ON THE ZEROS OF AN INTEGRAL FUNCTION REPRESENTED BY FOURIER'S INTEGRAL.

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We do not possess a general method for discussing the reality of zeros of an integral function represented by Fourier's integral (such a method would be available for Riemann's ξ-function.) I present here a special case where the discussion is not quite trivial, but may be carried out with the help of known results.

Consider the function

$$F_\alpha(z) = \int_0^\infty e^{-t^\alpha} \cos zt\, dt. \tag{1}$$

If $0 < \alpha < 1$, then $F_\alpha(z)$ is defined by this formula only for real values of z. We have

$$F_1(z) = \frac{1}{1+z^2}.$$

For $\alpha > 1$ we get

$$F_\alpha(z) = \frac{1}{\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{2n+1}{\alpha}\right)}{\Gamma(2n+1)} z^{2n}. \tag{2}$$

This development shows that $F_\alpha(z)$ is an integral function of order $\frac{\alpha}{\alpha - 1}$. In particular,

$$F_2(z) = \tfrac{1}{2}\sqrt{\pi}\, e^{-\frac{1}{4}z^2}. \tag{3}$$

Following the method employed by G. H. Hardy* to prove that Riemann's $\xi(t)$ has an infinite number of real zeros, F. Bernstein† proved the same thing for $F_4(z)$, $F_6(z)$, $F_8(z)$, ... Now it is easy to go further in the case of $F_\alpha(z)$ [though naturally not in the case of $\xi(t)$], and to prove the following results:

(I) *If* $\alpha = 2$, *then there are no zeros at all.*

(II) *If* $\alpha = 4, 6, 8, \ldots$, *then there are an infinite number of real zeros but no complex zeros.*

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* *Comptes Rendus*, 6 April, 1914.

† *Mathematische Annalen*, vol. lxxix. (1919), pp. 265–268.

For comments on this paper[81], see p. 423.

(III) *If* $\alpha > 1$, *and is not an even integer, then there are an infinite number of complex zeros and a finite number, not less than* $2\left[\frac{1}{2}\alpha\right]$, *of real zeros.*

The statement (I) needs no demonstration: compare (3). The proof of (II) is based on the following special case of a theorem of Laguerre: *

If $\Phi(z)$ *is an integral function of order less than* 2 *which assumes real values along the real axis and possesses only real negative zeros, then the zeros of the integral function*

$$\Phi(0) + \frac{\Phi(1)}{1!} z + \frac{\Phi(2)}{2!} z^2 + \ldots + \frac{\Phi(n)}{n!} z^n + \ldots$$

are also all real and negative.

Put

$$\Phi(z) = \frac{\Gamma\left(\frac{2z+1}{2k}\right)\Gamma(z+1)}{\Gamma(2z+1)}, \tag{4}$$

where k is a positive integer. The poles of the numerator

$$z = -\tfrac{1}{2},\ -\tfrac{1}{2}(2k+1),\ -\tfrac{1}{2}(4k+1),\ \ldots,\quad z = -\tfrac{2}{2},\ -\tfrac{4}{2},\ -\tfrac{6}{2},\ \ldots,$$

are absorbed by those of the denominator

$$z = -\tfrac{1}{2},\ -\tfrac{2}{2},\ -\tfrac{3}{2},\ -\tfrac{4}{2},\ -\tfrac{5}{2},\ \ldots .$$

Thus $\Phi(z)$ is an integral function satisfying the conditions required by the theorem of Laguerre, and consequently the zeros of

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma\left(\frac{2n+1}{2k}\right)\Gamma(n+1)}{\Gamma(2n+1)} = 2k F_{2k}(i\sqrt{z})$$

are all real and negative; we infer that the zeros of $F_{2k}(z)$ are all real.

The order of the integral function $F_{2k}(z)$ is $\frac{2k}{2k-1}$; if $k = 2, 3, 4, \ldots$, then $1 < \frac{2k}{2k-1} < 2$. Thus $F_{2k}(z)$ is not of integral order and consequently possesses an infinity of zeros; they are all real, and thus (II) is completely proved.

Suppose x is positive. Then we have, by partial integration,

$$x^{\alpha+1} F_\alpha(x) = x^\alpha \int_0^\infty \sin xt \,.\, \alpha t^{\alpha-1} e^{-t^\alpha} dt.$$

* *Oeuvres*, vol. i. (Paris, 1898), pp. 200–203.

Introduce the new variable $u=x^\alpha t^\alpha$; then we have

$$x^{\alpha+1}F_\alpha(x)=\mathfrak{I}\int_0^\infty \exp(iu^{1/\alpha}-ux^{-\alpha})\,du, \tag{5}$$

where $\mathfrak{I}A$ denotes the imaginary part of A. Choose as path of integration, not the positive real axis, but a straight line running from 0 to ∞ in the upper half-plane and making a sufficiently small angle with the positive real axis. With this path we have

$$\lim_{x=+\infty} x^{\alpha+1}F_\alpha(x)=\mathfrak{I}\int_0^\infty e^{iu^{1/\alpha}}du.$$

Rotating the path of integration in the positive direction until it reaches the position where $\arg z=\frac{1}{2}\pi\alpha$, we get finally

$$\lim_{x=+\infty} x^{\alpha+1}F_\alpha(x)=\mathfrak{I}\int_0^{+\infty} e^{-r^{1/\alpha}}e^{i\pi\alpha/2}dr \tag{6}$$
$$=\Gamma(\alpha+1)\sin(\pi\alpha/2).$$

If the limit (6) is different from 0, that is, if α is different from 2, 4, 6, ..., then $F_\alpha(z)$ possesses

(*a*) a finite number of real zeros and

(*b*) an infinite number of zeros.

Of these assertions, (*a*) is evident from (6). To prove (*b*) we make use of the theorem that an integral function of finite order having a finite number of zeros is of the form

$$P(z)\,e^{Q(z)}, \tag{7}$$

where $P(z)$, $Q(z)$ are polynomials. Now $F_\alpha(z)$ is certainly not of the form (7), since it converges to 0 when $z\to+\infty$ in the same manner as a negative power of z, as may be seen from (6). The statements (*a*), (*b*) just proved contain the first two parts of (III).

From (1) follows, by Fourier's theorem,

$$\frac{2}{\pi}\int_0^\infty F_\alpha(x)\cos xt\,dx=e^{-t^\alpha}=1-\frac{t^\alpha}{1!}+\dots\,.$$

Differentiating $2m$ times with respect to t, where

$$2m<\alpha<2m+2, \tag{8}$$

and then putting $t=0$, we get

$$\int_0^\infty F_\alpha(x)x^2dx=\int_0^\infty F_\alpha(x)x^4dx=\dots=\int_0^\infty F_\alpha(x)x^{2m}dx=0 \tag{9}$$

The convergence of the integrals (9) is assured by (6) and (8). It follows from (9) that

$$(10) \qquad \int_0^\infty F_\alpha(x)\, x^2 P(x^2)\, dx = 0,$$

where $P(z)$ denotes any polynomial in z of degree not exceeding $m-1$. Assume now, if possible, that $F_\alpha(x)$ changes sign at most $m-1$ times for $x>0$, *e.g.* at the points $x_1, x_2, \ldots, x_{m-1}$, where $0 < x_1 < x_2 < \ldots < x_{m-1}$; and put

$$P(x^2) = (x_1^2 - x^2)(x_2^2 - x^2)\ldots(x^2_{m-1} - x^2).$$

Then the integrand in (10) is never negative and our assumption leads to a contradiction. Thus $F_\alpha(x)$ changes sign at least $m = [\frac{1}{2}\alpha]$ times for $x > 0$. We have now proved the whole of Theorem (III).

The results we have obtained may be completed in many respects. If $a \geqq 0$ and k is an integer not less than 2, then the zeros of the integral function of z

$$\int_0^\infty e^{at^2 - t^{2k}} \cos zt\, dt$$

are all real; the asymptotic distribution of the zeros can be calculated by more laborious and more usual methods; and so on. The function $F_\alpha(z)$ has been considered in connection with questions arising in the theory of errors, especially by Cauchy*, and P. Lévy† proved that $F_\alpha(x) \geqq 0$ for $0 < \alpha \leqq 2$ and for real values of x. More recently W. R. Burwell‡ has discussed the asymptotic expansion of $F_\alpha(z)$ for $\alpha = 3, 4, 5, \ldots$, and has shown in particular that, when $\alpha = 4, 6, \ldots$, the number of complex zeros is finite. This result is included in Theorem (II) above. Finally we may add that $F_\alpha(z)$ is of much importance in Waring's problem.§

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* *Comptes Rendus*, vol. xxxvii. (1853), pp. 202–206, and *passim*.
† *Comptes Rendus*, vol. clxxvi. (1923), pp. 1118–1120.
‡ *Proc. Lond. Math. Soc.* (2), vol. xxii. (1923), pp. 57–72.
§ G. H. Hardy and J. E. Littlewood, *Göttinger Nachrichten* (1920), pp. 33–54.