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# 15

# The Hilbert Transform



## 15.1 The Hilbert Transform

### 15.1.1 Definition of Hilbert Transform

$$\upsilon(t) = \mathrm{H}\{x(t)\} = \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{x(\eta)}{\eta - t} d\eta = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{x(\eta)}{t - \eta} d\eta$$

$$x(t) = \mathrm{H}^{-1}\{\upsilon(t)\} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\upsilon(\eta)}{\eta - t} d\eta = \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{\upsilon(\eta)}{t - \eta} d\eta$$

where P stands for the Cauchy principal value of the integral.

**Convolution form representation**

$$v(t) = x(t) * \frac{1}{\pi t}$$

$$x(t) = -\upsilon(t) * \frac{1}{\pi t}$$

**Fourier transform of $\nu(t)$ and $x(t)$ and $1/\pi t$ (see Table 3.1.3)**

$$V(\omega) = X(\omega)[-j\,\mathrm{sgn}(\omega)]$$

$$\mathcal{F}^{-1}\{-j\,\mathrm{sgn}(\omega)\,X(\omega)\} = \upsilon(t)$$

$$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = -j\,\mathrm{sgn}(\omega)$$

**Example**

If $x(t) = \cos\omega t,$ then

$$\begin{aligned}\mathcal{H}\{\cos\omega t\} &= \upsilon(t)\\ &= \frac{-1}{\pi} P\int_{-\infty}^{\infty} \frac{\cos\omega\eta}{\eta - t}\,d\eta\\ &= \frac{-1}{\pi} P\int_{-\infty}^{\infty} \frac{\cos[\omega(y+t)]}{y}\,dy\\ &= \frac{-1}{\pi}\left\{\cos\omega t\, P\int_{-\infty}^{\infty}\frac{\cos\omega y}{y}\,dy - \sin\omega t\, P\int_{-\infty}^{\infty}\frac{\sin\omega y}{y}\,dy\right\}\\ &= \sin\omega t.\end{aligned}$$

The result is due to the fact that $\cos\omega y / y$ is an odd function and $P\int_{-\infty}^{\infty}\frac{\sin\omega y}{y}\,dy = \pi.$

**Example**

If $x(t) = p_a(t)$ then

$$\begin{aligned}v(t) = \mathcal{H}\{p_a(t)\} &= \frac{-1}{\pi} P\int_{-a}^{t-\varepsilon}\frac{d\eta}{\eta - t} - \frac{1}{\pi t} P\int_{t+\varepsilon}^{a}\frac{d\eta}{\eta - t}\\ &= \lim_{\varepsilon\to 0}\left[-\frac{1}{\pi}\ln(\eta - t)\Big|_{-a}^{t-\varepsilon} - \frac{1}{\pi}\ln(\eta - t)\Big|_{t+\varepsilon}^{a}\right] = \upsilon(t) = \frac{1}{\pi}\ln\left|\frac{t+a}{t-a}\right|\end{aligned}$$

**Example**

If $x(t) = a$ then

$$a\mathcal{H}\{1\} = a\lim_{a\to\infty}\frac{1}{\pi}\ln\left|\frac{t+a}{t-a}\right| = 0\,.$$

Hence, if $x_o$ = *constant* is the mean value of a function, then $x(t) = x_o + x_1(t)$. Therefore $\mathcal{H}\{x_o + x_1(t)\}$ = $\mathcal{H}\{x_1(t)\}$. This implies that the Hilbert transform cancels the mean value or the DC term in electrical engineering terminology.

## 15.1.2 Analytic Signal

A complex signal whose imaginary part is the Hilbert transform of its real part is called the *analytic signal*.

$$\psi(z) = \psi(t,\tau) = x(t,\tau) + j\upsilon(t,\tau), \quad x \text{ and } \upsilon \text{ are real functions}$$

$$z = t + j\tau$$

$$\upsilon(t,\tau) = \mathrm{H}\{x(t,\tau)\}$$

The function $\psi(z) = x(t,\tau) + j\upsilon(t,\tau)$ is analytic if the Cauchy-Riemann conditions

$$\frac{\partial x}{\partial t} = \frac{\partial \upsilon}{\partial \tau} \quad \text{and} \quad \frac{\partial x}{\partial \tau} = -\frac{\partial \upsilon}{\partial t}$$

are satisfied.

**Example**

The real and imaginary parts of the analytic function

$$\psi(z) = 1/(\alpha - jz) = \frac{\alpha + \tau}{(\alpha+\tau)^2 + t^2} + j\frac{t}{(\alpha+\tau)^2 + t^2}$$

satisfy Cauchy-Riemann conditions and, hence, they are Hilbert transform pairs.

$$x(t) = \frac{\psi(t) + \psi^*(t)}{2} \qquad \upsilon(t) = \frac{\psi(t) - \psi^*(t)}{2j} \quad (\tau = 0)$$

## 15.2 Spectra of Hilbert Transformation

### 15.2.1 One-Sided Spectrum of the Analytic Signal

$$x(t) = x_e(t) + x_o(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2}$$

$$X(\omega) = X_r(\omega) + jX_i(\omega) = \int_{-\infty}^{\infty} x_e(t)\cos\omega t\,dt + j\left(-\int_{-\infty}^{\infty} x_o(t)\sin\omega t\,dt\right)$$

$$V(\omega) = V_r(\omega) + jV_i(\omega) = \text{Spectrum of the Hilbert transform}$$

$$V_r(\omega) = -j\,\mathrm{sgn}(\omega)[jX_i(\omega)] = \mathrm{sgn}(\omega)\,X_i(\omega) \text{ (see also 15.1.1)}$$

$$V_i(\omega) = -\mathrm{sgn}(\omega)\,X_r(\omega)$$

**Example**

$\mathrm{H}\{\cos\omega t\} = \sin\omega t,\ \mathrm{H}\{\sin\omega t\} = -\cos\omega t$ and, therefore,

$$\mathrm{H}\{e^{j\omega t}\} = \sin\omega t - j\cos\omega t = -j\,\mathrm{sgn}(\omega)e^{j\omega t} = \mathrm{sgn}(\omega)e^{j(\omega t - \frac{\pi}{2})}$$

***Note:*** The operator $-j\,\mathrm{sgn}(\omega)$ provides a $\pi/2$ phase lag for all positive frequencies and $\pi/2$ lead for all negative frequencies.

### 15.2.2 Fourier Spectrum of the Analytic Signal

$$\mathrm{H}\{x(t)\} = \upsilon(t); \quad \mathrm{F}\{x(t)\} = X(\omega); \quad \mathrm{F}\{\upsilon(t)\} = -j\,\mathrm{sgn}(\omega)\,X(\omega)$$

$$\mathcal{F}\{\psi(t)\} = x(t) + j\upsilon(t) = \Psi(\omega) = X(\omega) + jV(\omega) = [1+\operatorname{sgn}(\omega)]X(\omega)$$

$$1+\operatorname{sgn}(\omega) = \begin{cases} 2 & \omega > 0 \\ 1 & \omega = 0 \\ 0 & \omega < 0 \end{cases}$$

***Note:*** The spectrum of the analytic signal is twice that of its Fourier transform at the positive frequency range $0 < \omega < \infty$.

**Example**

If $\psi(t) = \dfrac{1}{1+t^2} + j\dfrac{t}{1+t^2}$ then $\mathcal{F}\{\psi(t)\} = [1+\operatorname{sgn}(\omega)]\pi e^{-|\omega|}$ where

$$\mathcal{H}\{1/(1+t^2)\} = t/(1+t^2) \text{ and } \mathcal{F}\{1/(1+t^2)\} = \pi e^{-|\omega|}.$$

## 15.3 Hilbert Transform and Delta Function

### 15.3.1 Complex Delta Function

If we define $2\cdot 1(f) = 1(f) + \operatorname{sgn}(f)$, then the function (see Fourier transform properties [symmetry] and function, Chapter 3).

$$\psi_\delta(t) = \int_{-\infty}^{\infty} 2\cdot 1(f) e^{j\omega t}\, df = \int_{-\infty}^{\infty} 1(f) e^{j\omega t}\, df + \int_{-\infty}^{\infty} \operatorname{sgn}(f) e^{j\omega t}\, df$$

$$= \delta(t) + j\frac{1}{\pi t}$$

### 15.3.2 Hilbert Transform of the Delta Function

From (15.3.1) implies

$$\mathcal{H}\{\delta(t)\} = \frac{1}{\pi t}$$

## 15.4 Hilbert Transform of Periodic Signals

### 15.4.1 Hilbert Transform of Period Functions

A periodic function can be written in trigonometric form

$$x_p(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\omega_o t + \varphi_n), \quad \omega_o = 2\pi/T, \quad T = \text{period}$$

Therefore we obtain

$$v_p(t) = \mathcal{H}\{x_p(t)\} = \sum_{n=1}^{\infty} C_n \sin(n\omega_o t + \varphi_n)$$

because the Hilbert transform of a constant is zero (see 15.1.1).

A periodic function can also be written in *complex* form

$$x_p(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_o t}$$

Therefore,

$$v_p(t) = \mathrm{H}\{x_p(t)\} = \sum_{n=-\infty}^{\infty} \alpha_n \mathrm{H}\{e^{jn\omega_o t}\} = \sum_{n=-\infty}^{\infty} -j\,\mathrm{sgn}(n) e^{jn\omega_o t}$$

## 15.5 Hilbert Transform Properties and Pairs

### 15.5.1 Hilbert Transform Properties

**TABLE 15.1** Properties of the Hilbert transformation

| No. | Name | Original or Inverse Hilbert Transform | Hilbert Transform |
|---|---|---|---|
| 1 | Notations | $x(t)$ or $\mathrm{H}^{-1}[\upsilon]$ | $\upsilon(t)$ or $\hat{x}(t)$ or $\mathrm{H}[\upsilon]$ |
| 2 | Time domain definitions | $x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\upsilon(\eta)}{\eta - t}d\eta$; $x(t) = \frac{-1}{\pi t} * \upsilon(t)$ or | $\upsilon(t) = \frac{-1}{\pi}\int_{-\infty}^{\infty}\frac{x(\eta)}{\eta - t}d\eta$; $\upsilon(t) = \frac{1}{\pi t} * x(t)$ |
| 3 | Change of symmetry | $x(t) = x_{1e}(t) + x_{2o}(t)^*$; | $\upsilon(t) = \upsilon_{1o}(t) + \upsilon_{2e}(t)$ |
| 4 | Fourier spectra | | |
| | | $x(t) = \overset{F}{\Longleftrightarrow} X(\omega) = X_e(\omega) + jX_o(\omega)$; | $\upsilon(t) = \overset{F}{\Longleftrightarrow} V(\omega) = V_e(\omega) + jV_o(\omega)$ |
| | | $X(\omega) = j\,\mathrm{sgn}(\omega) V(\omega)$; | $V(\omega) = -j\,\mathrm{sgn}(\omega) X(\omega)$ |
| | For even functions the Hilbert transform is odd: | | |
| | | $X_e(\omega) = 2\int_o^{\infty} x_{1e}(t)\cos(\omega t)\,dt$ | $\upsilon_o(t) = 2\int_o^{\infty} X_e(\omega)\sin(\omega t)\,df$ |
| | For odd functions the Hilbert transform is even: | | |
| | | $X_o(\omega) = -2\int_o^{\infty} x_{2o}(t)\sin(\omega t)\,dt$ | $\upsilon_e(t) = 2\int_o^{\infty} X_o(\omega)\cos(\omega t)\,df$ |
| 5 | Linearity | $ax_1(t) + bx_2(t)$ | $a\upsilon_1(t) + b\upsilon_2(t)$ |
| 6 | Scaling and | $x(at)$; $a > 0$ | $\upsilon(at)$ |
| | time reversal | $x(-at)$ | $-\upsilon(-at)$ |
| 7 | Time shift | $x(t-a)$ | $\upsilon(t-a)$ |
| 8 | Scaling and time shift | $x(bt-a)$ | $\upsilon(bt-a)$ |
| | | | Fourier image |
| 9 | Iteration | $\mathrm{H}[x(t)] = \upsilon(t)$ | $-j\,\mathrm{sgn}(\omega) X(\omega)$ |
| | | $\mathrm{H}[\mathrm{H}[x]] = -x(t)$ | $[-j\,\mathrm{sgn}(\omega)]^2 X(\omega)$ |
| | | $\mathrm{H}[\mathrm{H}[\mathrm{H}[x]]] = -\upsilon(t)$ | $[-j\,\mathrm{sgn}(\omega)]^3 X(\omega)$ |
| | | $\mathrm{H}[\mathrm{H}[\mathrm{H}[\mathrm{H}[x]]]] = x(t)$ | $[-j\,\mathrm{sgn}(\omega)]^4 X(\omega)$ |

e = even; o = odd

**TABLE 15.1** Properties of the Hilbert transformation (continued)

| No. | Name | Original or Inverse Hilbert Transform | Hilbert Transform |
|---|---|---|---|
| | | | First option |
| 10 | Time derivatives | $\dot{x}(t) = \frac{-1}{\pi t} * \dot{\upsilon}(t)$ | $\dot{\upsilon}(t) = \frac{1}{\pi t} * \dot{x}(t)$ |
| | | | Second option |
| | | $\dot{x}(t) = \left[\frac{d}{dt}\frac{-1}{\pi t}\right] * \upsilon(t)$ | $\dot{\upsilon}(t) = \left[\frac{d}{dt}\frac{1}{\pi t}\right] * x(t)$ |
| 11 | Convolution | $\begin{cases} x_1(t) * x_2(t) = \\ -\upsilon_1(t) * \upsilon_2(t) \end{cases}$ | $x_1(t) * \upsilon_2(t) =$ $\upsilon_1(t) * x_2(t)$ |
| 12 | Autoconvolution equality | $\int x(\tau)\,x(t-\tau)\,d\tau = -\int \upsilon(\tau)\,\upsilon(t-\tau)\,d\tau$ for $\tau = 0$ energy equality | |
| 13 | Multiplication by $t$ | $t\,x(t)$ | $t\,\upsilon(t) - \int_{-\infty}^{\infty} x(\tau)\,d\tau$ |
| 14 | Multiplication of signals with non-overlapping spectra | $x_1(t)$ (low-pass signal) | $x_2(t)$ (high-pass signal) |
| | | $x_1(t)\,x_2(t)$ | $x_1(t)\,\upsilon_2(t)$ |
| 15 | Analytic signal | $\psi(t) = x(t) + j\,\mathsf{H}[x(t)]$ | $\mathsf{H}[\psi(t)] = -j\,\psi(t)$ |
| 16 | Product of analytic signals | $\psi(t) = \psi_1(t)\,\psi_2(t)$ | $\mathsf{H}[\psi(t)] = \psi_1(t)\mathsf{H}[\psi_2(t)]$ $= \mathsf{H}[\psi_1(t)]\psi_2(t)$ |
| 17 | Nonlinear transformations | $x(x)$ | $\upsilon(x)$ |
| 17a | $y = \frac{c}{bt+a}$ | $x_1(t) = x\left[\frac{c}{bt+a}\right]$ | $\upsilon_1(t) = \upsilon\left[\frac{c}{bt+a}\right] - \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{x(t)}{t}\,dt$ |
| 17b | $y = a + \frac{b}{t}$ | $x_1(t) = x\left[a + \frac{b}{t}\right]$ | $\upsilon_1(t) = \frac{b}{a}\left\{\upsilon\left[a + \frac{b}{t}\right] - \upsilon(a)\right\}$ |
| | Notice that the nonlinear transformation may change the signal $x(t)$ of finite energy to a signal $x_1(t)$ of infinite energy. $P$ is the Cauchy Principal Value. | | |
| 18 | Asymptotic value as $t \Rightarrow \infty$ for even functions of finite support: | | |
| | $x_e(t) = x_e(-t)$ | | $\lim_{t\Rightarrow\infty} \lvert \upsilon_o(t) \rvert = \frac{1}{\pi t}\int_S x_e(t)\,dt$ [a] |

[a] $S$ is support of $x_e(t)$

## 15.5.2 Iteration

- Iteration of the HT two times yields the original signal with reverse sign.
- Iteration of the HT four times restores the original signal
- In Fourier frequency domain, n-time iteration translates the n-time multiplication by $-j\mathrm{sgn}(\omega)$

## 15.5.3 Parseval's Theorem

$$v(t) = \mathsf{H}\{x(t)\}$$

$$\mathcal{F}\{\upsilon(t)\} = V(\omega) = -j\,\text{sgn}(\omega)\,X(\omega)$$

$$|V(\omega)|^2 = |-j\,\text{sgn}(\omega)\,X(\omega)|^2 = |X(\omega)|^2$$

since

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} |X(\omega)|^2\,df = \text{energy of } x(t)$$

$$E_\upsilon = \int_{-\infty}^{\infty} |V(\omega)|^2\,df = \int_{-\infty}^{\infty} |X(\omega)|^2\,df = E_x$$

### 15.5.4 Orthogonality

$$\int_{-\infty}^{\infty} \upsilon(t)\,x(t)\,dt = 0$$

### 15.5.5 Fourier Transform of the Autoconvolution of the Hilbert Pairs

$$\mathcal{F}\{x(t) * x(t)\} = X^2(\omega)$$

$$\mathcal{F}\{\upsilon(t) * \upsilon(t)\} = [-j\,\text{sgn}(\omega)\,X(\omega)]^2 = -X^2(\omega)$$

$$x(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\,x(t-\tau)\,d\tau = -\int_{-\infty}^{\infty} \upsilon(\tau)\,\upsilon(t-\tau)\,d\tau = -\upsilon(t) * \upsilon(t)$$

$$x_1(t) * x_2(t) = -\upsilon_1(t) * \upsilon_2(t)$$

### 15.5.6 Hilbert Transform Pairs

**TABLE 15.2** Selected Useful Hilbert Pairs

| No. | Name | Function | Hilbert Transform |
|---|---|---|---|
| 1 | sine | $\sin(\omega t)$ | $-\cos(\omega t)$ |
| 2 | cosine | $\cos(\omega t)$ | $\sin(\omega t)$ |
| 3 | Exponential | $e^{j\omega t}$ | $-j\,\text{sgn}(\omega)\,e^{j\omega t}$ |
| 4 | Square pulse | $\prod_{2a}(t)$ | $\frac{1}{\pi}\ln\left\|\frac{t+a}{t-a}\right\|$ |
| 5 | Bipolar pulse | $\prod_{2a}(t)\,\text{sgn}(t)$ | $-\frac{1}{\pi}\ln\left\|1-(a/t)^2\right\|$ |
| 6 | Double triangle | $t\prod_{2a}(t)\,\text{sgn}(t)$ | $-\frac{1}{\pi}\ln\left\|1-(a/t)^2\right\|$ |
| 7 | Triangle, tri$(t)$ | $1-\|t/a\|,\ \ \|t\|\le a$ | $\frac{-1}{\pi}\left\{\ln\left\|\frac{t-a}{t+a}\right\| + \frac{t}{a}\ln\left\|\frac{t^2}{t^2-a^2}\right\|\right\}$ |
| | | $0,\ \ \|t\| > a$ | |
| 8 | One-sided triangle | | $\frac{1}{\pi}\left\{(1-t/a)\ln\left\|\frac{t}{t-a}\right\| + 1\right\}$ |

**TABLE 15.2** Selected Useful Hilbert Pairs (continued)

| No. | Name | Function | Hilbert Transform |
|---|---|---|---|
| 9 | Trapezoid | | $\frac{-1}{\pi}\left\{\frac{b}{b-a}\ln\left\|\frac{(a+t)(b-t)}{(a-t)(b+t)}\right\|+\frac{t}{b-a}\ln\left\|\frac{a^2-t^2}{b^2-t^2}\right\|+\ln\left\|\frac{(a-t)}{(a+t)}\right\|\right\}$ |
| 10 | Cauchy pulse | $\frac{a}{a^2+t^2}$ | $\frac{t}{a^2+t^2}$ |
| 11 | Gaussian pulse | $e^{-\pi t^2}$ | $2\int_0^\infty e^{-\pi f^2}\sin(\omega t)\,df;\ \omega=2\pi f$ |
| 12 | Parabolic pulse | $1-(t/a)^2,\ \|t\|\le a$ | $\frac{-1}{\pi}\left\{\left[1-(t/a)^2\right]\ \ln\left\|\frac{t-a}{t+a}\right\|-\frac{2t}{a}\right\}$ |
| 13 | Symmetric exponential | $e^{-a\|t\|}$ | $2\int_0^\infty \frac{2a}{a^2-\omega^2}\sin(\omega t)\,df$ |
| 14 | Antisymmetric exponential | $\mathrm{sgn}(t)e^{-a\|t\|}$ | $-2\int_0^\infty \frac{2a}{a^2-\omega^2}\cos(\omega t)\,df$ |
| 15 | One-sided exponential | $1(t)e^{-a\|t\|}$ | $2\int_0^\infty \frac{a\sin(\omega t)-\omega\cos(\omega t)}{a^2-\omega^2}\,df$ |
| 16 | Sinc pulse | $\frac{\sin(at)}{at}$ | $\frac{\sin^2(at/2)}{(at/2)}=\frac{1-\cos(at)}{at}$ |
| 17 | Video test pulse | $\begin{cases}\cos^2(\pi t/2a); & \|t\|\le a\\ 0, & \|t\|>a\end{cases}$ | $2\int_0^\infty \frac{2a^2}{4a^2-\omega^2}\frac{\sin[\pi\omega/(2a)]}{\omega}\sin(\omega t)\,df$ |
| 18 | $\left\{\begin{array}{l}\text{Spectra of } a(t) \text{ and } \cos(\omega_0 t)\\ \text{overlapping}\end{array}\right.$ | $a(t)\cos(\omega_0 t)$ | $\left[a(t)*\frac{\sin(\omega_0 t)}{\pi t}\right]\sin(\omega_0 t)+\left[a(t)*\frac{\cos(\omega_0 t)}{\pi t}\right]\cos(\omega_0 t)$ |
| 19 | Bedrosian's theorem | $a(t)\cos(\omega_0 t)$ | $a(t)\sin(\omega_0 t)$ |
| 20 | A constant | $a$ | zero |

Hyperbolic Functions: Approximation by Summation of Cauchy Functions (see Hilbert Pairs No. 10 and 45)

| No. | Name | Function | Hilbert Transform |
|---|---|---|---|
| 21 | Tangent hyp. | $\tanh(t)=2\sum_{\eta=0}^{\infty}\frac{t}{(\eta+0.5)^2\pi^2+t^2}$ | $-2\pi\sum_{\eta=0}^{\infty}\frac{(\eta+0.5)}{(\eta+0.5)^2\pi^2+t^2}$ |
| 22 | Part of finite energy of tanh | $\mathrm{sgn}(t)-\tanh(t)$ | $\pi\delta(t)+2\pi\sum_{\eta=0}^{\infty}\frac{(\eta+0.5)}{(\eta+0.5)^2\pi^2+t^2}$ |
| 23 | Cotangent hyp. | $\coth(t)=\frac{1}{t}+2\sum_{\eta=1}^{\infty}\frac{t}{(\eta\pi)^2+t^2}$ | $-\pi\delta(t)+2\pi\sum_{\eta=1}^{\infty}\frac{\eta}{(\eta\pi)^2+t^2}$ |
| 24 | Secans hyp. | $\mathrm{sech}(t)=-2\pi\sum_{\eta=0}^{\infty}(-1)^{(\eta-1)}\frac{(\eta+0.5)}{(\eta+0.5)^2\pi^2+t^2}$ | $-2\sum_{\eta=0}^{\infty}(-1)^{(\eta-1)}\frac{t}{(\eta+0.5)^2\pi^2+t^2}$ |
| 25 | Cosecans hyp. | $\mathrm{cosech}(t)=\frac{1}{t}-2\sum_{\eta=1}^{\infty}(-1)^{(\eta-1)}\frac{t}{(\eta\pi)^2+t^2}$ | $-\pi\delta(t)+2\pi\sum_{\eta=1}^{\infty}(-1)^{(\eta-1)}\frac{\eta}{(\eta\pi)^2+t^2}$ |

**TABLE 15.2** Selected Useful Hilbert Pairs (continued)

| No. | Name | Function | Hilbert Transform |
|---|---|---|---|
| | | Hyperbolic Functions by Inverse Fourier Transformation; $\omega = 2\pi f$ | |
| 26 | | $\mathrm{sgn}(t) - \tanh(at/2)$ <br> $\mathrm{Re}\, a > 0$ | $2\int_0^\infty \left[\frac{2\pi}{a\sinh(\pi\omega/a)} - \frac{2}{\omega}\right]\cos(\omega t)\,df$ |
| 27 | | $\coth(t) - \mathrm{sgn}(t)$ | $2\int_0^\infty \left[\frac{2\pi}{a}\coth(\pi\omega/a) - \frac{2}{\omega}\right]\cos(\omega t)\,df$ |
| 28 | | $\mathrm{sec\,h}(at/2)$ | $2\int_0^\infty \frac{2\pi}{a\cosh(\pi\omega/(2a))}\sin(\omega t)\,df$ |
| 29 | | $\mathrm{cos\,ech}(at/2)$ | $-2\int_0^\infty \frac{2\pi}{a}\tanh(\pi\omega/(2a))\cos(\omega t)\,df$ |
| 30 | | $\mathrm{sec\,h}^2(at/2)$ | $2\int \frac{2\pi\omega}{a\sinh(\pi\omega/(2a))}\sin(\omega t)\,df$ |

Delta Distribution, $1/(\pi t)$ Distribution and its Derivatives: Derivation Using Successive Iteration and

| No. | Operation | Differentiation <br> If $x(t) \overset{H}{\Longleftrightarrow} v(t)$ then $\dot{x}(t) \overset{H}{\Longleftrightarrow} \dot{v}(t)$ <br> $x(t)$ | Iteration <br> $\mathsf{H}[v(t)] = \mathsf{HH}[u(t)] = -x(t)$ <br> $v(t)$ |
|---|---|---|---|
| 31 | | $\delta(t)$ | $1/(\pi t)$ |
| 32 | Iteration | $1/(\pi t)$ | $-\delta(t)$ |
| 33 | Differentiation | $\dot{\delta}(t)$ | $-1/(\pi t^2)$ |
| 34 | Iteration | $1/(\pi t^2)$ | $\dot{\delta}(t)$ |
| 35 | Differentiation | $\ddot{\delta}(t)$ | $2/(\pi t^3)$ |
| 36 | Iteration | $1/(\pi t^3)$ | $-0.5\ddot{\delta}(t)$ |
| 37 | Differentiation | $\dddot{\delta}(t)$ | $-6/(\pi t^4)$ |
| 38 | Iteration | $1/(\pi t^4)$ | $(1/6)\dddot{\delta}(t)$ |
| 39 | | $x(t)\delta(t)$ | $x(0)/(\pi t)$ |
| | | The procedure could be continued. | |
| | | Equality of Convolution | |
| 40 | | $\delta(t) * \delta(t) * \delta(t)$ | $\frac{1}{\pi t} * \frac{1}{\pi t} = -\delta(t)$ |
| 41 | | $\dot{\delta}(t) * \delta(t) = \dot{\delta}(t)$ | $\frac{1}{\pi t^2} * \frac{1}{\pi t} = \dot{\delta}(t)$ |
| 42 | | $\dot{\delta}(t) * \dot{\delta}(t) = \ddot{\delta}(t)$ | $\frac{1}{\pi t^2} * \frac{1}{\pi t^2} = -\ddot{\delta}(t)$ |
| 43 | | $\dddot{\delta}(t) * \delta(t) = \dddot{\delta}(t) = \ddot{\delta}(t) * \dot{\delta}(t)$ | $\frac{6}{\pi t^4} * \frac{1}{\pi t} = \dddot{\delta}(t) = \frac{2}{\pi t^3} * \frac{1}{\pi t^2}$ |
| | | Approximating Functions of Distributions (see No. 31 to 37 of this table) | |
| | | $x(t)$ | $v(t)$ |
| 44 | | $\int \delta(a,t)\,dt = \frac{1}{\pi}\tan^{-1}(t/a)$ | $\int \theta(a,t)\,dt = \frac{\ln(a^2 + t^2)}{2\pi}$ |

**TABLE 15.2** Selected Useful Hilbert Pairs (continued)

| No. | Name | Function | Hilbert Transform |
|---|---|---|---|
| 45 | | $\delta(a,t)=\frac{1}{\pi}\frac{a}{a^2+t^2}$ | $\theta(a,t)=\frac{1}{\pi}\frac{t}{a^2+t^2}$ |
| 46 | | $\dot{\delta}(a,t)=\frac{1}{\pi}\frac{-2at}{(a^2+t^2)^2}$ | $\dot{\theta}(a,t)=\frac{1}{\pi}\frac{a^2-t^2}{(a^2+t^2)^2}$ |
| 47 | | $\ddot{\delta}(a,t)=\frac{1}{\pi}\frac{6at^2-2a^2}{(a^2+t^2)^3}$ | $\ddot{\theta}(a,t)=\frac{1}{\pi}\frac{2t^2-6at^2}{(a^2+t^2)^3}$ |
| 48 | | $\dddot{\delta}(a,t)=\frac{1}{\pi}\frac{24a^3t-24at^2}{(a^2+t^2)^4}$ | $\dddot{\theta}(a,t)=\frac{1}{\pi}\frac{-6t^2+36a^2t^2-6a^4}{(a^2+t^2)^4}$ |

Derivation Using Successive Iteration and Differentiation (see the information above No. 31)

Trigonometric Expressions

| | Operation | $x(t)$ | $v(t)$ |
|---|---|---|---|
| 49 | | $\frac{\sin(at)}{t}$ | $\frac{1-\cos(at)}{t}=\frac{2\sin^2(at/2)}{t}$ |
| 50 | Iteration | $\frac{\cos(at)}{t}$ | $-\pi\delta(t)+\frac{\sin(at)}{t}$ |
| 51 | Differentiation | $\frac{\sin(at)}{t^2}$ | $-a\pi\delta(t)+\frac{1-\cos(at)}{t^2}$ |
| 52 | Iteration | $\frac{\cos(at)}{t^2}$ | $\pi\dot{\delta}(t)-\frac{a}{t}+\frac{\sin(at)}{t^2}$ |
| 53 | Differentiation | $\frac{\sin(at)}{t^3}$ | $\pi a\dot{\delta}(t)-\frac{a^2}{2t}+\frac{1-\cos(at)}{t^3}$ |
| 54 | Iteration | $\frac{\cos(at)}{t^3}$ | $-\frac{\pi}{2}\ddot{\delta}(t)+\frac{\pi a^2}{2}\delta(t)-\frac{a}{t^2}+\frac{\sin(at)}{t^3}$ |

Selected Useful Hilbert Pairs of Periodic Signals

| | Name | $x_p(t)$ | $\nu_p(t)$ |
|---|---|---|---|
| 55 | Sampling sequence | $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ | $\frac{1}{T}\sum_{n=-\infty}^{\infty}\cos[(\pi/T)(t-nT)]$ |
| 56 | Even square wave | $\operatorname{sgn}[\cos(\omega t)],\ \omega=2\pi/T$ | $(2/\pi)\ln\lvert\tan(\omega t/2+\pi/4)\rvert$ |
| 57 | Odd square wave | $\operatorname{sgn}[\sin(\omega t)],\ \omega=2\pi/T$ | $(2/\pi)\ln\lvert\tan(\omega t/2)\rvert$ |
| 58 | Squared cosine | $\cos^2(\omega t)$ | $0.5\sin(2\omega t)$ |
| 59 | Squared sine | $\sin^2(\omega t)$ | $-0.5\sin(2\omega t)$ |
| 60 | Cube cosine | $\cos^3(\omega t)$ | $\frac{3}{4}\sin(\omega t)+\frac{1}{4}\sin(3\omega t)$ |
| 61 | Cube sine | $\sin^3(\omega t)$ | $\frac{-3}{4}\cos(\omega t)+\frac{1}{4}\cos(3\omega t)$ |
| 62 | | $\cos^4(\omega t)$ | $\frac{1}{2}\sin(2\omega t)+\frac{1}{8}\sin(4\omega t)$ |
| 63 | | $\sin^4(\omega t)$ | $-\frac{1}{2}\sin(2\omega t)+\frac{1}{8}\sin(4\omega t)$ |
| 64 | | $\cos^5(\omega t)$ | $\frac{5}{8}\sin(2\omega t)+\frac{5}{16}\sin(3\omega t)+\frac{1}{16}\sin(5\omega t)$ |
| 65 | | $\cos^6(\omega t)$ | $\frac{15}{32}\sin(2\omega t)+\frac{6}{32}\sin(4\omega t)+\frac{1}{32}\sin(6\omega t)$ |
| 66 | | $\cos(at+\varphi)\cos(bt+\Psi)$<br>$0<a<b$<br>$\varphi,\Psi=$ constants | $\cos(at+\varphi)\sin(bt+\Psi)$ |

**TABLE 15.2** Selected Useful Hilbert Pairs (continued)

| No. | Name | Function | Hilbert Transform |
|---|---|---|---|
| 67 | Fourier Series | $X_o + \sum_{n=1}^{\infty} X_n \cos(n\omega t + \varphi_n)$ | $\sum_{n=1}^{\infty} X_n \sin(n\omega t + \varphi_n)$ |
| 68 | Any periodic function | $x_T$ = generating function | $x_T(t) \cdot \frac{1}{T} \sum_{k=-\infty}^{\infty} \cot[(\pi / T)(t - kT)]$ |
| | | $x_T(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT)$ | |

# 15.6 Differentiation of Hilbert Pairs

## 15.6.1 Differentiation Pairs

$$\mathsf{H}\{\dot{x}(t)\} = \dot{\mathrm{v}}(t)$$

$$\mathsf{H}\left\{\frac{d^n x(t)}{dt^n}\right\} = \frac{d^n \mathrm{v}(t)}{dt^n}$$

**Example**

$$\mathsf{H}\{\delta(t)\} = \frac{1}{\pi t};\ \mathsf{H}\{\dot{\delta}(t)\} = \frac{d}{dt}\left(\frac{1}{\pi t}\right) = -\frac{1}{\pi t^2}$$

## 15.6.2 Derivative of Convolution

$$\mathsf{H}\{x(t)\} = \mathsf{H}\left\{\frac{-1}{\pi t} * \mathrm{v}(t)\right\} \Rightarrow \mathrm{v}(t) = \frac{1}{\pi t} * x(t)$$

$$\mathsf{H}\{\dot{x}(t)\} = \mathsf{H}\left\{-\frac{d}{dt}\left(\frac{1}{\pi t}\right) * \mathrm{v}(t)\right\} \Rightarrow \dot{\mathrm{v}}(t) = \frac{d}{dt}\left(\frac{1}{\pi t}\right) * x(t) \quad \text{(see 15.6.1 and 15.5.5)}$$

$$= \left(-\frac{-1}{\pi t^2}\right) * x(t) = \frac{1}{\pi t^2} * \mathrm{v}(t)$$

$$\mathsf{H}\{\dot{x}(t)\} = \mathsf{H}\left\{-\frac{1}{\pi t} * \dot{\mathrm{v}}(t)\right\} \Rightarrow \dot{\mathrm{v}}(t) = \frac{1}{\pi t} * \dot{x}(t)$$

## 15.6.3 Fourier Transform of Hilbert Transform

$$\mathrm{v}(t) = \frac{1}{\pi t} * x(t), \qquad \mathsf{F}\{\mathrm{v}(t)\} = -j\,\mathrm{sgn}(\omega)\,X(\omega)$$

$$\mathsf{F}\{\dot{\mathrm{v}}(t)\} = j\omega[-j\,\mathrm{sgn}(\omega)\,X(\omega)] = \omega\,\mathrm{sgn}(\omega)\,X(\omega)$$

## 15.7 Hilbert Transform of Hermite Polynomials

### 15.7.1 Hermite Polynomials and their Hilbert Transform

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \qquad n = 0,1,2,\cdots, \ -\infty < t < \infty$$

$$H_n(t) = 2t\, H_{n-1}(t) - 2(n-1) H_{n-2}(t) \qquad n = 1,2,\cdots$$

$$\mathsf{F}\{e^{-t^2}\} = \sqrt{\pi}\, e^{-\pi^2 f^2} = \sqrt{\pi}\, e^{-\omega^2/4}$$

$$v(t) = \mathsf{H}\{x(t)\} \doteq \mathsf{H}\{e^{-t^2}\} = \mathsf{F}^{-1}\{V(\omega)\} = \int_{-\infty}^{\infty} -j\,\mathrm{sgn}(\omega)\sqrt{\pi}\, e^{-\pi^2 f^2} e^{j\omega t}\, df$$

$$= 2\sqrt{\pi}\int_{-\infty}^{\infty} e^{-\pi^2 f^2} \sin\omega t\, df$$

$$\mathsf{H}\{2te^{-t^2}\} = -2\sqrt{\pi}\int_{-\infty}^{\infty} \omega e^{-\pi^2 f^2} \cos\omega t\, df$$

### 15.7.2 Table of Hilbert Transform of Hermite Polynomials

**TABLE 15.3** Hilbert Transform of Weighted Hermite Polynomials [Notation: $x = \exp(-t^2)$]

| | Hermite Polynomial | Hilbert Transform | Energy |
|---|---|---|---|
| n | $H_n x$ | $\mathsf{H}(H_n x)$ | $E$ |
| 0 | $(1)x$ | $2\sqrt{\pi}\int_0^{\infty} \exp(-\pi^2 f^2)\sin(\omega t)\, df$ | $\sqrt{\pi/2}$ |
| 1 | $(2t)x$ | $-2\sqrt{\pi}\int_0^{\infty} \omega \exp(-\pi^2 f^2)\cos(\omega t)\, df$ | $\sqrt{\pi/2}$ |
| 2 | $(4t^2 - 2)x$ | $-2\sqrt{\pi}\int_0^{\infty} \omega^2 \exp(-\pi^2 f^2)\sin(\omega t)\, df$ | $3\sqrt{\pi/2}$ |
| 3 | $(8t^3 - 12t)x$ | $2\sqrt{\pi}\int_0^{\infty} \omega^3 \exp(-\pi^2 f^2)\cos(\omega t)\, df$ | $15\sqrt{\pi/2}$ |
| 4 | $(16t^4 - 48t^2 + 12)x$ | $2\sqrt{\pi}\int_0^{\infty} \omega^4 \exp(-\pi^2 f^2)\sin(\omega t)\, df$ | $105\sqrt{\pi/2}$ |
| 5 | $(32t^5 - 160t^3 + 120t)x$ | $-2\sqrt{\pi}\int_0^{\infty} \omega^5 \exp(-\pi^2 f^2)\cos(\omega t)\, df$ | $945\sqrt{\pi/2}$ |
| n | $H_n x = (-1)^n[2tH_{n-1}(t)$ $-2(n-1)H_{n-2}(t)]$ | $(-1)^n 2\sqrt{\pi}\int_0^{\infty} \omega^n \exp(-\pi^2 f^2)\sin\left(\omega t + \frac{n\pi}{2}\right) df$ | |

Energy = $\int_{-\infty}^{\infty} x^2 H_n^2\, dt = \int_{-\infty}^{\infty} [\mathsf{H}(xH_n)]^2\, dt = 1\times 3\times 5\times\cdots\times(2n-1)\times\sqrt{\pi/2}, \quad n \ge 1$

### 15.7.3 Hilbert Transform of Orthonormal Hermite Functions (see Chapter 22)

$$h_n(t) = (2^n n!)^{-1/2}\, \pi^{-1/2}\, e^{-t^2}\, H_n(t) \qquad n = 0,1,2,\cdots$$

$$\mathrm{H}\{h_n(t)\} = \nu_n(t)$$

$$= \left[\frac{2(n-1)!}{n!}\right]^{1/2} \left[t\,\nu_{n-1}(t) - \frac{1}{\pi}\int_{-\infty}^{\infty} h_{n-1}(\tau)\,d\tau\right] - (n-1)\left[\frac{(n-2)!}{n!}\right]^{1/2} \nu_{n-2}(t)$$

### 15.7.4 Hilbert Transform of Orthonormal Hermite Functions

**TABLE 15.4** Hilbert Transforms of Orthonormal Hermite Functions (Energy = 1).
Notations: $h_o(t), h_1(t), \cdots \Rightarrow h_o, h_1, \cdots;\qquad \upsilon_o(t), \cdots \Rightarrow \upsilon_o, \upsilon_1, \cdots$

$$g(t) = \int_0^{\infty} e^{-2\pi^2 f^2} \sin(2\pi f t)\, df\,; \qquad a = \pi^{-0.25} e^{-t^2/2}\,; \qquad b = \pi^{0.25}$$

| Hermite Functions $h_n(t)$ | Hilbert Transforms $\upsilon_n(t)$ |
|---|---|
| Recurrent Notation | |
| $h_0 = a$ | $\upsilon_0 = 2\sqrt{2}\; bg(t)$ |
| $h_1 = \sqrt{2}\; th_0$ | $\upsilon_1 = \sqrt{2}\left[t\upsilon_0 \dfrac{\sqrt{2}\,b}{\pi}\right]$ |
| $h_2 = th_1 - \sqrt{1/2}\, h_0$ | $\upsilon_2 = t\upsilon_1 - \sqrt{1/2}\;\upsilon_0$ |
| $h_3 = \sqrt{2/3}\;[th_2 - h_1]$ | $\upsilon_3 = \sqrt{2/3}\left[t\upsilon_2 - \dfrac{b}{\pi} - \upsilon_1\right]$ |
| $h_4 = \sqrt{1/2}\; th_3 - \sqrt{3/4}\; h_2$ | $\upsilon_4 = \sqrt{1/2}\; t\upsilon_3 - \sqrt{3/4}\;\upsilon_2$ |
| $h_5 = \sqrt{2/5}\; th_4 - \sqrt{4/5}\; h_3$ | $\upsilon_5 = \sqrt{2/5}\left[t\upsilon_4 - \dfrac{\sqrt{3}\,b}{2\pi}\right] - \sqrt{4/5}\;\upsilon_3$ |
| $h_n = \sqrt{\dfrac{2(n-1)!}{n!}}\; th_{n-1} +$ $(n-1)\sqrt{\dfrac{(n-2)!}{n!}}\; h_{n-2}$ | $\upsilon_n = \sqrt{\dfrac{2(n-1)!}{n!}}\;\; [t\upsilon_{n-1}$ $-\dfrac{1}{\pi}\int h_{n-1}(\tau)\,d\tau] - (n-1)\sqrt{\dfrac{(n-2)!}{n!}}\;\upsilon_{n-2}$ |
| Nonrecurrent Notation | |
| $h_0 = a1$ | $2\sqrt{2}\; bg(t)$ |
| $h_1 = \sqrt{2}\; at$ | $2b[2tg(t) - \pi^{-1}]$ |
| $h_2 = \dfrac{a}{\sqrt{8}}(4t^2 - 2)$ | $2b[(2t^2 - 1)\,g(t) - t\pi^{-1}]$ |
| $h_3 = \dfrac{a}{\sqrt{48}}(8t^3 - 12t)$ | $\sqrt{8/3b}\left[(2t^3 - 3t)\,g(t) - \dfrac{t^2}{\pi} + \dfrac{1}{2\pi}\right]$ |
| $h_4 = \dfrac{a}{\sqrt{384}}(16t^4 - 48t^2 + 12)$ | $\sqrt{4/3b}\left[(2t^4 - 6t^2 + 1.5)\,g(t) - \dfrac{t^3}{\pi} + \dfrac{2t}{2\pi}\right]$ |

**TABLE 15.4** Hilbert Transforms of Orthonormal Hermite Functions (Energy = 1). (continued)

Notations: $h_o(t), h_1(t), \cdots \Rightarrow h_o, h_1, \cdots;$ $\upsilon_o(t), \cdots \Rightarrow \upsilon_o, \upsilon_1, \cdots$

$$g(t) = \int_0^\infty e^{-2\pi^2 f^2} \sin(2\pi f t)\, df; \qquad a = \pi^{-0.25} e^{-t^2/2}; \qquad b = \pi^{0.25}$$

| Hermite Functions $h_n(t)$ | Hilbert Transforms $\upsilon_n(t)$ |
|---|---|
| $h_5 = \frac{a}{\sqrt{3840}}(32t^5 - 160t^3 + 120t)$ | $\sqrt{8/15}\, b \left[ (2t^5 - 10t^3 + 7.5) g(t) - \frac{(t^4 - 4t^2) + 1.75}{\pi} \right]$ |
| ........ | ........ |
| $h_n(t) = \frac{a}{\sqrt{2^n n!}} H_n(t),$ | $H_n(t) = 2t H_{n-1}(t) - 2(n-1) H_{n-2}(t)$ |

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | … |
|---|---|---|---|---|---|---|---|
| $\int_{-\infty}^{\infty} h_n(\tau)\, d\tau$ | $\sqrt{2}\, b$ | 0 | b | 0 | $\sqrt{3/4}\, b$ | 0 | … |

# 15.8 Hilbert Transform of Product of Analytic Signals

## 15.8.1 Hilbert Transform of Product of Analytic Signals:

From

$$\mathrm{H}\{\psi(t)\} = \mathrm{H}\{x(t) + j\upsilon(t)\} = \mathrm{H}\{x(t) + j\mathrm{H}\{x(t)\}\} = \mathrm{H}\{x(t)\} - jx(t)$$
$$= \upsilon(t) - jx(t) = -j(x(t) + j\upsilon(t)) = -j\psi(t)$$

we obtain $\mathrm{H}\{\psi_1(t)\psi_2(t)\} = -j\psi_1(t)\psi_2(t) = \psi_1(t)\mathrm{H}\{\psi_2(t)\} = \psi_2(t)\mathrm{H}\{\psi_1(t)\}$
since the product can be considered as an analytic function $\psi(t)$.

## 15.8.2 The n[th] Product of an Analytic Signal

$$\mathrm{H}\{\psi^2(t)\} = \psi(t)\mathrm{H}\{\psi(t)\} = -j\psi^2(t)$$

$$\mathrm{H}\{\psi^n(t)\} = \psi^{n-1}(t)\mathrm{H}\{\psi(t)\} = -j\psi^n(t)$$

**Example**

Because $\mathrm{H}\{(1-jt)^{-1}\} = -j(1-jt)^{-2}$, we obtain

$$\mathrm{H}\{(1-jt)^{-2}\} = (1-jt)^{-1}(-j(1-jt)^{-1}) = -j(1-jt)^{-2}$$

# 15.9 Hilbert Transform of Bessel Functions

## 15.9.1 Hilbert Transform of Bessel Function:

$$\mathrm{H}\{J_n(t)\} = \hat{J}_n(t) = \frac{1}{\pi}\int_0^\pi \sin(t\sin\varphi - n\varphi)\, d\varphi = \sum_{n=0}^{\infty} \frac{\hat{J}_n^{(n)}(t=0)}{n!} t^n$$

$$\hat{J}_0(t) = \frac{1}{\pi}\int_0^1 \frac{2}{(1-\omega^2)^{1/2}} \sin\omega t\, d\omega$$

$$\psi_0(t) = J_0(t) + j\hat{J}_0(t)$$

$$\hat{J}_0(0) = \frac{1}{\pi}\int_0^1 \frac{2\,d\omega}{(1-\omega^2)^{1/2}} \sin(0) = 0, \qquad \hat{J}_0^{(1)}(t) = \frac{1}{\pi}\int_0^1 \frac{2\,\omega\, d\omega}{(1-\omega^2)^{1/2}} \cos(0) = \frac{2}{\pi}$$

The parenthesis in the exponent indicates number of differentiations with respect to time.

### 15.9.2 Hilbert Transform Pairs of Bessel Functions:

**TABLE 15.5** Hilbert Transform of Bessel Functions of the First Kind

| Bessel Function | Fourier Transform | Hilbert Transform |
|---|---|---|
| $J_n(t)$ | $C_n(f)$ | $\hat{J}_n(t) = \mathrm{H}[J_n(t)]$ |
| $J_0(t)$ | $C_0 = \frac{2}{(1-\omega^2)^{0.5}}; \quad \lvert\omega\rvert < 1$ <br> $= 0; \quad \lvert\omega\rvert > 0$ | $\frac{1}{\pi}\int_0^1 C_0(f)\sin(\omega t)\,d\omega$ |
| $J_1(t)$ | $C_1 = -j\omega C_0$ | $-\frac{1}{\pi}\int_0^1 \lvert C_1(f)\rvert\cos(\omega t)\,d\omega$ |
| $J_2(t)$ | $C_2 = -(2\omega^2 - 1)C_0$ | $-\frac{1}{\pi}\int_0^1 \lvert C_2(f)\rvert\sin(\omega t)\,d\omega$ |
| $J_3(t)$ | $C_3 = j(4\omega^3 - 3\omega)C_0$ | $\frac{1}{\pi}\int_0^1 \lvert C_3(f)\rvert\cos(\omega t)\,d\omega$ |
| $J_4(t)$ | $C_4 = (8\omega^4 - 8\omega^2 + 1)C_0$ | $\frac{1}{\pi}\int_0^1 \lvert C_4(f)\rvert\sin(\omega t)\,d\omega$ |
| $J_5(t)$ | $C_5 = -j(16\omega^5 - 20\omega^3 + 5\omega)C_0$ | $-\frac{1}{\pi}\int_0^1 \lvert C_5(f)\rvert\cos(\omega t)\,d\omega$ |
| $J_6(t)$ | $C_6 = -(32\omega^6 - 48\omega^4 + 18\omega^2 - 1)C_0$ | $-\frac{1}{\pi}\int_0^1 \lvert C_6(f)\rvert\sin(\omega t)\,d\omega$ |
| ........ | ........ | ........ |
| $J_n(t)$ | $C_n = (-j)^n 2^{n-1} T_n(\omega) C_0$ | $\frac{(-1)^{n/2}}{\pi}\int_0^1 \lvert C_n(f)\rvert\sin(\omega t)\,d\omega$ <br> for $n = 0,2,4,\ldots$ <br> $\frac{(-1)^{(n+1)/2}}{\pi}\int_0^1 \lvert C_n(f)\rvert\cos(\omega t)\,d\omega$ <br> for $n = 1,3,5,\ldots$ |

$T_n(\omega) = \cos[n\cos^{-1}(\omega)]$ is the Chebyshev polynomial

## 15.10 Instantaneous Amplitude, Phase, and Frequency

### 15.10.1 Instantaneous Angular Frequency

$$\psi(t) = x(t) + j\upsilon(t) = A(t)e^{j\varphi(t)} = A(t)\cos\varphi(t) + A(t)\sin\varphi(t)$$

$$A(t) = \sqrt{x^2(t) + \upsilon^2(t)}, \quad \varphi(t) = \tan^{-1}\frac{\upsilon(t)}{x(t)}$$

$$\dot{\varphi}(t) = \Omega(t) = 2\pi F(t) \equiv \text{instantaneous angular frequency}$$

$$F(t) = \text{instantaneous frequency} = \frac{\Omega(t)}{2\pi} = \frac{\dot{\varphi}(t)}{2\pi}$$

$$\Omega(t) = \frac{d}{dt}\tan^{-1}\frac{\upsilon(t)}{x(t)} = \frac{x(t)\dot{\upsilon}(t) - \upsilon(t)\dot{x}(t)}{x^2(t) + \upsilon^2(t)}$$

## 15.11 Hilbert Transform and Modulation

### 15.11.1 Modulated Signal (see 15.10.1)

$$\psi(t) = A_o\,\gamma(t)e^{j\Phi_0}\,e^{j\Omega_0 t}$$

$$\psi_x(t) = x(t) + j\hat{x}(t)$$

$$x(t) = \frac{\psi_x(t) + \psi_x^*(t)}{2}$$

### 15.11.2 Instantaneous Amplitude and Angular Frequency (see 15.10.1)

$$A(t) = \frac{m}{2}\left|\psi_x(t)\right| = \frac{m}{2}[x^2(t) + \hat{x}^2(t)]^{1/2}$$

$$\omega_x(t) = \pm\frac{d}{dt}\tan^{-1}\left[\frac{\hat{x}(t)}{x(t)}\right]$$

### 15.11.3 High-Frequency Analytic Signals ($\Phi_0 = 0$)

$$\psi_{upper}(t) = \text{upper sideband} = \psi_x(t)e^{j\Omega_0 t}$$

$$\psi_{lower}(t) = \text{lower sideband} = \psi_x^*(t)e^{j\Omega_0 t}$$

$$x_{SSB}(t) = x(t)\cos\Omega_0 t \mp \hat{x}(t)\sin\Omega_0 t$$

where $x(t)\cos(\Omega_0 t)$ and $\hat{x}(t)\sin\Omega_0 t$ represent double sideband (DSB) compressed carrier AM signals.

## 15.12 Hilbert Transform and Transfer Functions of Linear Systems

### 15.12.1 Causal Systems

$$H(s) = A(\alpha,\omega) + jB(\alpha,\omega), \qquad \sigma = \alpha + j\omega$$

$$A(\omega) = -\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{B(\lambda)}{\lambda - \omega}d\lambda$$

$$B(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{A(\lambda)}{\lambda - \omega} d\lambda$$

### 15.12.2 Minimum Phase Transfer Function

$$H(j\omega) = H_{\varphi}(j\omega) H_{ap}(j\omega)$$

$$H_{\varphi}(j\omega) = \text{minimum phase transfer function}$$

$$H_{ap}(j\omega) = \text{all-pass transfer function}$$

$$H_{\varphi}(j\omega) = |H(j\omega)| e^{j\varphi(\omega)} = A_{\varphi}(\omega) + jB_{\varphi}(\omega)$$

$H_{\varphi}(j\omega)$ has all the zeros lying in the left half-plane of the s-plane. The minimum phase transfer function is analytic and its real and imaginary parts for a Hilbert pair

$$\mathsf{H}\{A(\omega)\} = -B_{\varphi}(\omega)$$

## 15.13 The Discrete Hilbert Filter

### 15.13.1 Discrete Hilbert Filter

$$H(k) = \begin{cases} -j & k = 1, 2, \cdots, \frac{N}{2} - 1 \\ 0 & k = 0 \text{ and } k = \frac{N}{2} \\ j & k = \frac{N}{2} + 1, \frac{N}{2} + 2, \cdots, N - 1 \end{cases} \qquad (N = \text{even})$$

$$H(k) = -j\,\text{sgn}\left(\frac{N}{2} - k\right)\text{sgn}(k), \qquad k = 0, 1, \cdots, N-1 \quad (N = \text{even})$$

### 15.13.2 Impulse Response of the Hilbert Filter

$$h(i) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) e^{jw} = \frac{1}{N}\sum_{k=0}^{N-1} -j\,\text{sgn}\left(\frac{N}{2} - k\right)\text{sgn}(k) e^{jw}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \sin(w) = \frac{2}{N}\sin^2\left(\frac{\pi i}{2}\right)\cot\left(\frac{\pi i}{N}\right), \qquad i = 0, 1, \cdots, N-1, \ w = \frac{2\pi k i}{N} \quad (N \text{ even})$$

### 15.13.3 DHT of a Sequence $x(i)$ in the Form of Convolution

$$\upsilon(i) = -x(i) \otimes h(i) = -x(i) \otimes \frac{2}{N}\sin^2\left(\frac{\pi i}{2}\right)\cot\left(\frac{\pi i}{N}\right), \qquad i = 0, 1, \cdots, N-1$$

$$\otimes = \text{circular convolution}$$

$$\upsilon(i) = \sum_{r=0}^{N-1} h(i - r) x(r), \qquad i = 0, 1, \cdots, N-1 \quad (N \text{ even})$$

### 15.13.4 DHT of a Sequence $x(i)$ via DFT

$$\mathsf{F}_D\{x(i)\} = X(k)$$

$$V(k) = -j\,\mathrm{sgn}\left(\frac{N}{2} - k\right)\mathrm{sgn}(k)\,X(k)$$

$$\upsilon(i) = \mathsf{F}_D^{-1}\{V(k)\}, \qquad i,k = 0,1,2,\cdots,N-1 \ (N \text{ even})$$

$\mathsf{F}_D \equiv$ discrete Fourier transform, $\mathsf{F}_D^{-1} \equiv$ inverse discrete Fourier transform

### 15.13.5 Discrete Hilbert Filter when N is odd

$$H(k) = \begin{cases} -j & k = 1,2,\cdots,\frac{N-1}{2} \\ 0 & k = 0 \\ j & k = \frac{N}{2}+1, \frac{N}{2}+2,\cdots,N-1 \end{cases}$$

$$h(i) = \frac{2}{N}\sum_{k=1}^{(N-1)/2} \sin(2\pi ki/N), \qquad i = 0,1,\cdots,N-1$$

Also

$$h(i) = \frac{1}{N}\left[1 - \frac{\cos(\pi i)}{\cos(\pi i/N)}\cot\left(\frac{\pi i}{N}\right)\right]$$

## 15.14 Properties of Discrete Hilbert Transform

### 15.14.1 Parseval's Theorem

$$E\{x(i)\} = \sum_{i=0}^{N-1} |x(i)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

$$E\{x(i)\} \neq E\{\upsilon(i)\}$$

The reason is that the DC term (average value of $x(i)$) is eliminated in the DHT.

$$x_{DC} = \frac{1}{N}\sum_{i=0}^{N-1} x(i) = X(0)$$

### 15.14.2 Discrete Hilbert Transform

$$\mathsf{H}_D\{x_{AC}(i)\} = \upsilon(i)$$

$$x_{AC}(i) = x(i) - x_{DC}$$

where $x_{AC}(i)$ is the alternating part of $x(i)$.

## 15.14.3 Energies (powers) of $x_{AC}$ and $\upsilon(i)$

$$\sum_{i=0}^{N-1} |x_{AC}(i)|^2 = \sum_{i=1}^{N-1} |\upsilon(i)|^2 + \frac{|X(N/2)|^2}{N} \qquad (N \text{ even})$$

where the special term $X\left(\frac{N}{2}\right)$ is zero, the two energies are equal.

**Example**

If $x(i) = \delta(i)$ and $N = 8$ we obtain (see 15.13.3)

$$\upsilon(i) = -\delta(i) \otimes \tfrac{1}{4}\sin^2(\pi i/2)\cot(\pi i/N)$$

Figure 15.1 shows the desired components and transforms. The $x_{DC} = 1/8 = 0.125$ and the energies are: $E\{x(i)\} = 1,\ E\{x_{AC}(i)\} = 1 - \frac{1^2}{N} = 0.875,$ and $E\{\upsilon(i)\} = 1 - \frac{1^2}{N} - \frac{1^2}{N} = 1 - \frac{2}{8} = 0.75.$

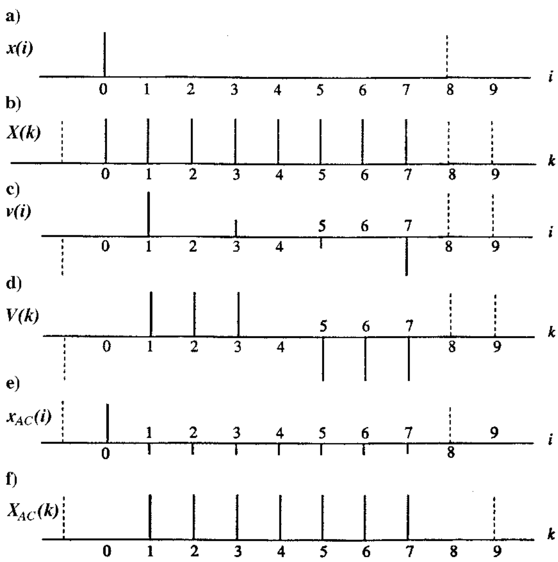


**FIGURE 15.1** (a) The sequence $x(i)$ consisting of a single sample $\delta(i)$, (b) its spectrum $X(k)$ given by the DFT, (c) the samples of the discrete Hilbert transform, (d) the corresponding spectrum $V(k)$, (e) the samples of the AC component of $x(i)$, and (f) the corresponding spectrum $X_{AC}(k)$.

## 15.14.4 Shifting Property:

$$\mathsf{F}_D\{x(i \pm m)\} = e^{\pm j2\pi mk/N} X(k)$$

See 15.13.4

$$\upsilon(i) = \mathsf{F}_D^{-1}\{-j\,\mathrm{sgn}\left(\frac{N}{2} - k\right)\mathrm{sgn}(k) e^{\pm j2\pi mk/N} X(k)\}$$

### 15.14.5 Linearity:

$$\mathsf{H}_D\{ax_1(i) + bx_2(i)\} = a\upsilon_1(i) + b\upsilon_2(i)$$

### 15.14.6 Complex Analytic Discrete Sequence:

$$\psi(i) = x(i) + j\upsilon(i), \qquad \upsilon(i) = \mathsf{H}_D\{x(i)\}$$

$$\mathsf{H}_D\{\psi(i)\} = X(k) + j[-j\,\mathrm{sgn}\left(\frac{N}{2} - k\right)\mathrm{sgn}(k)]X(k), \qquad k = 0,1,\cdots,N-1 \quad (N \text{ even})$$

## 15.15 Hilbert Transformers (continuous)

### 15.15.1 Hilbert Transformer (quadratic filter)

$$H(jf) = \mathsf{F}\left\{\frac{1}{\pi t}\right\} = |H(f)|e^{j\varphi(f)} = -j\,\mathrm{sgn}\,f$$

$$H(jf) = \begin{cases} -j & f > 0 \\ 0 & f = 0 \\ j & f < 0 \end{cases}$$

$$\varphi(f) = \arg H(jf) = -\frac{\pi}{2}\mathrm{sgn}\,f$$

### 15.15.2 Phase-Splitter Hilbert Transformers

Analog Hilbert transformers are mostly implemented in the form of a phase splitter consisting of two parallel all-pass filters with a common input pot and separated output ports, each having the following transfer function respectively.

$$Y_1(jf) = e^{j\varphi_1(f)}, \qquad Y_2(jf) = e^{j\varphi_2(f)}$$

with

$$\delta(f) = \varphi_1(f) - \varphi_2(f) = -\pi/2 \quad \text{for all} \quad f > 0$$

### 15.15.3 All-Pass Filters

$$H(j\omega) = \frac{R - jX(\omega)}{R + jX(\omega)} \qquad \omega = 2\pi f$$

$$\varphi(\omega) = \arg\{(R - jX(\omega))^2\} = \tan^{-1}\left[\frac{-2RX(\omega)}{R^2 - X^2(\omega)}\right]$$

See Figure 15.2a.

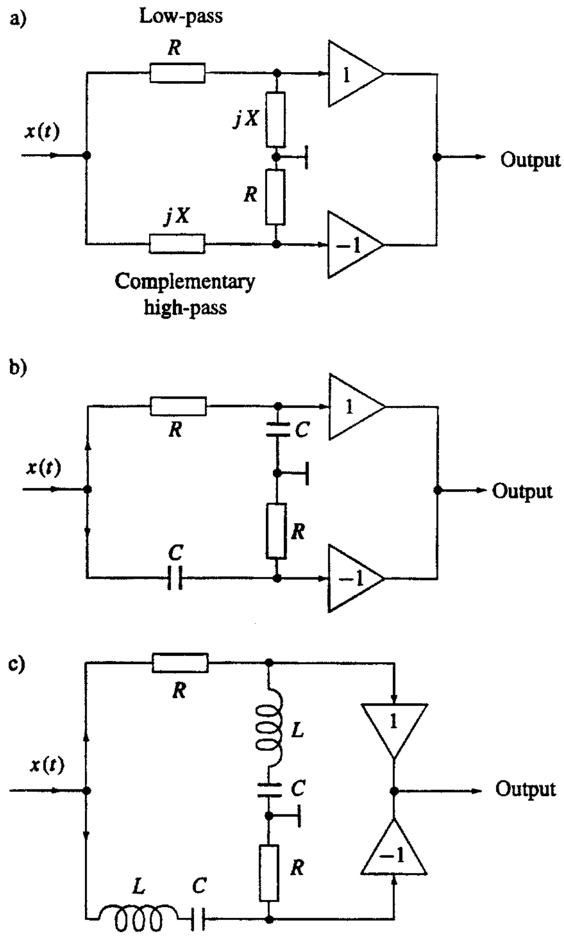


**FIGURE 15.2** An all-pass consisting of (a) a low-pass and a complementary high-pass, (b) a first-order RC low-pass and complementary CR high-pass, and (c) a second-order RLC low-pass and complementary RLC high-pass.

If $X(\omega) = \dfrac{1}{\omega C}$, then (see Figure 15.2b)

$$\varphi(y) = \tan^{-1}\left[\frac{-2y}{1-y^2}\right], \qquad y = \omega RC = \omega\tau$$

If $X(\omega) = \omega L - 1/\omega C$, then (see Figure 15.2c)

$$\varphi(y) = \tan^{-1}\left[\frac{2(1-y^2)qy}{(1-y^2)^2 - q^2y^2}\right], \qquad y = \omega/\omega_r, \qquad \omega_r = 1/\sqrt{LC}$$

$$q = \omega_r RC = R\sqrt{C/L}$$

### 15.15.4 Design Hilbert Phase Splitters

**Example**

Filter with two first-order all-pass filters in each branch. The phase function for the first branch is (see Figure 15.3)

$$\varphi_1(f) = \tan^{-1}\left[\frac{-2y}{y^2-1}\right] + \tan^{-1}\left[\frac{-2ay}{a^2y^2-1}\right], \qquad y = 2\pi fRC$$

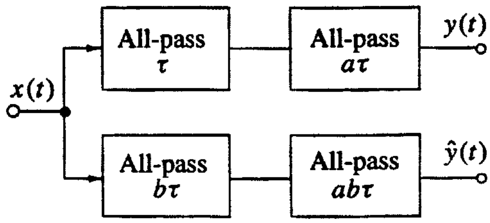


**FIGURE 15.3** Phase Hilbert splitter with two all-pass filters.

Find $a$ to get the best linearity of $\varphi_1(f)$ in the logarithmic scale. Small changes of a introduce a trade-off between the RMS phase error and the pass-band of the Hilbert transformer. Find shift parameter $b$ to yield the minimum RMS phase error

$$\varphi_2(f) = \tan^{-1}\left[\frac{2by}{b^2y^2-1}\right] + \tan^{-1}\left[\frac{2aby}{a^2b^2y^2-1}\right]$$

Figure 15.4 shows an example with a = 0.08 and b = 0.24 giving the normalized edge frequencies $y_1 = 1.6$ and $y_2 = 30$ $(f_2/f_1 = 18.75,$ or more than 4 octaves) with $\varepsilon_{RMS} = 0.016$.

## 15.16 Digital Hilbert Transformers

### 15.16.1 Digital Hilbert Transformers

Ideal discrete-time Hilbert transformer is defined as an all-pass with a pure imaginary transfer function.

$$H(e^{j\psi}) = H_r(\psi) + jH_i(\psi)$$

$$H_r(\psi) = 0 \qquad \text{for all } f$$

$$H(e^{j\psi}) = jH_i(\psi) = \begin{cases} -j & 0 < \psi < \frac{\pi}{2} \\ 0 & \psi = 0,\ |\psi| = \pi \\ j & -\pi < \psi < 0 \end{cases}$$

**Equivalent Notation**

$$H(e^{j\psi}) = -j\,\mathrm{sgn}(\sin\psi) = -\mathrm{sgn}(\sin\psi)e^{j\pi/2} = |H(\psi)|e^{j\arg H(\psi)}$$

$$|H(\psi)| = |\mathrm{sgn}(\sin\psi)| = \begin{cases} 1 & 0 < |\psi| < \pi \\ 0 & \psi = 0,\ |\psi| = \pi \end{cases}$$

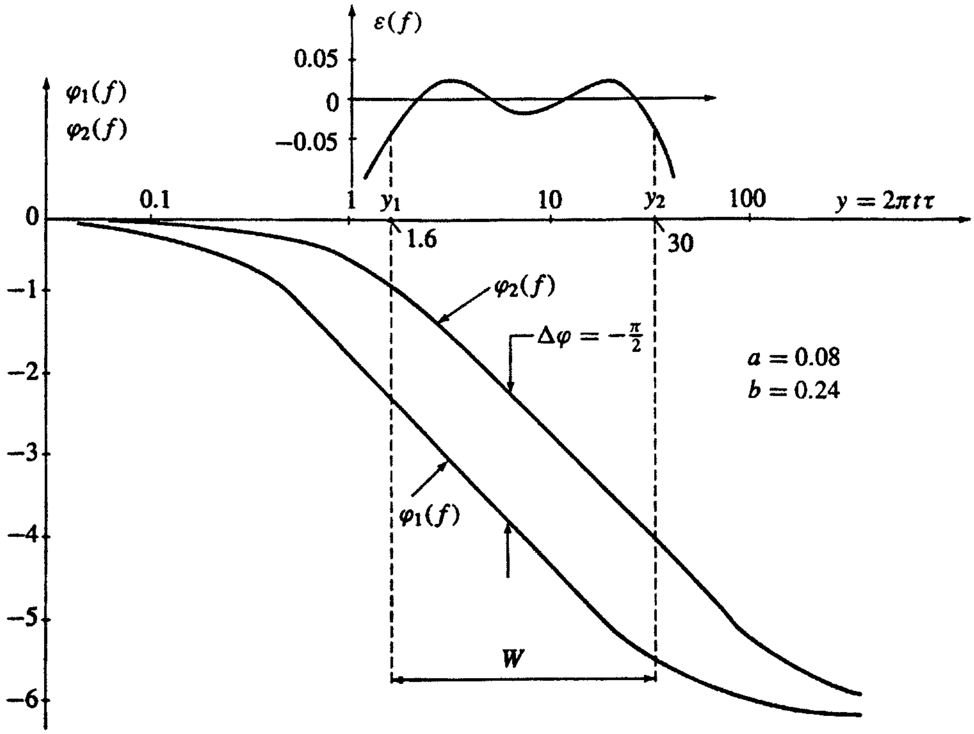


**FIGURE 15.4** The phase functions and the phase error of the Hilbert transformer of Figure 15.3.

$$\arg[H(\psi)] = -\frac{\pi}{2}\operatorname{sgn}(\sin\psi)$$

$$\psi = 2\pi f_n, \qquad f_n = f/f_s, \qquad f_s = \text{sampling frequency}$$

Noncausal impulse response of the ideal Hilbert transformer is

$$h(i) = \frac{2}{\pi i}\sin^2\left(\frac{i\pi}{2}\right) \qquad i = 0, \pm 1, \pm 2, \cdots$$

## 15.16.2 Ideal Hilbert Transformer With Linear Phase Term

$$H(e^{j\psi}) = \begin{cases} -je^{j\psi\tau} & 0 < \psi < \pi \\ 0 & \psi = 0, \ |\psi| = \pi \\ je^{-j(\psi - 2\pi)\tau} & \pi < \psi < 2\pi \end{cases}$$

$$h(i) = \frac{2}{\pi}\frac{\sin^2\left[\frac{\pi}{2}(i-\tau)\right]}{i-\tau} \qquad i = 0, \pm 1, \pm 2, \cdots$$

$$h(i) = -h(-i) \qquad i = 0, 1, 2, \cdots$$

### 15.16.3 FIR Hilbert Transformers:

Figure 15.5 shows a noncausal impulse response Hilbert transformer and its truncated and shifted version so that a causal one is generated.

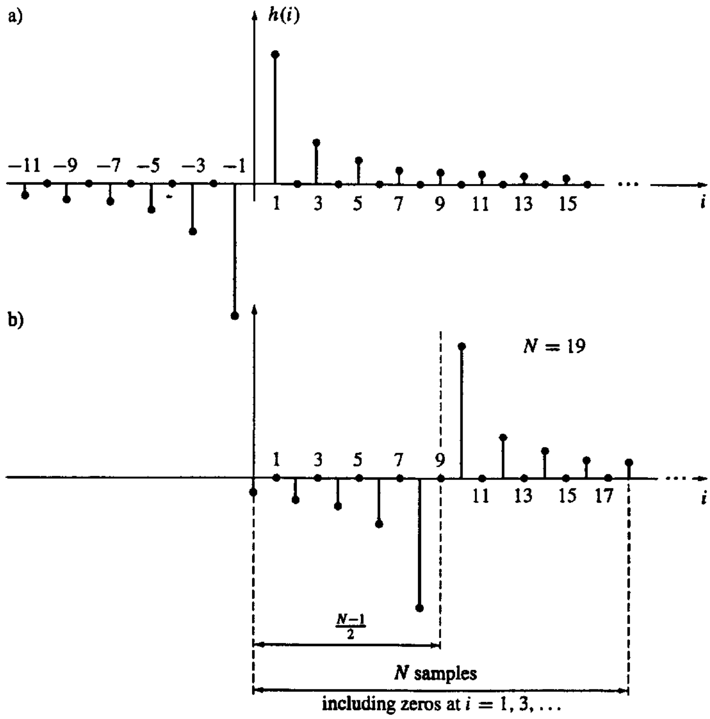


**FIGURE 15.5** Impulse responses of (a) the ideal discrete time Hilbert transformer (see 15.16.1) and (b) a FIR Hilbert transformer given by the truncation and shifting of the impulse response shown in (a).

### Causal Filter Impulse Response

$$H(i_1) = \sum_{i_1=0}^{N-1} h_1(i_1) z^{-i_1}$$

$$h_1\left(i + \frac{N-1}{2}\right) = h(i) \qquad i_1 = i + \tfrac{N-1}{2}, \quad i = -\tfrac{N-1}{2}, \cdots 0, \cdots, \tfrac{N-1}{2}$$

Transfer function =

$$H(e^{j\psi}) = e^{-j\psi\frac{N-1}{2}} \sum_{i=-\frac{N-1}{2}}^{N-1} h(i) e^{-j\psi i} = e^{-j\psi\frac{N-1}{2}} \sum_{i=1}^{\frac{N-1}{2}} -j2h(i)\sin(\psi i), \quad \psi = \frac{2\pi f}{f_s}$$

### Amplitude of Hilbert Transformer (see Figure 15.6)

$$G(e^{j\psi}) = -\sum_{i=1}^{\frac{N-1}{2}} 2h(i)\sin(\psi i)$$

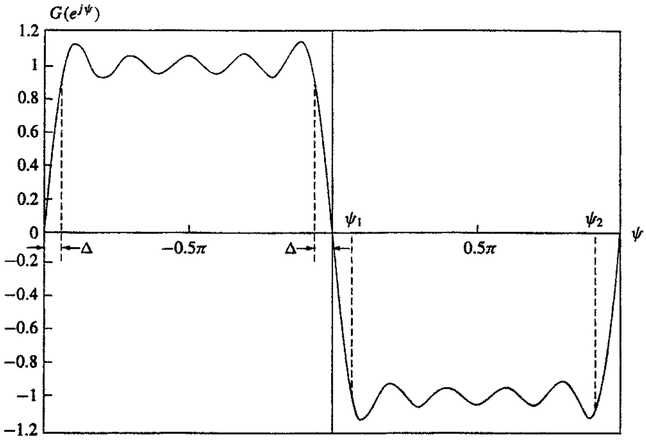


**FIGURE 15.6** The $G(e^{j\psi})$ function of an FIR Hilbert transformer (amplitude).

**Normalized Dimensionless Pass-band Hilbert Transformer**

$$W_\psi = \psi_2 - \psi_1 = \pi - 2\Delta, \quad \psi_1, \psi_2 = \text{edge frequencies}$$

$$W_f[H_z] = \frac{\pi - 2\Delta}{2\pi} f_s$$

## 15.17 IIR Hilbert Transformers

### 15.17.1 IIR Ideal Hilbert Transformer (see Figure 15.7)

$$H_{HB}(z) = 1 + z^{-1}G(z^2) \equiv \text{ideal half-band filter} \quad (\text{see Figure 15.7a})$$

$$G(z^2) = \text{all-pass filter with unit magnitude}$$

$$H_H(z) = z^{-1}G(-z^2) \equiv \text{ideal IIR Hilbert transformer}$$

$$F(z) = z^{-1}G(z^2), \quad z = e^{j\psi} \qquad (\text{see Figure 15.7b})$$

$$F(e^{j\psi}) = e^{-j\psi} e^{j\Phi_G(\psi)} = e^{j\Phi(\psi)}$$

$$\Phi(\psi) = 0.5\pi[\operatorname{sgn}(\sin(2\psi)) - \operatorname{sgn}\psi]$$

$$\Phi_G(\psi) = \Phi(\psi) + \psi \qquad (\text{see Figure 15.7e})$$

$$H_H(e^{j\psi}) = e^{-j\psi} e^{j\Phi_G(0.5\pi+\psi)}, \quad z^2 = e^{j2\psi}, \; -z^2 = e^{j2(0.5\pi+\psi)}$$

$$\arg\{z^{-1}G(-z^2)\} = -\psi + \Phi_G(0.5\pi + \psi) \qquad (\text{see Figure 15.7g})$$

IIR Hilbert transformer has an equi-ripple phase function and exact amplitude. A noncausal transfer function may have the form

$$H(z) = z^{-1}\sum_{i=1}^{N} \frac{1 - a_i z^2}{z^2 - a_i}$$

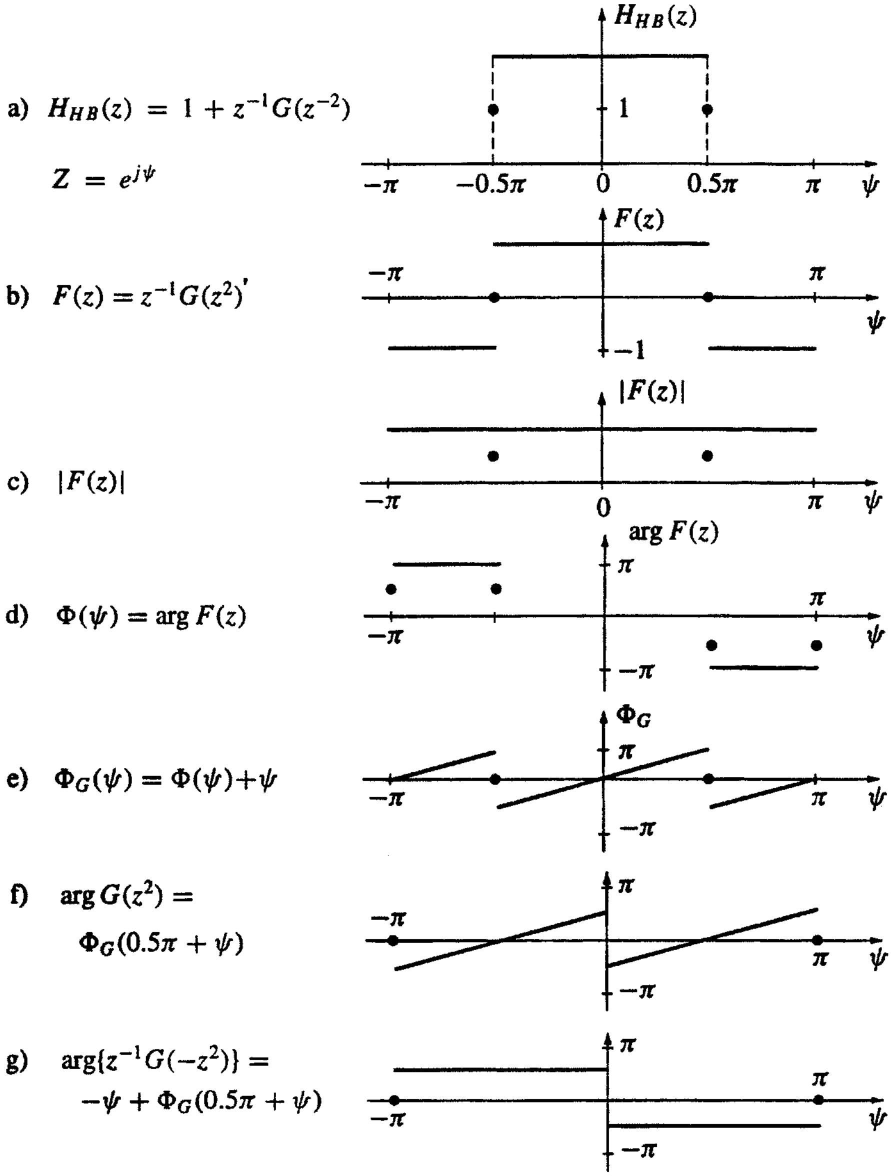


**FIGURE 15.7** Step-by-step derivation of the IIR transfer function of a Hilbert transformer $Z^{-1}G(-z^2)$, starting from the transfer function of the ideal half-band filter given by $1 + Z^1G(z^2)$

## Example

Let $\psi_1 = 0.02\pi \equiv$ low-frequency edge, $\psi_2 = 0.98\pi$ = high-frequency edge ($\Delta = 0.02\pi$), phase equiripple amplitude $|\Delta\Phi| \leq 0.01\pi$. Because $\delta = \sin(0.5\Delta\Phi)$, $\delta = 0.0157$. Using the procedure from Ansari (1985), we find $a(1) = 5.36078$, $a(2) = 1.2655$, $a(3) = 0.94167$, and $a(4) = 0.53239$. Inserting $a_i$'s, in $H(z)$ above, we find the phase function.

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