

**A Bessel function based proof
that the Euler-Mascheroni constant γ is irrational**

Klaus Braun
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euler-mascheroni-constant.de

Dedicated to my son Mario
on the occasion of his 30th birthday

We shall consider generalized Bessel functions in the forms

$$B_\nu(x) := \frac{J_\nu(2x)}{x^\nu} - \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^k (1-x^k)}{k! \Gamma(\nu+k+1)}$$

where

$$\frac{J_\nu(2x)}{x^\nu} := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k}}{\Gamma(\nu+k+1)}.$$

Remark (SiC): for $\nu \in \mathbb{C}$, and α algebraic with $\alpha \neq \pm \frac{1}{2}, -1, \pm \frac{3}{2}, -2 \dots$, $J_\nu(\alpha)$ are transcendental.

Lemma: it holds

- i) $\gamma = \int_0^\infty [J_0(2x) - J_0(2\sqrt{x})] \frac{dx}{x} = \int_0^\infty B_0(x) \frac{dx}{x}$
- ii) $\int_0^\infty x^\mu B_\nu(x) \frac{dx}{x} = \frac{1}{\mu} \left[\frac{\Gamma(1+\frac{\mu}{2})}{\Gamma(1+\nu-\frac{\mu}{2})} - \frac{\Gamma(1+\mu)}{\Gamma(1+\nu-\mu)} \right]$ for $0 < \operatorname{Re}(\mu) < \operatorname{Re}(\nu) + \frac{3}{2}$
- iii) $B_\beta(x) - B_\alpha(x) > 0$ for $0 \leq \alpha < \beta$, $0 \leq x < 1$.

Proof: lemmata 1-3 below.

Corollary: Putting $\mu := \nu = \frac{1}{n}$ and

$$a_n := \int_0^\infty x^{1/n} B_{1/n}(x) \frac{dx}{x} = \int_0^1 \left[x^{1/n} B_{1/n}(x) + x^{-1/n} B_{1/n}\left(\frac{1}{x}\right) \right] \frac{dx}{x}$$

it follows

- i) $a_1 = 0$, $a_\infty = \int_0^\infty B_0(x) \frac{dx}{x} = \gamma$
- ii) $a_n = n \left[1 - \Gamma\left(1 + \frac{1}{n}\right) \right] < a_{n+1}$ (*)
- iii) all a_n are transcendental for $n > 1$.

(*) (AbM) p. 255: for $1 \leq x \leq 2$ it holds

- i) $\max\{\Gamma(x)\} = \Gamma(1) = \Gamma(2) = 1$,
- ii) $\Gamma(x)$ is convex with minimum at $\Gamma(3/2) \sim 0,88622\dots = 1 - 0,11378\dots$
- iii) $1/\Gamma(x)$ is concave with maximum at $\frac{1}{\Gamma(3/2)} \sim 1 + 0,11378\dots$

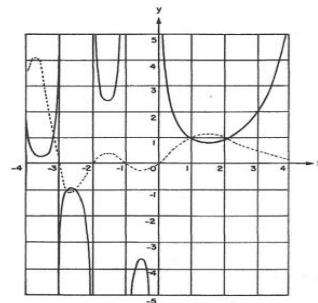


FIGURE 6.1. Gamma function. *
——, $y = \Gamma(x)$, - - - - , $y = 1/\Gamma(x)$

Lemma 1: It holds $\gamma = \int_0^\infty [J_0(2x) - J_0(2\sqrt{x})] \frac{dx}{x}$.

Proof: In (BrR) it is proven that $\int_0^\infty [e^{-t} - J_0(2t)] \frac{dt}{t} = 0$. The formula is derived from the related Mellin transforms formula, (WaG) 13-24, (WeH),

$$\int_0^\infty t^{\mu-1} J_0(2t) dt = \frac{\frac{\mu}{2} \Gamma(\frac{\mu}{2})}{\Gamma(1-\frac{\mu}{2})} \text{ for } 0 < \text{Re}(\mu) < 3/2$$

in combination with the asymptotics

$$\Gamma(\mu) - \frac{\frac{1}{2} \Gamma(\frac{\mu}{2})}{\Gamma(1-\frac{\mu}{2})} = \frac{1}{\mu} \left[\Gamma(1+\mu) - \frac{\Gamma(1+\frac{\mu}{2})}{\Gamma(1-\frac{\mu}{2})} \right] = \frac{1}{\mu} \left[(1-\gamma\mu) - \frac{(1-\frac{\gamma\mu}{2})}{(1+\frac{\gamma\mu}{2})} + O(\mu^2) \right] = O(\mu) \rightarrow_{\mu \rightarrow 0^+} 0.$$

Analogue it follows that $\int_0^\infty [e^{-t} - J_0(2\sqrt{t})] \frac{dt}{t} = \gamma$. The corresponding asymptotics formula is given by

$$\Gamma(\mu) - \frac{\Gamma(\mu)}{\Gamma(1-\mu)} = \frac{1}{\mu} \left[\Gamma(1+\mu) - \frac{\Gamma(1+\mu)}{\Gamma(1-\mu)} \right] = \frac{1}{\mu} \left[(1-\gamma\mu) - \frac{(1-\gamma\mu)}{(1+\gamma\mu)} + O(\mu^2) \right] = \gamma + O(\mu) \rightarrow_{\mu \rightarrow 0^+} \gamma,$$

which proves lemma 1.

The Mellin transforms of $B_\nu(x)$ are given by

Lemma 2: it holds

$$\int_0^\infty x^\mu B_\nu(x) \frac{dx}{x} = \frac{1}{\mu} \left[\frac{\Gamma(1+\frac{\mu}{2})}{\Gamma(1+\nu-\frac{\mu}{2})} - \frac{\Gamma(1+\mu)}{\Gamma(1+\nu-\mu)} \right] \text{ for } 0 < \text{Re}(\mu) < \text{Re}(\nu) + \frac{3}{2}.$$

Proof: Applying the formula, (BrR), (WaG) 13-24, (WeH),

$$\int_0^\infty x^\mu \frac{J_\nu(x)}{x^\nu} \frac{dx}{x} = \frac{\Gamma(\frac{\mu}{2})}{2^{\nu-\mu+1} \Gamma(1+\nu-\frac{\mu}{2})} \text{ for } 0 < \text{Re}(\mu) < \text{Re}(\nu) + \frac{3}{2}$$

provides the following representations for the two terms of $B_\nu(x) = \frac{J_\nu(2x)}{x^\nu} - \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}}$:

$$i) \quad \int_0^\infty x^\mu \frac{J_\nu(2x)}{x^\nu} \frac{dx}{x} = \frac{\frac{1}{2} \Gamma(\frac{\mu}{2})}{\Gamma(\nu+1-\frac{\mu}{2})} = \frac{1}{\mu} \frac{\Gamma(1+\frac{\mu}{2})}{\Gamma(1+\nu-\frac{\mu}{2})}$$

$$ii) \quad \frac{1}{2} \int_0^\infty x^{\mu/2} \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}} \frac{dx}{x} = \frac{\frac{1}{2} \Gamma(\frac{\mu}{2})}{\Gamma(1+\nu-\frac{\mu}{2})} \text{ resp. } \int_0^\infty x^\mu \frac{J_\nu(2\sqrt{x})}{x^{\nu/2}} \frac{dx}{x} = \frac{\Gamma(\mu)}{\Gamma(1+\nu-\mu)} = \frac{1}{\mu} \frac{\Gamma(1+\mu)}{\Gamma(1+\nu-\mu)}.$$

From the representation

$$B_\nu(x) = \sum_{k=0}^\infty (-1)^{k+1} \frac{x^k (1-x^k)}{k! \Gamma(\nu+k+1)}$$

one gets

Lemma 3: for $0 \leq \alpha < \beta$, $0 \leq x < 1$ it holds $B_\beta(x) - B_\alpha(x) > 0$.

Proof: lemma 3 follows from the equations

$$\begin{aligned} B_\beta(x) - B_\alpha(x) &= \sum_{k=0}^\infty (-1)^{k+1} \frac{x^k (1-x^k)}{k! \Gamma(\beta+k+1)} - \sum_{k=0}^\infty (-1)^{k+1} \frac{x^k (1-x^k)}{k! \Gamma(\alpha+k+1)} \\ &= \sum_{k=0}^\infty (-1)^k \frac{x^k (1-x^k)}{k!} \left[\frac{1}{\Gamma(\alpha+k+1)} - \frac{1}{\Gamma(\beta+k+1)} \right] > 0. \end{aligned}$$

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