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DISPERSIVE ESTIMATE FOR THE WAVE EQUATION WITH THE INVERSE-SQUARE POTENTIAL

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Abstract. We prove that spherically symmetric solutions of the Cauchy problem for the linear wave equation with the inverse-square potential satisfy a modified dispersive inequality that bounds the L^∞ norm of the solution in terms of certain Besov norms of the data, with a factor that decays in t for positive potentials. When the potential is negative we show that the decay is split between t and r, and the estimate blows up at $r = 0$. We also provide a counterexample showing that the use of Besov norms in dispersive inequalities for the wave equation are in general unavoidable.

1. **Introduction.** Consider the following linear wave equation

$$\begin{cases} \Box_n u + \frac{a}{|x|^2} u = 0 \\ u(0,x) = f(x) \\ \partial_t u(0,x) = g(x) \end{cases} \tag{1.1}$$

where $\Box_n = \partial_t^2 - \Delta_n$ is the D'Alembertian in $\mathbb{R}^{n+1}$ and a is a real number. The interest in this equation comes from the potential term being homogeneous of degree -2 and therefore scaling the same way as the D'Alembertian term. This implies that perturbation methods alone cannot be used in studying the effect of this potential. In particular, the value of the constant a is important.

In [8] we showed that in the radial case, i.e. when the data – and thus the solution – are radially symmetric, the solution to (1.1) satisfies generalized space-time Strichartz estimates as long as $a > -(n-2)^2/4$. In this paper we continue the study of the radial case by proving a dispersive estimate, i.e. a decay-in-time estimate for the L^∞ norm of the solution. We note that for non-radial data, the same estimate can be proven in the same way for each component in the spherical

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harmonics expansion of the solution, but we do not know if it is possible to re-sum these to obtain the estimate in the general case.

It is well-known that in the free case $a = 0$ the following dispersive inequality holds

$$|u(x,t)| \leq \frac{C}{t^{\frac{n-1}{2}}} \left(\|f\|_{\dot{W}^{\frac{n+1}{2},1}} + \|g\|_{\dot{W}^{\frac{n-1}{2},1}} \right), \tag{1.2}$$

where $\dot{W}^{k,p}(\mathbb{R}^n)$ is the homogeneous Sobolev space on $\mathbb{R}^n$ which is defined to be the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\dot{W}^{k,p}(\mathbb{R}^n)} = \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^p(\mathbb{R}^n)}.$$

When n is even, the $\dot{W}^{k,1}$ norms on the right need to be replaced with the Besov norms $\dot{B}_1^{k,1}$, see [10] for details. Even when n is odd, the estimate

$$\|u(t)\|_{L^\infty(\mathbf{R}^n)} \leq \frac{C}{t^{(n-1)/2}} \left(\|(-\Delta)^{\frac{n+1}{4}} f\|_{L^1(\mathbf{R}^n)} + \|(-\Delta)^{\frac{n-1}{4}} g\|_{L^1(\mathbf{R}^n)} \right)$$

is false for all n, as we show in an appendix to this paper. This shows that one cannot replace the $\dot{W}^{k,1}$ norms with $\dot{H}^{k,1}$ in the dispersive inequality.

For the wave equation with a compactly supported potential, Beals [1] has shown that (1.2) still holds. We would like to show that a similar estimate holds for the solution of (1.1). Because of the special form of the potential, this equation is scale-invariant for any a, and thus it is enough to prove the above estimate (which is also scale-invariant) for $t = 1$, i.e., boundedness implies the decay. It is important to note however, that such an estimate is false if $a < 0$: It was shown in [8] that, generically, the solution of the wave equation with a negative inverse-square potential blows up at $x = 0$ even if the data is smooth.

Since negative potentials do appear in nonlinear applications of the above, we want to prove a modified dispersive estimate which allows for blowup at the origin. Let

$$\lambda := \frac{n-2}{2}$$

and for $a > -\lambda^2$ let

$$\nu := \sqrt{\lambda^2 + a}.$$

The following is the main result of this paper:

Theorem 1.1. *Let $n \geq 3$, λ and ν defined as above, and assume $\nu \neq \frac{n-1}{2}, \frac{n-3}{2}$ if n is even, and $\nu \neq \frac{n-3}{2}, \frac{n-4}{2}$ if n is odd. Let u be the radial solution of* (1.1). *Then u satisfies the following dispersive estimate:*

$$|u(r,t)| \leq \begin{cases} \frac{C}{r^{\lambda-\nu} t^{\nu+\frac{1}{2}}} \left(\|f\|_{\dot{B}_1^{\frac{n+1}{2},1}} + \|g\|_{\dot{B}_1^{\frac{n-1}{2},1}} \right) & \text{if } -\lambda^2 < a < 0 \\ \frac{C}{t^{\frac{n-1}{2}}} \left(\|f\|_{\dot{B}_1^{\frac{n+1}{2},1}} + \|g\|_{\dot{B}_1^{\frac{n-1}{2},1}} \right) & \text{if } a \geq 0. \end{cases} \tag{1.3}$$

Here $\dot{B}_1^{s,1}$ is the homogeneous Besov space on $\mathbb{R}^n$ (see [2]). We note that if $a < -\lambda^2$, then no dispersive estimate is possible at all, since the operator $-\Delta_n + \frac{a}{|x|^2}$ will have negative spectrum in that case, and thus for general data we would expect exponential growth even in L^2.

For the proof of (1.3) we consider two separate cases: $t < 2r$, the *exterior* region, and $t \geq 2r$, the *interior* one. In the exterior case we use the representation of the solution of (1.1) in frequency space using the Hankel transform and the spectral decomposition of the operator $-\Delta + ar^{-2}$. In the interior case we use the representation of the solution in physical space via Legendre functions, and integration by parts using the Legendre operator.

1.1. **Estimate in the Exterior Region.** A radially symmetric function on $\mathbb{R}^n$ is a function of the form $f(x) = F(|x|)$ where F is a function on the positive reals $\mathbb{R}_+$. Throughout this paper we will follow the standard practice of ignoring the distinction between f and F. We will, in particular, identify the radially symmetric subspace $L^p_{\text{rad}}(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$ with the space of functions on $\mathbb{R}_+$ for which the norm

$$||f||_{L^p(\mathbb{R}^n)} = \begin{cases} (\int_0^\infty f(r)^p c_n r^{n-1}\, dr)^{1/p} & \text{if } 1 \le p < \infty \\ \text{ess sup} f & \text{if } p = \infty \end{cases}$$

is finite. Here c_n is the surface area of the unit sphere in $\mathbb{R}^n$. Similar remarks apply to Sobolev and Besov norms, though the explicit representation of these norms for functions on $\mathbb{R}_+$ is more complicated.

For any real $\nu > 1$ we define the Hankel transform of order ν as follows. If f is continuous and supported between two spheres then

$$(\mathcal{H}_\nu f)(\omega) = c_n^{-1} \int_0^\infty (r\omega)^{-\lambda} J_\nu(r\omega) f(r) c_n r^{n-1}\, dr. \tag{1.4}$$

Here λ is as in the introduction and J_ν is the usual Bessel function. Recall that the Hankel transform thus defined has a unique extension to an isometry of L^2, which we will again denote by $\mathcal{H}_\nu$. Furthermore, $\mathcal{H}_\nu$ is self-adjoint, and hence an involution. In [8] we have shown that

$$\Delta - ar^{-2} = -\mathcal{H}_\nu \omega^2 \mathcal{H}_\nu \tag{1.5}$$

where ω^2 denotes the corresponding multiplication operator. We will take (1.5) as a definition. In other words $\Delta - ar^{-2}$ is defined on the dense subspace

$$\text{Dom}(\Delta - ar^{-2}) = \mathcal{H}_\nu \,\text{Dom}\, \omega^2$$

of $L^2_{\text{rad}}(\mathbb{R}^n)$ and is given there by the equation above. The domain is clearly dilation-invariant. It can be shown that these are the only extensions of $\Delta - ar^{-2}$ from smooth functions supported in a compact set not containing the origin to a dense, dilation-invariant domain in $L^2_{\text{rad}}(\mathbb{R}^n)$. See [8] and the references cited there for a more extensive discussion.

With these definitions, if u satisfies the initial value problem 1.1 with data f and g then $\mathcal{H}_\nu u$ satisfies the initial value problem

$$\begin{cases} \left(\partial_t^2 + \omega^2\right) \mathcal{H}_\nu u = 0 \\ \mathcal{H}_\nu u(0,x) = \mathcal{H}_\nu f(x) \\ \partial_t \mathcal{H}_\nu u(0,x) = \mathcal{H}_\nu g(x) \end{cases} \tag{1.6}$$

In what follows we are going to assume for simplicity that $f \equiv 0$. Solving the ODE in the above we then obtain

$$u(t,r) = \mathcal{H}_\nu\Big(\frac{\sin \omega t}{\omega}(\mathcal{H}_\nu g)(\omega)\Big)(r) \tag{1.7}$$

$$= \int_0^\infty \int_0^\infty (rs)^{-\lambda} J_\nu(r\omega) J_\nu(s\omega) \sin \omega t g(s)\; d\omega\; s^{n-1} ds. \tag{1.8}$$

We will use this formula, Littlewood-Paley decomposition in frequency ω, and the following simple estimate for the Bessel function

$$|J_\nu(z)| \le C_\nu \min\{|z|^\nu, |z|^{-1/2}\},$$

to prove the estimate (1.2) in the region $t < 2r$. To this end, let us fix a function $\beta \in C_0^\infty(\mathbb{R})$ which is supported in $[\frac{1}{2}, 2]$ and satisfies $\sum_{j=-\infty}^{\infty} \beta(2^j\omega) = 1$ for $\omega > 0$. For $\Omega \in 2^{\mathbb{Z}}$ we let $\beta_\Omega(\omega) := \beta(\omega/\Omega)$, and define the dyadic parts of a radial function u as

$$u_\Omega = P_\Omega u := \mathcal{H}_\nu \beta_\Omega \mathcal{H}_\nu u,$$

so that $u = \sum_\Omega u_\Omega$. We also note that, on the support of β_Ω, $\beta_{\Omega/2} + \beta_\Omega + \beta_{2\Omega} = 1$. From (1.7) we thus have $u = u_{-1} + u_0 + u_1$, where

$$u_i(t,\cdot) = \sum_\Omega \mathcal{H}_\nu \frac{\sin \omega t}{\omega} \mathcal{H}_\nu \beta_{2^i\Omega} g_\Omega$$

and it is enough to estimate u_0, the other two being similar. Hence

$$\begin{aligned}
|u_0(t,r)| &= \left| \sum_\Omega \int_0^\infty \int_{\Omega/2}^{2\Omega} (rs)^{-\lambda} J_\nu(r\omega) J_\nu(s\omega) \sin \omega t \; \beta_\Omega(\omega) g_\Omega(s) s^{n-1} d\omega ds \right| \\
&\leq r^{-\lambda} \sum_\Omega \int_{\Omega/2}^{2\Omega} |J_\nu(r\omega)| \int_0^\infty |J_\nu(s\omega)||g_\Omega(s)| s^{-\lambda} s^{n-1} ds d\omega \\
&\leq C r^{-\lambda-\frac{1}{2}} \sum_\Omega \left\{ \int_{\Omega/2}^{2\Omega} \omega^{\nu-\frac{1}{2}} \int_0^{1/\omega} s^{\nu-\lambda} |g_\Omega(s)| s^{n-1} ds \, d\omega \right. \\
&\qquad \left. + \int_{\Omega/2}^{2\Omega} \omega^{-1} \int_{1/\omega}^\infty s^{-\lambda-\frac{1}{2}} |g_\Omega(s)| s^{n-1} ds \, d\omega \right\} \\
&\leq C \min\{t^{-\lambda-\frac{1}{2}}, r^{-\lambda+\nu} t^{-\nu-\frac{1}{2}}\} \left[\sum_{\Omega \in 2^{\mathbb{Z}}} \Omega^{\frac{n-1}{2} - \frac{n}{p'}} |g_\Omega|_{L^p(\mathbb{R}^n)} \right] \qquad (1.9)
\end{aligned}$$

This last inequality is valid for $n \geq 2$, $t < 2r$, $\nu > -\frac{1}{2}$ and for conjugate exponents p and p' such that

$$\max\{1, \frac{2n}{n+2+2\nu}\} < p < \frac{2n}{n+1}. \qquad (1.10)$$

The bracketed expression in (1.9) is nothing but the homogeneous Besov norm based on the dyadic decomposition relative to the operator $-\Delta + ar^{-2}$. We need to use the almost orthogonality of projections, proved in the next section, in order to compare this with the standard Besov norm, which is based on the dyadic decomposition with respect to $-\Delta$. For $\Lambda \in 2^{\mathbb{Z}}$, Let Δ_Λ denote the standard projection operator onto functions with Fourier transform supported in the annulus $\Lambda/2 \leq |\xi| \leq 2\Lambda$. We then have, using the embedding theorem for homogeneous Besov spaces,

$$\begin{aligned}
\max\{t^{\lambda+\frac{1}{2}}, r^{\lambda-\nu} t^{\nu+\frac{1}{2}}\} |u_0(r,t)| &\leq C \sum_{\Lambda \in 2^{\mathbb{Z}}} \Lambda^{\frac{n-1}{2} - \frac{n}{p'}} |\Delta_\Lambda g|_{L^p(\mathbb{R}^n)} \\
&= C \|g\|_{\dot{B}_p^{\frac{n-1}{2} - \frac{n}{p'}, 1}} \\
&\leq C \|g\|_{\dot{B}_1^{\frac{n-1}{2}, 1}},
\end{aligned}$$

which is the desired result.

1.2. **Almost orthogonality of projections.** The goal of this section is to prove that, at least for radial functions, one may replace frequency localization with respect to the operator $-\Delta + a/r^2$ by the usual (Fourier) frequency localization. For this we rely heavily on properties of the operator $\mathcal{K}_{\lambda,\nu}$ introduced in [8] (and denoted there by $\mathcal{K}^0_{\lambda,\nu}$). Recall that this operator intertwines the usual Laplacian and our modified Laplacian with a potential:

$$-\Delta \mathcal{K}_{\lambda,\nu} = \mathcal{K}_{\lambda,\nu}(-\Delta + \frac{a}{r^2}).$$

Recall also that we defined $\mathcal{K}_{\lambda,\nu}$ as being $\mathcal{H}_\lambda \mathcal{H}_\nu$. Hence we very easily obtain

Lemma 1.2. *Let Δ_Ω be the usual frequency localization operator, which in our context is nothing but $\mathcal{H}_\lambda \beta_\Omega \mathcal{H}_\lambda$. Then we have*

$$P_\Omega = \mathcal{K}_{\nu,\lambda} \Delta_\Omega \mathcal{K}_{\lambda,\nu}, \tag{1.11}$$

and P_Ω is continuous on L^p_{rad} provided that

$$\max\{0, \frac{\lambda - \nu}{n}\} < \frac{1}{p} < \min\{1, \frac{\lambda + \nu + 2}{n}\}, \tag{1.12}$$

and the continuity is uniform with respect to Ω.

The first part follows directly from the definitions of the operators. The continuity property in turn follows from the continuity property of $\mathcal{K}_{\lambda,\nu}$ and its inverse $\mathcal{K}_{\nu,\lambda}$ (see corollary 1 in [8]). It follows from the above Lemma that

$$\|P_\Omega \Delta_\Lambda f\|_{L^p} \leq C \|\Delta_\Lambda f\|_{L^p} \leq C \|f\|_{L^p}, \tag{1.13}$$

where C is independent of Ω and Λ.

Now we would like to obtain almost-orthogonality between Δ_Λ and P_Ω for Λ and Ω well separated. In order to achieve this we will use the continuity properties of the $\mathcal{K}_{\cdot,\cdot}$ on the Sobolev spaces $\dot{H}^\sigma$. Theorem 3.2 in [8] gives us $\varepsilon > 0$ such that both $\mathcal{K}_{\cdot,\cdot}$ are continuous on $\dot{H}^{\pm\varepsilon}$. Then we obtain

Lemma 1.3. *The operators P_Ω and Δ_Λ are almost orthogonal in the L^2 sense, and*

$$\|P_\Omega \Delta_\Lambda f\|_{L^2} \lesssim \inf(\frac{\Omega}{\Lambda}, \frac{\Lambda}{\Omega})^\varepsilon \|f\|_{L^2}. \tag{1.14}$$

Let f, g be two (radial) test functions. Suppose that $\Lambda < \Omega$ (the other case would be obtained by switching the sign of ε). Then we can write

$$\begin{aligned}
\int (P_\Omega \Delta_\Lambda f) g \, dx &= \int \Delta_\Lambda f P_\Omega g \, dx \\
&\leq \|\Delta_\Lambda f\|_{\dot{H}^\varepsilon} \|P_\Omega g\|_{\dot{H}^{-\varepsilon}} \\
&\lesssim \Lambda^\varepsilon \|\Delta_\Lambda f\|_{L^2} \|\mathcal{K}_{\nu,\lambda} \Delta_\Omega \mathcal{K}_{\lambda,\nu} g\|_{\dot{H}^{-\varepsilon}} \\
&\lesssim \Lambda^\varepsilon \|\Delta_\Lambda f\|_{L^2} \|\Delta_\Omega \mathcal{K}_{\lambda,\nu} g\|_{\dot{H}^{-\varepsilon}} \\
&\lesssim \Lambda^\varepsilon \Omega^{-\varepsilon} \|\Delta_\Lambda f\|_{L^2} \|\Delta_\Omega \mathcal{K}_{\lambda,\nu} g\|_{L^2} \\
&\lesssim \Lambda^\varepsilon \Omega^{-\varepsilon} \|f\|_{L^2} \|g\|_{L^2},
\end{aligned}$$

which is the desired result by duality. We are then in a position to state the desired result, which follows by interpolating between (1.13) and (1.14):

Proposition 1.4. *The operators P_Ω and Δ_Λ are almost orthogonal in the L^p sense, for p satisfying* (1.12), *and we have*

$$\|P_\Omega \Delta_\Lambda f\|_{L^p} \lesssim \inf(\frac{\Omega}{\Lambda}, \frac{\Lambda}{\Omega})^{\tilde{\varepsilon}} \|f\|_{L^p}. \tag{1.15}$$

As a corollary we obtain the estimate used in the previous section

$$\sum_{\Omega\in 2^{\mathbb{Z}}} \Omega^{\frac{n-1}{2}-\frac{n}{p'}} |g_\Omega|_{L^p(\mathbb{R}^n)} \lesssim \sum_{\Lambda\in 2^{\mathbb{Z}}} \Lambda^{\frac{n-1}{2}-\frac{n}{p'}} |\Delta_\Lambda g|_{L^p(\mathbb{R}^n)}. \tag{1.16}$$

Indeed, one may write

$$g_\Omega = \sum_\Lambda \Delta_\Lambda P_\Omega g,$$

use the above proposition, switch the sums and sum the exponentially decaying tails in Ω, provided that the range of p given by the proposition and by (1.10) overlap, which is true since the upper restriction on $1/p$ is the same in both cases.

1.3. **Estimate in the Interior Region.** In the radial case, there is an integral formula for the solution to the Cauchy problem (1.1) which goes back to Lamb [5]: Let

$$\mu := \frac{r^2+s^2-t^2}{2rs}, \qquad \nu := \sqrt{\lambda^2+a}. \tag{1.17}$$

Then

$$u(r,t) = \int_0^\infty \frac{1}{(rs)^{\frac{n-1}{2}}} \left\{ L_\nu(\mu) g(s) + \frac{d}{dt} L_\nu(\mu) f(s) \right\} s^{n-1} ds \tag{1.18}$$

where

$$L_\nu(\mu) := \begin{cases} 0 & \text{if } 1<\mu \\ \frac{1}{2} P_{\nu-\frac12}(\mu) & \text{if } -1<\mu<1 \\ \frac{\cos \pi\nu}{\pi} Q_{\nu-\frac12}(-\mu) & \text{if } \mu<-1 \end{cases} \tag{1.19}$$

Here P_m and Q_m denote the *Legendre functions of degree m of the first and second kind* respectively. They can be defined for any $m \in \mathbb{C}$ in terms of the hypergeometric function F:

$$P_m(\mu) := F(-m, m+1, 1, \frac{1-\mu}{2}) \tag{1.20}$$

$$Q_m(\mu) := B(\frac12, m+1)\frac{1}{(2\mu)^{m+1}} F(\frac{m+1}{2}, \frac{m+2}{2}, m+\frac32, \frac{1}{\mu^2}) \tag{1.21}$$

Formula (1.18) can be obtained by performing a (not quite justifiable) change of order of integration in (1.8) and using MacDonald's formula [6]. For a rigorous derivation see [3].

Legendre functions of degree m of the first and second kind are solutions of the *Legendre's equation of degree m*:

$$\frac{d}{d\mu}((1-\mu^2)\frac{df}{d\mu}) + m(m+1)f = 0 \tag{1.22}$$

They satisfy many recurrence relations, one of which will be used here:

$$(2m+1)(\mu^2-1)P'_m = m(m+1)(P_{m+1}-P_{m-1}) \tag{1.23}$$

$$(2m+1)(\mu^2-1)Q'_m = m(m+1)(Q_{m+1}-Q_{m-1}) \tag{1.24}$$

We note that the function L_ν defined above is in fact a *weak solution* of Legendre's equation, i.e. it satisfies the equation (1.22), with $m=\nu-\frac12$, on the intervals $(-\infty,-1)$ and $(-1,1)$, while $(1-\mu^2)\frac{dL_\nu}{d\mu}$ is continuous across $\mu=-1$ and vanishes at $\mu=1$ and $\mu=-\infty$. This allows us to integrate by parts in (1.18) using the

Legendre operator without getting boundary terms. It is therefore convenient to define, for $\kappa \neq \pm\frac{1}{2}$ and $\mu \neq \pm 1$,

$$A_\kappa(\mu) := \frac{1}{\kappa^2 - \frac{1}{4}}(\mu^2 - 1)L'_\kappa(\mu) \tag{1.25}$$

The following properties of A_κ are then not hard to deduce from the properties of Legendre functions:

Proposition 1.5. 1. *For $\kappa \neq \pm\frac{1}{2}$,*

$$L_\kappa(\mu) = A'_\kappa(\mu). \tag{1.26}$$

2. *A_κ extends as a continuous function to $(-\infty, \infty)$, i.e.,*

$$\lim_{\mu\to -1^+} A_\kappa(\mu) = \lim_{\mu\to -1^-} A_\kappa(\mu) = \frac{-\cos\nu\pi}{\pi\kappa^2 - \pi/4}, \qquad \lim_{\mu\to 1} A_\kappa(\mu) = 0. \tag{1.27}$$

   Moreover, $A_\kappa \in C^\alpha(\mathbb{R})$ for any $0 < \alpha < 1$.
3. *A_κ has the following asymptotic behavior:*

$$A_\kappa(\mu) = O(|\mu|^{-\kappa+\frac{1}{2}}) \text{ as } \mu \to -\infty. \tag{1.28}$$

   In particular, A_κ is bounded for $\kappa \geq \frac{1}{2}$.
4. *For $\kappa \neq 0, \pm\frac{1}{2}$,*

$$A_\kappa = \frac{1}{2\kappa}\left(L_{\kappa+1} - L_{\kappa-1}\right) \tag{1.29}$$

Proof. (1.26) is the Legendre equation (1.22). (1.27) follows from the known expansions of P and Q near $\mu = 1$ and $\mu = -1$ (See [4, §3.9.2, pp. 163–164]). Hölder continuity of A_κ is assured since its derivative, L_κ has a logarithmic singularity at $\mu = -1$. (1.28) is a consequence of (1.21), and (1.29) follows from the recurrence formulae for P and Q, (1.23), (1.24). □

The idea for obtaining the dispersive estimate (1.3) is to integrate by parts in (1.18), first using (1.26) and then the recursion (1.29), to put the required number of derivatives on the data, before estimating the resulting kernel in L^∞.

Remark 1.6. *We note that the requirement κ not being zero or $\pm\frac{1}{2}$ means that there are some exceptional cases that need a separate argument, in particular the dispersive estimate for the wave equation without a potential cannot be obtained in this way! It is possible to treat these exceptional cases by other methods, such as using the fact that in the even dimensional case the Legendre functions involved for the exceptional values of ν are in fact polynomials. We have chosen not do so here in the interest of brevity.*

To do the integration by parts, the variable of integration in (1.18) should first be changed to μ. From (1.17) we have $s(\mu, r, t) = r\mu + \sqrt{r^2(\mu^2 - 1) + t^2}$ and therefore

$$\frac{\partial s}{\partial \mu} = r + \frac{r^2\mu}{\sqrt{r^2(\mu^2-1)+t^2}} = \frac{rs}{\sqrt{r^2(\mu^2-1)+t^2}}$$

For simplicity we first assume $f \equiv 0$. Then letting

$$G(\mu, r, t) := g(s)s^{\frac{n-1}{2}}\frac{\partial s}{\partial \mu}$$

we have from (1.18) and the above Proposition that

$$\begin{aligned} r^{\frac{n-1}{2}}u(r,t) &= \int_{-\infty}^{\infty} L_\nu(\mu) G(\mu,r,t)\ d\mu \\ &= -\int A_\nu(\mu)\frac{\partial G}{\partial \mu}\ d\mu \\ &= \frac{1}{2\nu}\int \{A_{\nu+1}(\mu) - A_{\nu-1}(\mu)\}\frac{\partial^2 G}{\partial \mu^2}\ d\mu \\ &= \frac{(-1)^3}{4\nu}\int \{\frac{A_{\nu+2}(\mu)}{\nu+1} - \frac{2\nu A_\nu(\mu)}{\nu^2-1} + \frac{A_{\nu-2}(\mu)}{\nu-1}\}\frac{\partial^3 G}{\partial \mu^3}\ d\mu \\ &= \dots \\ &= \int \frac{\partial^j G}{\partial \mu^j} B_j(\mu) d\mu \end{aligned}$$

where

$$B_j(\mu) := \sum_{i=1}^{j} (-1)^{i+1} c_{ji} A_{\nu+j+1-2i}(\mu)$$

where c_{ji} are constants depending on ν. Thus $B_1 = A_\nu$ and $B_j' = B_{j-1}$. As a result, $B_j \in C^{j-1,\alpha}$ for all $0 < \alpha < 1$. We also note that the above integration by parts is valid as long as

$$\nu - j + 1 \notin \{\frac{1}{2}, 0, -\frac{1}{2}\}. \tag{1.30}$$

This gives us the exceptional cases mentioned in Theorem 1.1.

For $k = 1, 2, \dots$ let

$$s_k := \frac{\partial^k s}{\partial \mu^k}.$$

Then we have, by repeated application of Leibniz and chain rule, that

$$\frac{\partial^j G}{\partial \mu^j} = \sum_{i=0}^{j} \left\{ \partial_s^{j-i} g\ s^{\frac{n-1}{2}-i} \sum_{\alpha\in\mathcal{A}} c_\alpha^i s^{\alpha_0} s_1^{\alpha_1} \dots s_{j+1}^{\alpha_{j+1}} \right\}, \tag{1.31}$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{j+1})$ is a multi-index with non-negative integer components,

$$\mathcal{A} := \{\alpha | \sum_{k=0}^{j+1} \alpha_k = \sum_{k=0}^{j+1} k\alpha_k = j+1\}, \tag{1.32}$$

and c_α^i are numerical constants that can be determined recursively. In particular, we note for future use that the first term $i = 0$ in (1.31), which contains the highest order derivative, is simply

$$\partial_s^j g\ s^{\frac{n-1}{2}} s_1^{j+1} \tag{1.33}$$

Now setting

$$\tau := \frac{t}{r},$$

we have that

$$s_1 = \frac{\partial s}{\partial \mu} = (\mu^2 + \tau^2 - 1)^{-1/2} s, \tag{1.34}$$

so that s_1/s is a function of (μ, τ). By differentiating the above with respect to μ we see that the same holds for s_k/s, $k \geq 2$. Moreover,

$$|s_k| \leq C(\mu^2 + \tau^2 - 1)^{-k/2} s. \tag{1.35}$$

Set

$$\delta := \min\{\nu, \lambda\} + \frac{1}{2}.$$

Thus δ is the amount of t-decay in (1.3). We have, using (1.32), that

$$\begin{aligned} r^{\frac{n-1}{2}-\delta} t^\delta u(r,t) &= \tau^\delta \int \frac{\partial^j G}{\partial \mu^j} B_j(\mu) s_1^{-1} ds && (1.36) \\ &= \sum_{i=0}^{j} \int s^{-i}\ \partial_s^{j-i} g\ s^{j-\frac{n-1}{2}}\ h_{i,j}(\tau,\mu)\ s^{n-1} ds, && (1.37) \end{aligned}$$

where

$$h_{i,j} := \tau^\delta (\mu^2 + \tau^2 - 1)^{1/2} B_j(\mu) \sum_{\alpha \in \mathcal{A}} c_\alpha^i \left(\frac{s_1}{s}\right)^{\alpha_1} \cdots \left(\frac{s_{j+1}}{s}\right)^{\alpha_{j+1}}.$$

We then have the following

Lemma 1.7. *For all integers $j \geq \delta$, there is a constant C (depending on n and ν) such that*

$$\sup_{\substack{2 \leq \tau < \infty \\ -\infty < \mu \leq 1}} |h_{i,j}(\tau,\mu)| \leq C.$$

for all $0 \leq i \leq j$.

Proof. We have, by virtue of (1.32) and (1.35), that

$$|h_{i,j}| \leq C\tau^\delta B_j(\mu)(\mu^2 + \tau^2 - 1)^{-j/2}.$$

From (1.28) and the definition of B_j we have that $|B_j(\mu)| \leq C$ for $\mu \geq -2$, while $B_j(\mu) = O(|\mu|^{-\nu+j-\frac{1}{2}})$ as $\mu \to -\infty$. Hence, for $\mu \geq -2$ the result follows from $j \geq \delta$, while for $\mu < -2$,

$$\begin{aligned} |h_{i,j}| &\leq C\tau^\delta (\mu^2 + \tau^2 - 1)^{-\delta/2} (\mu^2 + \tau^2 - 1)^{-j/2+\delta/2} |\mu|^{-\nu+j-1/2} \\ &\leq C|\mu|^{\min\{\lambda,\nu\}-\nu} \\ &\leq C \end{aligned}$$

again using $j \geq \delta$. □

From (1.36) and by the above lemma, we have that, if n is odd, and thus $j = \frac{n-1}{2}$,

$$\begin{aligned} r^{\frac{n-1}{2}-\delta} t^\delta |u(r,t)| &\leq \left| \int \partial_s^j g\ h_{0,j} s^{n-1} ds \right| + \sum_{i=1}^{j} \left| \int s^{-i} h_{i,j} \partial_s^{j-i} g\ s^{n-1} ds \right| \\ &\leq C\|g\|_{\dot{B}_1^{j,1}(\mathbb{R}^n)} + C \sum_{i=1}^{j} \|s^{-i}\|_{L^{n/i,\infty}(\mathbb{R}^n)} \|\partial_s^{j-i} g\|_{L^{n/(n-i),1}(\mathbb{R}^n)} \\ &\leq C\|g\|_{\dot{B}_1^{j,1}(\mathbb{R}^n)}, \end{aligned}$$

where we have used the Hölder inequality for Lorenz spaces due to O'Neil [7] and the embedding of Besov into Lorenz spaces [2]. This establishes the dispersive estimate (1.3) for n odd and at least 3, in the region $\tau \geq 2$.

Suppose now that n is even and at least 4, and set $j = \frac{n}{2}$, and note that $j \geq \delta + \frac{1}{2}$. From (1.36) we then have

$$r^{\frac{n-1}{2}-\delta} t^\delta |u(r,t)| \leq C \sum_{i=0}^{j} \left| \int_0^\infty s^{-i}\ h_{i,j} s^{1/2}\ \partial_s^{j-i} g\ s^{n-1} ds \right| \qquad (1.38)$$

Consider first the term $i = 0$:

$$I_0 = \int_0^{t+r} h_{0,j}(\mu, \tau) s^{1/2} \partial_s^j g(s) s^{n-1} ds.$$

From (1.33) we have that

$$h_{0,j} = \tau^\delta (\mu^2 + \tau^2 - 1)^{-j/2} B_j(\mu),$$

and by Lemma 1.7, $h_{0,j}$ is bounded uniformly in τ.

Let $\phi_0 : \mathbb{R}^+ \to [0,1]$ be a smooth cut-off function which is zero for $x \geq 1$ and $\phi_0(x) \equiv 1$ for $0 \leq x \leq \frac{1}{2}$. Let $\phi(s) := \phi_0(s/s_0)$, where s_0 is the value of s corresponding to $\mu = -\tau$, i.e. $s_0 := r(\sqrt{2\tau^2 - 1} - \tau)$. Thus we have

$$I_0 = \int_0^{t+r} (1 - \phi) h_{0,j} \sqrt{s} \partial_s^j g s^{n-1} ds + \int_0^{s_0} \phi h_{0,j} \sqrt{s} \partial^j g s^{n-1} ds := I_1 + I_2.$$

Consider first I_1. We need to shave half a derivative off the data. The idea is to exploit the duality of Besov spaces $\dot{B}_\infty^{\frac{1}{2},\infty}$ (which is the same as $\dot{C}^{\frac{1}{2}}$) and $\dot{B}_1^{-\frac{1}{2},1}$. We thus need the $\frac{1}{2}$-Hölder norm of $(1 - \phi) h_{0,j} \sqrt{s}$ to be bounded on the interval $[s_0/2, t + r]$, uniformly in τ for $\tau \geq 2$. An additional complication is that the radial derivative ∂_s is not well-behaved on negative regularity Besov spaces, so we need to rewrite it in terms of the gradient operator. For the other piece I_2, the plan is to first integrate by parts back in s once to reduce the number of derivatives on the data by one, and then use the duality between $\dot{B}_1^{\frac{1}{2},1}$ and $\dot{B}_\infty^{-\frac{1}{2},\infty}$ to put half a derivative more on the data.

We begin by observing

$$\begin{aligned} \|(1-\phi)h_{0,j}\sqrt{s}\|_{\dot{C}^{1/2}} &\leq \|h_{0,j}\|_{L^\infty} \left\{ \|\sqrt{s}\|_{\dot{C}^{1/2}} + \|1-\phi\|_{\dot{C}^{1/2}} \|\sqrt{s}\|_{L^\infty} \right\} \\ &\quad + \|h_{0,j}\|_{\dot{C}^{1/2}} \|\sqrt{s}\|_{L^\infty} \\ &:= J_1 + J_2, \end{aligned}$$

where all the norms are on the annular region $s_0/2 \leq s \leq t + r$. We obviously have $\|\sqrt{s}\|_{\dot{C}^{1/2}} \leq C$, $\|1 - \phi\|_{\dot{C}^{1/2}} \leq C/\sqrt{s_0} \leq Cr^{-1/2}\tau^{-1/2}$, and $\|\sqrt{s}\|_{L^\infty} = \sqrt{t+r} \leq Cr^{1/2}\tau^{1/2}$. Thus from Lemma 1.7 we have that $J_1 < C$. On the other hand

$$\begin{aligned} \|h_{0,j}\|_{\dot{C}^{1/2}} &\leq \tau^\delta \Big\{ \|B_j(\mu)\|_{\dot{C}^{1/2}} \|(\mu^2 + \tau^2 - 1)^{-j/2}\|_{L^\infty} \\ &\qquad + \|B_j(\mu)\|_{L^\infty} \|(\mu^2 + \tau^2 - 1)^{-j/2}\|_{\dot{C}^{1/2}} \Big\} \\ &:= \tau^\delta (K_1 + K_2). \end{aligned}$$

Therefore $J_2 \leq Cr^{1/2}\tau^{\delta + 1/2}(K_1 + K_2)$. We have

$$\begin{aligned} \|B_j(\mu(\cdot, r, t))\|_{\dot{C}^{1/2}([s_0/2, t+r])} &\leq \|\mu(\cdot, r, t)\|_{\dot{C}^1([s_0/2, t+r])}^{1/2} \|B_j(\cdot)\|_{\dot{C}^{1/2}([-\tau, 1])} \\ &\leq Cr^{-1/2} \max\{1, \tau^{-\nu + j - 1}\}, \end{aligned}$$

and thus $K_1 < Cr^{-1/2}\tau^{-j} \max\{1, \tau^{-\nu+j-1}\}$, while similarly

$$\begin{aligned} &\|(\mu^2(\cdot, r, t) + \tau^2 - 1)^{-j/2}\|_{\dot{C}^{1/2}([s_0/2, t+r])} \leq \\ &\qquad Cr^{-1/2} \|(\mu^2 + \tau^2 - 1)^{-j/2}\|_{\dot{C}^{1/2}([-\tau, 1])} \leq Cr^{-1/2}\tau^{-j-1/2}, \end{aligned}$$

and

$$\|B_j(\mu)\|_{L^\infty} \leq C \max\{1, \tau^{-\nu + j - 1/2}\}$$

so that $K_2 \le Cr^{-1/2}\tau^{-j-1/2}\max\{1,\tau^{-\nu+j-1/2}\}$. Hence

$$J_2 \le C\tau^{\delta+1/2-j}\max\{1,\tau^{-\nu+j-1}\} \le C\max\{\tau^{\delta-\lambda-1/2},\tau^{\delta-\nu-1/2}\} \le C.$$

and we have shown that

$$\|(1-\phi)h_{0,j}\sqrt{s}\|_{\dot{B}^{1/2,\infty}_{\infty}(A^{t+r}_{s_0/2})} \le C$$

where $A^{t+r}_{s_0/2}$ is the annular region $s_0/2 \le s \le t+r$ in $\mathbb{R}^n$. We note that $\frac{t+r}{s_0/2} \le 10$ and thus in the following we can use scaling to compute the norms on a fixed annulus $A = A_1^{10}$. Therefore,

$$\begin{aligned} |I_1| &\le Cs_0^{(n+1)/2}\|\partial_s^j g\|_{\dot{B}_1^{-1/2,1}(A)} \\ &= Cs_0^{(n+1)/2}\|\frac{x^i}{s}\partial_i\partial_s^{j-1}g\|_{\dot{B}_1^{-1/2,1}(A)} \\ &\le Cs_0^{(n+1)/2}\|\partial_i\partial_s^{j-1}g\|_{\dot{B}_1^{-1/2,1}(A)} \\ &\le Cs_0^{(n+1)/2}\|\partial_s^{j-1}g\|_{\dot{B}_1^{1/2,1}(A)} \le C\|g\|_{\dot{B}_1^{\frac{n-1}{2},1}(\mathbb{R}^n)}. \end{aligned}$$

Here we have used that multiplication by x^i/s is a bounded operator on the Besov space $\dot{B}_1^{-1/2,1}(A)$, as can be easily checked by duality.

We next consider I_2. Integrating by parts once, we have

$$\begin{aligned} I_2 &= \int_0^{s_0}\partial_s^j g(s)\phi h_{0,j}s^{n-1/2}ds \\ &= \int_0^{s_0}\partial_s^{j-1}g(s)\frac{1}{\sqrt{s}}H(\tau,\mu)s^{n-1}ds \end{aligned}$$

where

$$\begin{aligned} H(\tau,\mu) := &-\tau^\delta(\mu^2+\tau^2-1)^{-j/2+1/2}\Big\{[-j\phi\mu(\mu^2+\tau^2-1)^{-1} \\ &+((n-\frac{1}{2})\phi+s\phi')(\mu^2+\tau^2-1)^{-1/2}]B_j(\mu)+\phi B_{j-1}(\mu)\Big\}. \end{aligned}$$

On the region of integration, $\mu \le -\tau$, and the expression inside the braces is $O(|\mu|^{-\nu+j-3/2})$. We thus have that

$$\begin{aligned} |H(\tau,\mu)| &\le C\tau^\delta(\mu^2+\tau^2-1)^{-j+1/2}|\mu|^{-\nu+j-3/2} \\ &\le C\tau^{1/2}|\mu|^{\delta-\nu-1} \le C. \end{aligned}$$

Since $s^{-1/2} \in L^{2n,\infty}(\mathbb{R}^n)$ and $H \in L^\infty$, we have $s^{-1/2}H \in L^{2n,\infty}$, and hence

$$\begin{aligned} |I_2| &\le C\|\partial_s^{j-1}g\|_{L^{2n/(2n-1),1}(\mathbb{R}^n)} \\ &\le C\|\partial_s^{j-1}g\|_{\dot{B}_1^{1/2,1}(\mathbb{R}^n)} \\ &\le C\|g\|_{\dot{B}_1^{\frac{n-1}{2},1}(\mathbb{R}^n)}. \end{aligned}$$

Finally, for the terms $i \ge 1$ in (1.38) we again make use of the O'Neil inequality to bound them by the same right hand side as above. This establishes the dispersive estimate in the $n \ge 4$ even case, and concludes the proof of Theorem 1.1.

Appendix: A counterexample for the free wave equation. The dispersive estimate is quite sensitive to the choice of norms. In particular the estimate

$$\|u(t)\|_{L^\infty(\mathbf{R}^n)} \le \frac{C}{t^{(n-1)/2}} \left(\|(-\Delta)^{\frac{n+1}{4}} f\|_{L^1(\mathbf{R}^n)} + \|(-\Delta)^{\frac{n-1}{4}} g\|_{L^1(\mathbf{R}^n)} \right)$$

is false in all dimensions n for the *free* wave equation initial value problem

$$-\partial_t^2 u + \Delta u = 0, \qquad u(0) = f, \qquad \partial_t u(0) = g.$$

In fact the estimate is false even in the case of radial symmetry. In order to construct a counter-example we will use the function

$$h_\delta(r) := \begin{cases} 0 & \text{if } 0 < r < 1 - 2\delta, \\ r\frac{1}{c_n \delta r^{n-1}} & \text{if } 1 - 2\delta \le r \le 1 - \delta, \\ 0 & \text{if } 1 - \delta < r < \infty. \end{cases}$$

Here $c_n = 2\pi^{n/2}\Gamma(\frac{n}{2})^{-1}$ is the volume of the unit sphere in $\mathbf{R}^n$. We note that

$$\|h_\delta\|_{L^1(\mathbf{R}^n)} = 1.$$

The construction of a counterexample depends on the congruence class of n modulo 4. If $n \not\equiv 3$ then we set

$$f_\delta := (-\Delta)^{-\frac{n+1}{4}} h_\delta, \qquad g = 0.$$

Then

$$\|(-\Delta)^{\frac{n+1}{4}} f_\delta\|_{L^1(\mathbf{R}^n)} = 1, \qquad \|(-\Delta)^{\frac{n-1}{4}} g_\delta\|_{L^1(\mathbf{R}^n)} = 0$$

We will now show that for any C there is a $\delta > 0$ such that

$$\|u_\delta(1, \cdot)\|_{L^\infty(\mathbf{R}^n)} > C$$

We begin by observing that

$$u_\delta(t, r) = \int_0^\infty \int_0^\infty r^{-\lambda} s^{-\lambda} \omega^{-\frac{n+1}{2}} \cos(\omega t) J_\lambda(r\omega) J_\lambda(s\omega) h_\delta(s) s^{n-1} ds\, \omega\, d\omega$$

satisfies the given initial value problem.[1] We claim that $u_\delta(1, r)$ is continuous, and in fact smooth, for $r < \delta$. Indeed by elliptic regularity f_δ is smooth except on the spheres of radius $1 - 2\delta$ and $1 - \delta$. By propagation of singularities $u_\delta(1, \cdot)$ is smooth except possibly on the spheres of radius δ, 2δ, $2 - 2\delta$, and $2 - \delta$. It then follows that

$$\|u_\delta(1, \cdot)\|_{L^\infty(\mathbf{R}^n)} > |u(1, 0)|.$$

Since

$$\lim_{r \to 0+} r^{-\lambda} J_\lambda(r\omega) = 2^{-\lambda}\Gamma(\frac{n}{2})^{-1} \omega^\lambda,$$

we find that

$$u_\delta(1, 0) = 2^{-\lambda}\Gamma(\frac{n}{2})^{-1} \int_0^\infty \int_0^\infty s^{-\lambda} \omega^{-1/2} \cos(\omega) J_\lambda(s\omega) h_\delta(s) s^{n-1} ds\, d\omega$$

or, since

$$\cos(\omega) = \sqrt{\frac{\pi\omega}{2}} J_{-1/2}(\omega),$$

$$u_\delta(1, 0) = 2^{-\lambda + \frac{1}{2}} \pi^{1/2} \Gamma(\frac{n}{2})^{-1} \int_0^\infty \int_0^\infty s^{-\lambda} J_{-1/2}(\omega) J_\lambda(s\omega) h_\delta(s) s^{n-1} ds\, d\omega.$$

[1]This will satisfy the wave equation only in the sense of distributions. It is not a classical solution. It is possible, by choosing a more complicated h, to construct a counterexample which is a classical solution.

We now switch the order of integration and recall the definition of h_δ,

$$u_\delta(1,0) = \frac{1}{(2\pi)^{\frac{n-1}{2}}}\frac{1}{\delta}\int_{1-2\delta}^{1-\delta}\int_0^\infty s^{-\lambda}J_{-1/2}(\omega)J_\lambda(s\omega)\,ds\,d\omega.$$

We then use the following special case of Schafheitlin's integral [9]

$$\int_0^\infty J_{-1/2}(\omega)J_{\frac{n-2}{2}}(s\omega)\,d\omega = \frac{\Gamma(\frac{n-1}{4})}{\Gamma(1-\frac{n+1}{4})\Gamma(\frac{n}{2})}s^{\frac{n-2}{2}}F(\frac{n-1}{4},\frac{n+1}{4};\frac{n}{2};s^2)$$

for $s^2 < 1$ and standard identities for the Gamma function to obtain the representation

$$u_\delta(1,0) = \frac{1}{(2\pi)^{\frac{n-1}{2}}}\Gamma(\frac{n}{2})^{-1}\sin((n+1)\frac{\pi}{4})\frac{1}{\delta}\int_{1-2\delta}^{1-\delta}\int_0^\infty F(\frac{n-1}{4},\frac{n+1}{4};\frac{n}{2};s^2)\,ds.$$

Using the singular case of the transit relations for hypergeometric functions we obtain the representation

$$\begin{aligned} u_\delta(1,0) \quad = \quad & \frac{1}{(2\pi)^{\frac{n-1}{2}}}\sin((n+1)\frac{\pi}{4})\frac{1}{\delta}\int_{1-2\delta}^{1-\delta}\sum_{k=0}^{\infty}\frac{(\frac{n-1}{4})_k(\frac{n+1}{4})_k}{(k!)^2}\Big[\log(1-s^2) \\ & + \psi(k+\frac{n-1}{4}) + \psi(k+\frac{n+1}{4}) - 2\psi(k+1)\Big]\,ds \end{aligned}$$

($\psi(z)$ is the logarithmic derivative of the Gamma function). All the terms other than $k = 0$ contribute nothing to the limit $\delta \to 0$ and we see that

$$\begin{aligned} \lim_{\delta\to 0}\Big\{ u_\delta(1,0) - & \frac{1}{(2\pi)^{\frac{n-1}{2}}}\sin((n+1)\frac{\pi}{4})\Big[\log\delta + 3\log 2 \\ & -1 + \psi(\frac{n-1}{4}) + \psi(\frac{n+1}{4}) - 2\psi(1)\Big]\Big\} \quad = \quad 0. \end{aligned}$$

It follows that

$$\lim_{\delta\to 0}\|u(1,\cdot)\|_{L^\infty(\mathbf{R}^n)} = \infty.$$

A similar argument applies to the case $n \not\equiv 1$ modulo 4. In this case we take initial conditions

$$f_\delta = 0, \qquad g_\delta = (-\Delta)^{-\frac{n-1}{4}}h,$$

and proceed as above.

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