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RESEARCH ARTICLE

Distributional Wavelet Transform

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Abstract In this paper, we define the wavelet transform for a class of distributions in $G'_{\alpha,\beta}(\mathbb{R})$. The corresponding inversion formula is established by interpreting convergence in the weak distributional sense.

Keywords Integral transform · Wavelet transform · Distribution space · Generalized functions

Mathematics Subject Classification 46E15 · 46E40 · 46F12

1 Introduction

By dilation and translation of the basic function ψ, the wavelet $\psi_{b,a}(t)$ is defined by ([1, p. 63]):

$$\psi_{b,a}(t) := |a|^{-\rho}\psi\left(\frac{t-b}{a}\right), \quad t \in \mathbb{R},\ b \in \mathbb{R}, a \in \mathbb{R}_0 \qquad (1)$$
$$\mathbb{R}_0 = \mathbb{R}\backslash\{0\}, \rho > 0.$$

If $\rho = \frac{1}{2}$, then the mapping $\psi \to \psi_{b,a}$ is a unitary operator from $L^2(\mathbb{R})$ onto itself. Sometimes, to simplify analysis it is assumed that $a > 0$ and $\rho = 1$.

The wavelet transform $W(b,a)$ of f with respect to the wavelet $\psi_{b,a}(t)$ is defined by

$$W(b,a) := \int_{\mathbb{R}} f(t)\overline{\psi_{b,a}(t)}dt, \qquad (2)$$

provided the integral exists. If $\rho = \frac{1}{2}$ and $\psi \in L^2(\mathbb{R})$, then the wavelet transform maps each L^2-function f on $\mathbb{R}$ to a function W on $\mathbb{R} \times \mathbb{R}_0$. From Eq. (2) it follows that

$$W(b,a) = (f * \theta_{a,0})(b), \qquad (3)$$

where $\theta(x) := \overline{\psi(-x)}$.

If $f \in L^p(\mathbb{R})$ and $\psi \in L^q(\mathbb{R})$ then by ([2, p. 122]),

$$f * \theta_{a,0}(b) \in L^r(\mathbb{R}), \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Now, applying Fourier transform:

$$\hat{f}(\omega) := \mathcal{F}(f)(\omega) = \int_{-\infty}^{\infty} f(x)e^{-ix\omega}dx, \qquad (4)$$

to Eq. (3) and using convolution property, we get

$$W(b,a) = \frac{1}{2\pi}|a|^{-\rho}\int_{\mathbb{R}} e^{ib\omega}\hat{f}(\omega)\overline{\hat{\psi}(a\omega)}d\omega. \qquad (5)$$

Moreover, if $f \in L^2(\mathbb{R})$ and $\psi \in L^2(\mathbb{R})$ satisfies the following admissibility condition:

$$C_\psi := \int_{\mathbb{R}} \frac{|\hat{\psi}(w)|^2}{|w|}dw \leq \infty, \qquad (6)$$

then the following inversion formula for the wavelet transform (2) with $\rho = \frac{1}{2}$, holds:

$$\frac{1}{C_\psi}\int_{\mathbb{R}}\int_{\mathbb{R}_0} \frac{1}{\sqrt{|a|}} W(b,a)\psi\left(\frac{x-b}{a}\right)\frac{dbda}{a^2} = f(x). \qquad (7)$$

The existent applications of wavelet methods in mathematical analysis are rich. Wavelet analysis is also used to provide intrinsic characterizations of function and distribution spaces [3]. The requirements of modern mathematics, mathematical physics and engineering, need to

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incorporate ideas from distribution theory to wavelet analysis.

From Eqs. (2) and (5) it is clear that there are two ways for investigating properties of the wavelet transform. Using representation (5) with $\rho = 1$ and $a > 0$, the wavelet transform has been extended to certain tempered distributions of Schwartz and inversion formulae have been established in distribution setting by Pathak ([4], [5]), Pathak et al. [6] using duality arguments. This form of the wavelet transform has also been studied on certain Gel'-fand–Shilov spaces of type S and wavelet transform of certain ultradifferentiable function by Pathak and Singh [7]. The Shannon wavelet transform has been extended to Schwartz distributions by Pandey [8].

In the present work, the wavelet transform defined by Eq. (2) with $\rho = \frac{1}{2}$ and $a \in \mathbb{R}_0$, is investigated using kernel method.

2 Testing Function Space $G_{\alpha,\beta}(\mathbb{R})$ and Its Dual

Let us recall the definition of the space $G_{\alpha,\beta}(\mathbb{R})$ from [9, pp. 48–49]. Assume that a positive and continuous function $\zeta_{\alpha,\beta}(t)$ on $\mathbb{R}$ is given by

$$\zeta_{\alpha,\beta}(t) = \begin{cases} e^{\alpha t} & 0 \le t < \infty \\ e^{\beta t} & -\infty < t < 0, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$.

Then $G_{\alpha,\beta}(\mathbb{R})$ denotes the space of all complex-valued smooth functions $\psi(t)$ on $-\infty < t < \infty$ such that for each $k = 0, 1, 2, \ldots$,

$$\gamma_{\alpha,\beta,k}(\psi) = \sup_{t\in\mathbb{R}} \left|\zeta_{\alpha,\beta}(t) D^k \psi(t)\right| < \infty,$$

where $D^k = \left(\frac{d}{dt}\right)^k$, $k = 0, 1, 2, \ldots$.

$G_{\alpha,\beta}$ is a vector space. The topology over $G_{\alpha,\beta}$ is generated by the sequence of seminorms $\{\gamma_k\}_{k=0}^{\infty}$ [9]. A sequence $\{\psi_\nu\}_{\nu=1}^{\infty}$ is a Cauchy sequence in $G_{\alpha,\beta}$ if for each non-negative integer k, $\gamma_k(\psi_\mu - \psi_\nu) \to 0$ as $\mu, \nu \to \infty$ independently of each other. The space $G_{\alpha,\beta}$ is a sequentially complete space and therefore it is a complete countably multinormed space and so a Fréchet space. $\mathcal{D}$ is the space of smooth functions on $\mathbb{R}$ having compact support. The topology of $\mathcal{D}$ is that which makes its dual the space $\mathcal{D}'$ of Schwartz distributions on $\mathbb{R}$. Since $\mathcal{D} \subset G_{\alpha,\beta}$ and the topology of $\mathcal{D}$ is stronger than that induced on $\mathcal{D}$ by $G_{\alpha,\beta}$, it follows that the restriction of any $f \in G'_{\alpha,\beta}$ to $\mathcal{D}$ is in $\mathcal{D}'$. For details, see ([9, 10]).

Lemma 1 If $\psi \in G_{\alpha,\beta}$, then $\psi(\frac{t-b}{a}) \in G_{\alpha,\beta}$ for $\alpha \le 0$ and $\beta \ge 0$ when $|a| \ge 1$ and $\psi(\frac{t-b}{a}) \in G_{\alpha,\beta}$ for $\alpha \ge 0$ and $\beta \le 0$ when $0 < |a| < 1$.

Proof Let a and b be fixed real numbers. Then for $k = 0, 1, 2, \ldots$,

$$\begin{aligned} \sup_{-\infty<t<\infty} \left|\zeta_{\alpha,\beta}(t) D^k \psi\left(\frac{t-b}{a}\right)\right| &= \sup_{-\infty<t<\infty} \left|\zeta_{\alpha,\beta}\left(\frac{t-b}{a}\right) \times \psi^{(k)}\left(\frac{t-b}{a}\right)\left(\frac{1}{a^k}\right)\right| \left|\frac{\zeta_{\alpha,\beta}(\frac{t}{a})}{\zeta_{\alpha,\beta}(\frac{t-b}{a})} \cdot \frac{\zeta_{\alpha,\beta}(t)}{\zeta_{\alpha,\beta}(\frac{t}{a})}\right| \\ &= \sup_{-\infty<t<\infty} \left|\frac{\zeta_{\alpha,\beta}(\frac{t}{a})}{\zeta_{\alpha,\beta}(\frac{t-b}{a})} \cdot \frac{\zeta_{\alpha,\beta}(t)}{\zeta_{\alpha,\beta}(\frac{t}{a})}\right| \left(\frac{1}{|a|^k}\right) \gamma_k(\psi). \end{aligned}$$

Here, $\left|\frac{\zeta_{\alpha,\beta}(\frac{t}{a})}{\zeta_{\alpha,\beta}(\frac{t-b}{a})}\right|$ is bounded on $-\infty < t < \infty$ [9]. Thus, our Lemma is proven if we show that the positive function

$$\Im(t,a) = \left|\frac{\zeta_{\alpha,\beta}(t)}{\zeta_{\alpha,\beta}(\frac{t}{a})}\right|$$

is bounded on the (t, a) plane.

Step A. For $t > 0$,

(i) if $\alpha \le 0$ and $|a| > 1$, $\Im(t,a) = e^{\alpha t(1-\frac{1}{|a|})} < \infty$
(ii) if $\alpha \ge 0$ and $|a| < 1$, $\Im(t,a) = e^{\alpha t(1-\frac{1}{|a|})} < \infty$.

Step B. For $t < 0$,

(i) if $\beta \le 0$ and $|a| < 1$, $\Im(t,a) = e^{\beta t(1-\frac{1}{|a|})} < \infty$
(ii) if $\beta \ge 0$ and $|a| > 1$, $\Im(t,a) = e^{\beta t(1-\frac{1}{|a|})} < \infty$.

Thus, $\Im(t,a)$ is bounded for $\alpha \le 0, \beta \ge 0$ when $|a| \ge 1$ and for $\alpha \ge 0, \beta \le 0$ when $0 < |a| < 1$, and $\forall\, t \in \mathbb{R}$.

This completes the proof of the Lemma. □

3 Distributional Wavelet Transform

We assume $\psi \in G_{\alpha,\beta}(\mathbb{R})$ is the basic function generating the wavelet $\psi_{b,a}(t)$ given in Eq. (1). Since function $\psi(\frac{t-b}{a})$ belongs to $G_{\alpha,\beta}$ for fixed b and $a \neq 0$ as a function of t under conditions of Lemma 1, for $f \in G'_{\alpha,\beta}$ the wavelet transform $W(b,a)$ of f is defined by

$$W(b,a) = \frac{1}{\sqrt{|a|}} \left\langle f(t),\ \psi\left(\frac{t-b}{a}\right)\right\rangle, \quad a \in \mathbb{R}_0,\ b \in \mathbb{R}. \tag{8}$$

For convenience, in what follow, we shall deal with

$$\widetilde{W}(b,a) = \left\langle f(t),\ \psi\left(\frac{t+b}{a}\right)\right\rangle, \quad a \in \mathbb{R}_0,\ b \in \mathbb{R}, \tag{9}$$

instead of $W(b,a)$.

Theorem 1 *Let $f \in G'_{\alpha,\beta}$, $\psi \in G_{\alpha,\beta}$ and $\widetilde{W}(b,a)$ be defined by Eq. (9). Then $\widetilde{W}(b,a)$ is smooth and*

$$D_b^k \widetilde{W}(b,a) = \left\langle f(t),\ D_b^k \psi\left(\frac{t+b}{a}\right)\right\rangle, \quad k = 1, 2, 3 \ldots$$

and

$$D_a^k \widetilde{W}(b,a) = \left\langle f(t),\ D_a^k \psi\left(\frac{t+b}{a}\right)\right\rangle, \quad k = 1,2,3\ldots$$

Proof Assuming at first that $b \geq 0$ and $|a| > 0$. For a fixed a and $h \neq 0$, we have

$$\frac{1}{h}\left[\widetilde{W}(b+h,a) - \widetilde{W}(b,a)\right] - \left\langle f(t),\ D_b\psi\left(\frac{t+b}{a}\right)\right\rangle = \left\langle f(t), \vartheta_h\left(\frac{t+b}{a}\right)\right\rangle,$$

where

$$\vartheta_h\left(\frac{t+b}{a}\right) = \frac{1}{h}\left[\psi\left(\frac{t+b+h}{a}\right) - \psi\left(\frac{t+b}{a}\right)\right] - D_b\psi\left(\frac{t+b}{a}\right).$$

To prove differentiability of $\widetilde{W}(b,a)$ with respect to b we show that $\vartheta_h\left(\frac{t+b}{a}\right) \to 0$ in $G_{\alpha,\beta}$ as $h \to 0$. Let $\frac{t+b}{a} = u$, where b and a are fixed, and $\psi^{(p)}(x)$ denote $D_x^p\psi(x)$. By using Taylor's formula with remainder, we write

$$\psi^{(p)}\left(u+\frac{h}{a}\right) = \psi^{(p)}(u) + \frac{h}{a}\psi^{(p+1)}(u) + \int_0^{\frac{h}{a}}\left(\frac{h}{a} - y\right)\psi^{(p+2)}(u+y)dy.$$

Therefore,

$$\begin{aligned}\vartheta_h^{(p)}(u) &= \frac{1}{h}[\psi^{(p)}(u) + \frac{h}{a}\psi^{(p+1)}(u) + \int_0^{\frac{h}{a}}\left(\frac{h}{a} - y\right) \\ &\quad \times \psi^{(p+2)}(u+y)dy - \psi^{(p)}(u)] \\ &\quad - \frac{d}{du}\psi^{(p)}(u)\frac{1}{a} \\ &= \frac{1}{h}\int_0^{\frac{h}{a}}\left(\frac{h}{a} - y\right)\psi^{(p+2)}(u+y)dy.\end{aligned}$$

Since $\psi\left(\frac{t-b}{a}\right) \in G_{\alpha,\beta}(\mathbb{R})$, using the technique of proof of Lemma 1 it is shown that

$$\zeta_{\alpha,\beta}(t) \sup_{|y| < \frac{|h|}{a}}\left|\psi^{(p+2)}\left(\frac{t+b}{a} + y\right)\right|$$

is bounded by a constant $C = C(p,a,b)$. Therefore, we have

$$\left|\zeta_{\alpha,\beta}(t)\vartheta_h^{(p)}\left(\frac{t+b}{a}\right)\right| \leq C\int_0^{\frac{h}{a}}\left(\frac{h}{a} - y\right)dy = \frac{1}{2}C\frac{|h|^2}{a^2} \to 0$$

as $h \to 0$. Thus ϑ_h converges in $G_{\alpha,\beta}$ to zero as $h \to 0$. This proves differentiability of ψ with respect to b. Similarly, we can prove differentiability of $\widetilde{W}(b,a)$ with respect to a.

Remark 1 Using change of variables and following the above technique differentiability of $W(b,a)$ is also be proved.

Theorem 2 *For real b and $a \in \mathbb{R}_0$ let $W(b,a)$ be defined as in Eq. (8), then under conditions of Lemma 1,*

$$W(b,a) = O\left(\frac{1}{|a|^{k+\frac{1}{2}}}\right), \quad |a| \to 0, \quad \textit{for some } k \in \mathbb{N}.$$

Proof By the boundedness property of generalized functions there exist a constant $C > 0$ and a non-negative integer r depending on ψ such that

$$\begin{aligned}|W(b,a)| &= \left|\frac{1}{\sqrt{|a|}}\left\langle f(t),\ \psi\left(\frac{t-b}{a}\right)\right\rangle\right| \\ &\leq \frac{C}{\sqrt{|a|}}\max_{0\leq k\leq r}\sup_{b,t\in\mathbb{R}}\left|\zeta_{\alpha,\beta}(t)D_t^k\psi\left(\frac{t-b}{a}\right)\right| \\ &= \frac{C}{\sqrt{|a|}}\max_{0\leq k\leq r}\sup_{b,t\in\mathbb{R}}|\zeta_{\alpha,\beta}\left(\frac{t-b}{a}\right) \\ &\quad \times \psi^{(k)}\left(\frac{t-b}{a}\right)\left(\frac{1}{a^k}\right)|\left|\frac{\zeta_{\alpha,\beta}(\frac{t}{a})}{\zeta_{\alpha,\beta}(\frac{t-b}{a})}\cdot\frac{\zeta_{\alpha,\beta}(t)}{\zeta_{\alpha,\beta}(\frac{t}{a})}\right| \\ &= C\left[\sup_{b,t\in\mathbb{R}}\left|\frac{\zeta_{\alpha,\beta}(\frac{t}{a})}{\zeta_{\alpha,\beta}(\frac{t-b}{a})}\cdot\frac{\zeta_{\alpha,\beta}(t)}{\zeta_{\alpha,\beta}(\frac{t}{a})}\right|\right] \\ &\quad \times \left(\frac{1}{|a|^{k+\frac{1}{2}}}\right)\max_{0\leq k\leq r}\gamma_{\alpha,\beta,k}(\psi)\end{aligned}$$

by using boundedness property, as given in the Lemma 1. This gives the required result. □

4 Inversion of the Distributional Wavelet Transform

In order to derive inversion formula for the distributional wavelet transform, we construct a structure formula for the distribution $f \in G'_{\alpha,\beta}$ for $\alpha, \beta > 0$ ([11, pp. 272–274]).

If $f \in G'_{\alpha,\beta}$ and $\phi \in G_{\alpha,\beta}$, then by boundedness property of distributions, there exist a $C > 0$ and a non-negative integer m satisfying

$$|\langle f, \phi\rangle| \leq C\max_{0\leq k\leq m}\gamma_{\alpha,\beta,k}(\phi). \tag{10}$$

Then from Eq. (10) we have

$$\begin{aligned}|\langle f,\phi\rangle| &\leq C\max_{0\leq k\leq m}\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t}\left|\frac{d}{dt}\left[\zeta_{\alpha,\beta}(t)D_t^k\phi(t)\right]dt\right|\right| \\ &\leq C\sum_{k=0}^{m}\int_0^{\infty}\left|e^{\alpha t}D_t^{k+1}\phi(t) + \alpha e^{\alpha t}D_t^k\phi(t)\right|dt \\ &\quad + C\sum_{k=0}^{m}\int_{-\infty}^{0}\left|e^{\beta t}D_t^{k+1}\phi(t) + \beta e^{\beta t}D_t^k\phi(t)\right|dt \\ &\leq C'\sum_{k=0}^{m}\int_{-\infty}^{\infty}[|\zeta_{\alpha,\beta}(t)D_t^{k+1}\phi(t)| + |\zeta_{\alpha,\beta}(t)D_t^k\phi(t)|]dt \\ &\leq C''\sum_{k=0}^{m}[||\zeta_{\gamma,\delta}(t)D_t^{k+1}\phi(t)||_2 + ||\zeta_{\gamma,\delta}(t)D_t^k\phi(t)||_2].\end{aligned}$$

Where $\alpha<\gamma, \delta<\beta$ [9, p. 49].

Now, using Hahn–Banach theorem and the Riesz representation theorem we get g_k belonging to the space $L^2(\mathbb{R})$ satisfying

$$\langle f,\phi\rangle=\sum_{k=0}^{m}\left[\left\langle g_{1,k}(t),\zeta_{\gamma,\delta}(t)D_t^{k+1}\phi(t)\right\rangle + \left\langle g_{2,k}(t),\zeta_{\gamma,\delta}(t)D_t^{k}\phi(t)\right\rangle\right].$$

Therefore our structure formula is

$$f=\sum_{k=0}^{m}(-1)^{k+1}\{D_t^{k+1}[\zeta_{\gamma,\delta}(t)g_{1,k}(t)] - D_t^{k}[\zeta_{\gamma,\delta}(t)g_{2,k}(t)]\}. \tag{11}$$

We now establish the inversion formula for the distributional wavelet transform using Eq. (7).

Theorem 3 *Assume that the wavelet transform $W(b,a)$ of $f\in G'_{\alpha,\beta}$ is given by Eq. (8). Then*

$$\lim_{\substack{N\to\infty\\ R\to\infty}}\left\langle\frac{1}{C_\psi}\int_{-R}^{R}\int_{-N}^{N}W(b,a)\psi_{b,a}(x)\frac{dbda}{a^2},\ \phi(x)\right\rangle = \langle f,\phi\rangle, \tag{12}$$

for each $\phi\in\mathcal{D}$, $a\in\mathbb{R}_0$ and $b\in\mathbb{R}$, where $\psi_{b,a}(x)$ is defined by Eq. (1) with $\rho=1$.

Proof Using the structure formula for f as given in Eq. (11), we have

$$W(b,a)=\langle\sum_{k=0}^{m}(-1)^{k+1}\{D_t^{k+1}[\zeta_{\gamma,\delta}(t)g_{1,k}(t)] - D_t^{k}[\zeta_{\gamma,\delta}(t)g_{2,k}(t)]\},\psi_{b,a}(t)\rangle. \tag{13}$$

Moreover,

$$J\equiv\lim_{\substack{N\to\infty\\ R\to\infty}}\left\langle\frac{1}{C_\psi}\int_{-R}^{R}\int_{-N}^{N}W(b,a)\psi_{b,a}(x)\frac{dbda}{a^2},\ \phi(x)\right\rangle$$

$$=\lim_{\substack{N\to\infty\\ R\to\infty}}\left\langle\frac{1}{C_\psi}\int_{-R}^{R}\int_{-N}^{N}[\int_{-\infty}^{\infty}\{\sum_{k=0}^{m}\zeta_{\gamma,\delta}(t)[g_{1,k}(t)D_t^{k+1}\overline{\psi_{b,a}(t)} +g_{2,k}(t)D_t^{k}\overline{\psi_{b,a}(t)}]\}\psi_{b,a}(x)dt]\frac{dbda}{a^2},\ \phi(x)\right\rangle$$

$$=\lim_{\substack{N\to\infty\\ R\to\infty}}\left\langle\frac{1}{C_\psi}\int_{-R}^{R}\int_{-N}^{N}[\int_{-\infty}^{\infty}\{\sum_{k=0}^{m}\zeta_{\gamma,\delta}(t)(-1)^k[-g_{1,k}(t) \times D_b^{k+1}\overline{\psi_{b,a}(t)}+g_{2,k}(t)D_b^{k}\overline{\psi_{b,a}(t)}]\}\psi_{b,a}(x)dt]\frac{dbda}{a^2},\phi(x)\right\rangle,$$

$$\text{as}\, D_t\psi_{b,a}(t)=-D_b\psi_{b,a}(t).$$

Thus

$$J=\lim_{\substack{N\to\infty\\ R\to\infty}}\left\langle\frac{1}{C_\psi}\int_{-R}^{R}\int_{-N}^{N}\int_{-\infty}^{\infty}\sum_{k=0}^{m}\zeta_{\gamma,\delta}(t)\overline{\psi_{b,a}(t)}[g_{1,k}(t) \times D_b^{k+1}\psi_{b,a}(x)+g_{2,k}(t)D_b^{k}\psi_{b,a}(x)]\frac{dtdbda}{a^2},\ \phi(x)\right\rangle$$

[by integration by parts]

$$=\lim_{\substack{N\to\infty\\ R\to\infty}}\left\langle\frac{1}{C_\psi}\int_{-R}^{R}\int_{-N}^{N}\int_{-\infty}^{\infty}\sum_{k=0}^{m}\zeta_{\gamma,\delta}(t)\overline{\psi_{b,a}(t)}(-1)^k \times[-g_{1,k}(t)D_x^{k+1}\psi_{b,a}(x)+g_{2,k}(t)D_x^{k}\psi_{b,a}(x)]\frac{dtdbda}{a^2},\ \phi(x)\right\rangle$$

$$=\lim_{\substack{N\to\infty\\ R\to\infty}}\frac{1}{C_\psi}\sum_{k=0}^{m}\int_{-R}^{R}\int_{-N}^{N}\int_{-\infty}^{\infty}\Big\langle\zeta_{\gamma,\delta}(t)\overline{\psi_{b,a}(t)}\psi_{b,a}(x), g_{1,k}(t)D_x^{k+1}\phi(x)+g_{2,k}(t)D_x^{k}\phi(x)\Big\rangle\frac{dtdbda}{a^2} \tag{14}$$

The integrand

$$\left\{(g_{2,k}(t)+g_{1,k}(t)D_x)D_x^k\phi(x)\right\}\overline{\psi_{b,a}(t)}\psi_{b,a}(x)\frac{\zeta_{\gamma,\delta}(t)}{a^2}$$

is absolutely integrable with respect to x and t in the (x,t)-plane and so Fubini's theorem is applicable with respect to integration by x and t. Therefore Eq. (14) yields

$$J=\frac{1}{C_\psi}\sum_{k=0}^{m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(g_{2,k}(t)+g_{1,k}(t)D_x)D_x^k\phi(x)\} \times\overline{\psi_{b,a}(t)}\psi_{b,a}(x)\zeta_{\gamma,\delta}(t)\frac{dxdbdadt}{a^2}$$

$$=\frac{1}{C_\psi}\sum_{k=0}^{m}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\overline{W_\psi\{D_x^{k+1}\phi(x)\}(b,a)\psi_{b,a}(t)}} \times\frac{dbda}{a^2}[g_{1,k}(t)\zeta_{\gamma,\delta}(t)]dt$$

$$+\frac{1}{C_\psi}\sum_{k=0}^{m}\int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{W_\psi\{D_x^k\phi(x)\}(b,a)\psi_{b,a}(t)\frac{dbda}{a^2}}$$
$$\times[g_{2,k}(t)\zeta_{\gamma,\delta}(t)]dt$$

[invoking Fubini's theorem]

$$=\sum_{k=0}^{m}\left\{\int_{-\infty}^{\infty}\overline{D_t^{k+1}\phi(t)}g_{1,k}(t)\zeta_{\gamma,\delta}(t)dt + \int_{-\infty}^{\infty}\overline{D_t^k\phi(t)}g_{2,k}(t)\zeta_{\gamma,\delta}(t)dt\right\}$$

[by inversion formula (7)]

$$=\sum_{k=0}^{m}\int_{-\infty}^{\infty}(g_{2,k}(t)+g_{1,k}(t)D_t)D_t^k\overline{\phi(t)}\zeta_{\gamma,\delta}(t)dt$$
$$=\sum_{k=0}^{m}[\langle g_{1,k}(t)\zeta_{\gamma,\delta}(t),\ D_t^{k+1}\phi(t)\rangle + \langle g_{2,k}(t)\zeta_{\gamma,\delta}(t),\ D_t^k\phi(t)\rangle] \qquad \text{[using duality]}$$
$$=\left\langle\sum_{k=0}^{m}\{(-1)^{k+1}D_t^{k+1}[\zeta_{\gamma,\delta}(t)g_{1,k}(t)] + (-1)^k D_t^k[\zeta_{\gamma,\delta}(t)g_{2,k}(t)]\},\ \phi(t)\right\rangle$$
$$=\langle f,\phi\rangle \qquad \text{[by structure formula (11)]}.$$

This completes the proof of the Theorem. □

Example 1 Let us consider the Mexican hat wavelet, which is an even wavelet defined by the second derivative of a Gaussian function as

$$\psi(t)=(1-t^2)\exp(-t^2/2)=-\frac{d^2}{dt^2}\exp(-t^2/2),$$

and its Fourier transform is given by

$$\hat{\psi}(w)=\sqrt{2\pi}w^2\exp(-w^2/2).$$

It is a C^∞-function and well localized in time and frequency domains. The kth derivative of Mexican hat wavelet given by

$$D^k\psi(t)=\sum_{r=0}^{k}\binom{k}{r}D^{(r)}(1-t^2)D^{(k-r)}\exp(-t^2/2).$$

Using property of Hermite polynomial [12] we write the last expression as

$$D^k\psi(t)=\sum_{r=0}^{k}\binom{k}{r}D^{(r)}(1-t^2)\exp(-t^2/2)H_{k-r}(t)$$
$$=k(1-t^2)\exp(-t^2/2)H_k(t) + \binom{k}{r}(-2t)\exp(-t^2/2)H_{k-1}(t),$$

where $H_k(t)$ denotes the Hermite polynomial. Therefore, $\psi(t)\in G_{\alpha,\beta}(\mathbb{R})$ and the Mexican hat wavelet transform of $f\in G'_{\alpha,\beta}$ is then defined by Eq. (8).

5 Conclusions

A suitable testing function space $G_{\alpha,\beta}(\mathbb{R})$ containing the wavelet $\psi_{b,a}(t)$ is constructed. The wavelet transform of a generalized function f belonging to the corresponding generalized function space $G'_{\alpha,\beta}$ is investigated and inversion formula is established by interpreting convergence in $\mathcal{D}'$. In most of the cases, in classical analysis, we study wavelets belonging to the space $L^2(\mathbb{R})$ and establish inversion formula in certain $L^2(\mathbb{R}_0\times\mathbb{R})$-space, where $\mathbb{R}_0=\mathbb{R}\backslash\{0\}$. Thus, our approach is based on the properties of the wavelet whereas classical approach imposes conditions on the wavelet as per requirement of $L^2(\mathbb{R})$. The aforesaid analysis can be applied to develop the theory of n-dimensional wavelet transform of generalized functions.

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