tfrac{1}{2}b\rho\cos\gamma\}).$$

If $D = 0$ then for $\gamma = \frac{1}{4}\pi$,

$$G(\rho e^{\pm i\gamma}) = O(e^{-\delta\rho\cos\gamma}), \qquad \{\delta = \tfrac{1}{4}\min(a, b)\}.$$

If $D > 0$, let

$$\gamma = \tan^{-1}\frac{a}{4\pi D}.$$

Then $\pi D\sin\gamma = \frac{1}{4}a\cos\gamma$ and

$$G(\rho e^{\pm i\gamma}) = O(\exp\{-\tfrac{1}{4}a\rho\cos\gamma\} + \exp\{-\tfrac{1}{2}b\rho\cos\gamma\}),$$

and again

(21.10) $$G(\rho e^{\pm i\gamma}) = O(e^{-\delta\rho\cos\gamma}).$$

Thus for $\delta > 0$,

(21.11) $$e^{\delta w}G(w) = O(1), \qquad |w| \to \infty,\ \operatorname{am} w = \pm\gamma.$$

By (21.08) there exists an A such that for $|w - \lambda_k| \geqq \frac{1}{4}c$, $(k > 0)$,

$$|H(w)| \leqq Ae^{A|w|}.$$

CHAPTER VI

A DENSITY THEOREM OF PÓLYA

21. The density theorem. In this chapter we shall prove what might be called the classical density theorem although the theorem started as a gap result.

If a sufficient number of the coefficients of a power series are zero, every point on its circle of convergence is a singular point. This was first proved by Hadamard for the power series

$$\sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

where

$$\lambda_{n+1}/\lambda_n \geqq c > 1.$$

An example of such a series is

$$F(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

The points

$$\{k/2^m\}, \qquad 0 < k < 2^n, m = 1, 2, \dots,$$

are everywhere dense on (0, 1). Let us consider the series $F(z)$ as $z \to e^{2\pi i k/2^m}$ along a radius from $z = 0$. Clearly here $\lim |F(z)| = \infty$. Since this is the case for an everywhere dense set of points on the unit circle, the unit circle is a natural boundary for $F(z)$.

This result has been extended and improved considerably. Let us consider the following series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^{3n}.$$

$f(z)$ must possess at least one singular point on its circle of convergence. Let this point be z_0. Then the points $z_0 e^{2\pi i/3}$ and $z_0 e^{4\pi i/3}$ must also be singular points. Thus if $\lambda_n = 3n$, $f(z)$ will have at least one singular point on every arc on its circle of convergence of circular measure exceeding $2\pi/3$. This result remains true if $\lambda_n = 3n$ is replaced by the much less restrictive condition

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{1}{3}.$$

This follows from the following theorem which is stated for Dirichlet series and therefore includes power series as a special case.

THEOREM XXIX.[1] *Let*

$$(21.01) \qquad f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

where

$$(21.02) \qquad \lim_{n\to\infty} \frac{n}{\lambda_n} = D$$

and for some $c > 0$

$$(21.03) \qquad \lambda_{n+1} - \lambda_n \geqq c.$$

Then on the abscissa of convergence there is at least one singularity in every interval of length exceeding $2\pi D$.

In particular if $D = 0$, the abscissa of convergence is a natural boundary. Theorems involving a sequence $\{\lambda_n\}$ of density D are very often proved by using the entire function

$$F(z) = \prod_1^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

In other words an important way to make use of the properties of $\{\lambda_n\}$ is by means of the corresponding properties of $F(z)$. For example $n/\lambda_n \to D$ implies that

$$\lim_{|y|\to\infty} \frac{\log F(iy)}{|y|} = \pi D.$$

In certain respects this last result is a much easier one to make use of than $n/\lambda_n \to D$. Thus the proof of Theorem XXIX makes use of the following theorem.

THEOREM XXX.[2] *If* $\{\lambda_n\}$ *satisfies* (21.02) *and* (21.03) *and if*

$$(21.04) \qquad F(z) = \prod_1^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right),$$

then as $r \to \infty$, *for* $\epsilon > 0$

$$(21.05) \qquad F(re^{i\theta}) = O(\exp\{\pi D r |\sin\theta| + \epsilon r\}),$$

$$(21.06) \qquad \frac{1}{F(re^{i\theta})} = O(\exp\{-\pi D r |\sin\theta| + \epsilon r\}), \qquad |re^{i\theta} - \lambda_n| \geqq \tfrac{1}{2}c,$$

[1] Pólya, *Ueber die Existenz unendlich vieler singulärer Punkte auf der Konvergenzgeraden gewisser Dirichletscher Reihen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1923, pp. 45–50.

[2] F. Carlson, *Über Potenzreihen mit endlich vielen verschiedenen Koeffizienten*, Mathematische Annalen, vol. 79 (1919), pp. 237–245.

AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS
VOLUME XXVI

GAP AND DENSITY THEOREMS

BY
NORMAN LEVINSON
ASSISTANT PROFESSOR OF MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Chapter VI & VII

PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
531 West 116th Street, New York City
1940